Canad. Math. Bull. Vol. 15 (2), 1972

IDENTITIES IN CATEGORIES

BY
H. HERRLICH AND C. M. RINGEL

In [4] Hatcher introduced the notion of an identity in an arbitrary category and proved a characterization of quasivarietal subcategories which is similar to Birkhoff's characterization of varietal subcategories in universal algebra. The aim of this note is to show that the theorem of Hatcher as well as the categorical generalization of Birkhoff's theorem are special cases of a "relative" theorem, formulated with respect to a projective structure.

We recall some definitions:

(1) (Hatcher). If X and Y are objects of a category **C** and Y belongs to a class **Y** of objects, a pair (k_1, k_2) of morphisms in $\mathbf{C}(X, Y)$ is called a **Y**-*identity* (or also an identity). An object A is said to *satisfy the identity* (k_1, k_2) if for every morphism $a: Y \to A$ we have the equality $ak_1 = ak_2$. If **K** is a class of identities, the *subcategory defined by* **K** is the full subcategory of all objects of **C** which satisfy the identities of **K**.

(2) (Birkhoff, Malcev). A full subcategory **A** of **C** is called *quasivarietal* if **A** is closed under products and subobjects; **A** is called *verital* if **A** is quasivarietal and closed under regular quotients.

(3) (Eilenberg and Moore, Maranda). If **P** is a class of objects, **E** a class of morphisms, the pair (**P**, **E**) is a *projective structure*, if the following conditions are satisfied:

(i) for $P \in \mathbf{P}$ and $e \in \mathbf{E}$, $\mathbf{C}(P, e)$ is surjective,
(ii) if $\mathbf{C}(X, e)$ is surjective for every $e \in \mathbf{E}$, then X belongs to **P**,
(iii) if $\mathbf{C}(P, y)$ is surjective for every $P \in \mathbf{P}$, then y belongs to **E**,
(iv) for every object C of **C** there is a morphism $P \to C$ in **E** with $P \in \mathbf{P}$.

An example of a projective structure is given by $(|\mathbf{C}|, \mathbf{R})$, where $|\mathbf{C}|$ is the class of all objects of **C**, and **R** is the class of all retractions. In many categories we get another example considering the class of regular epimorphisms:

(4) The category **C** *has regular projectives* if $(\mathbf{P}_r, \mathbf{E}_r)$ is a projective structure, where $\mathbf{E}_r$ is the class of regular epimorphisms and $\mathbf{P}_r$ is the class of regular projective objects, i.e., objects X for which $\mathbf{C}(X, -)$ preserves regular epimorphisms.

Each "varietal" category in the sense of Linton has regular projectives. More general each "algebraic" category (i.e. each category **C** which is complete and cocomplete and has a regular projective regular separator) has regular projectives (see [7], [6], [3], and [5]).

(5) If **Y** is a class of morphisms, a subcategory **A** is *closed under* **Y**-*quotients* if for every morphism $y: A \to B$ in **Y**, $A \in$ **A** implies $B \in$ **B**. If **Y** is the class of all epimorphisms (regular epimorphisms, retractions) then the assertion that **A** is closed under **Y**-quotients means that **A** is closed under quotients (regular quotients, retracts).

Let **C** denote a complete and extremally-cowell-powered category in which every morphism has a factorization in a regular epimorphism followed by a monomorphism.

THEOREM. *Let* (**P**, **E**) *be a projective structure on* **C** *and* **A** *a full subcategory.* **A** *is defined by a class of* **P**-*identities iff* **A** *is quasivarietal and closed under* **E**-*quotients.*

Proof. Assume that **A** is defined by a class **K** of **P**-identities. It is easy to see that **A** is a quasivariety, so we only show that **A** is closed under **E**-quotients. Let $e: A \to B$ be a morphism in **E** with $A \in$ **A**. If (k_1, k_2) is a **P**-identity, say $k_i: X \to P$ with $P \in$ **P**, and $b: P \to B$ is arbitrary, we find a morphism a with $ea=b$ (because $P \in$ **P** and $e \in$ **E**). Now A belongs to **A**, so $ak_1=ak_2$, but this implies

$$bk_1 = eak_1 = eak_2 = bk_2,$$

so $B \in$ **A**.

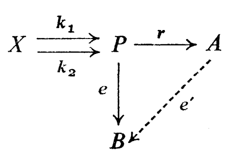
For the converse, assume that **A** is quasivarietal and closed under **E**-quotients. Let **K** be the class of all **P**-identities satisfied by all objects of **A**. If **B** is the subcategory defined by **K**, then **A**$\subseteq$**B**. So we have to show that any object B of **B** belongs to **A**. For B there exists a morphism $e: P \to B$ in **E** with $P \in$ **P**. **A** is a quasivariety, so **A** is a reflective subcategory and the reflection morphisms are regular epimorphisms. If r: P $\to A$ is the reflection morphism of P into **A**, and (k_1, k_2) is the congruence-relation (=kernel pair) of r, then (k_1, k_2) belongs to **K**: for given a morphism $a: P \to A'$ with A' in **A**, we can factorize a through r, say $a=a'r$. But the equality $rk_1=rk_2$ then implies

$$ak_1 = a'rk_1 = a'rk_2 = ak_2.$$

We have assumed that B belongs to **B**, so (k_1, k_2) is satisfied for B, that is, we have the equality

$$ek_1 = ek_2. \tag{*}$$

But r is a regular epimorphism, so (r, A) is the coequalizer of (k_1, k_2), and (*) implies that e factors through r, $e=e'r$.



But with e also e' belongs to $\mathbf{E}$, so X is the codomain of the morphism $e' : A \to B$ in $\mathbf{E}$ and $A \in \mathbf{A}$ implies $B \in \mathbf{A}$. This proves the theorem.

For the projective structure $(|\mathbf{C}|, \mathbf{R})$ we get:

COROLLARY 1 (Hatcher). *A full subcategory* $\mathbf{A}$ *is defined by a class of identities iff* $\mathbf{A}$ *is quasivarietal.*

If $(\mathbf{P}_r, \mathbf{E}_r)$ is a projective structure, we get:

COROLLARY 2. *If* $\mathbf{C}$ *has regular projectives, a full subcategory* $\mathbf{A}$ *is defined by a class of* $\mathbf{P}_r$*-identities iff* $\mathbf{A}$ *is varietal.*

Since any algebraic category satisfies the hypotheses of the last corollary (including the assumptions on $\mathbf{C}$) we get the following generalization of Birkhoff's theorem:

COROLLARY 3. *In an algebraic category a full subcategory* $\mathbf{A}$ *is defined by a class of* $\mathbf{P}$*-identities iff* $\mathbf{A}$ *is varietal.*

REFERENCES

1. G. Birkhoff, *On the structure of abstract algebras*, Proc. Cambridge Philos. Soc. **31** (1935), 433–454.

2. S. Eilenberg and J. C. Moore, *Foundation of relative homological algebra*, Memoirs Amer. Math. Soc., **55**, 1965.

3. W. Felscher, *Kennzeichnung von primitiven und quasiprimitiven Kategorien von Algebren*, Arch. Math. **19** (1968), 390–397.

4. W. S. Hatcher, *Quasiprimitive subcategories*, Math. Ann. **190** (1970), 93–96.

5. H. Herrlich, *Algebraic categories, An axiomatic approach*, Manuscript.

6. J. R. Isbell, *Normal completions of categories*, Reports of the Midwest Category Seminar. Lecture Notes **47** (1967), 110–155.

7. F. W. Lawvere, *Functorial semantics of algebraic theories*, Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 869–872.

8. F. E. J. Linton, *Some aspects of equational categories*, Proc. Conf. categ. algebra, La Jolla, 1965; Berlin (1966), 84–94.

9. A. Malcev, *On the general theory of algebraic systems* (Russian), Mat. Sb. **35** (1954), 3–20.

10. J. Maranda, *Injective structures*, Trans. Amer. Math. Soc. **110** (1964), 98–135.

MATHEMATISCHES INSTITUT DER UNIVERSITÄT,
D. 48 BIELEFELD, GERMANY

CARLETON UNIVERSITY,
OTTAWA, ONTARIO