Canad. Math. Bull. Vol. **28** (2), 1985

# ZERO MULTIPLIERS OF BERGMAN SPACES

BY
SHELDON AXLER

ABSTRACT. This paper proves that if $p < s$, then 0 is the only function that multiplies a Bergman $L^p$ space into a Bergman $L^s$ space.

Fix a positive integer $N$, and let $G$ be an open, connected, nonempty subset of $\mathbb{C}^N$. Let $dA$ denote the usual Lebesgue measure on $\mathbb{C}^N$, normalized so that the unit ball has measure 1. Let $w$ be a positive continuous function defined on $G$, and we consider the Lebesgue spaces $L^p(G, w\,dA)$ of complex valued functions $g$ defined on $G$ such that

$$\int_G |g|^p \, w \, dA < \infty.$$

For $0 < p \leq \infty$, the Bergman space $L^p_a(G, w\,dA)$ is defined by

$$L^p_a(G, w\,dA) = \{g \in L^p(G, w\,dA)\colon g \text{ is analytic on } G\}.$$

Let $p < s$. Then the only function on $G$ which multiplies $L^p(G, w\,dA)$ into $L^s(G, w\,dA)$ is 0 (see Proposition 1). If $g$ is a function on $G$ which only multiplies the Bergman space $L^p_a(G, w\,dA)$ into $L^s(G, w\,dA)$, then $g$ need not be zero (for precise conditions on $g$ for the case where $G$ is a polydisk, see the Theorem in [3]). But what if $g$ is also required to be analytic? Can we conclude that the only analytic multiplier of $L^p_a(G, w\,dA)$ into $L^s(G, w\,dA)$ is zero? Clearly we need to eliminate the possibility that the spaces involved are trivial, so from now on we assume that $G$ and $w$ are such that $L^p_a(G, w\,dA)$ has dimension greater than 1 for each $0 < p < \infty$. The main result of this paper (Theorem 4) is that 0 is the only analytic function multiplying $L^p_a(G, w\,dA)$ into $L^s(G, w\,dA)$.

A major tool used in the proof of Theorem 4 is the Fredholm alternative from operator theory. Except for the case where $G$ has a very smooth boundary and $w$ is well behaved, I have been unable to prove Theorem 4 without using operator theory. It seems that using the Fredholm alternative allows one to avoid dealing with the problems that arise from the geometry of $G$.

If $p \geq 1$, then $L^p(G, w\,dA)$ is a Banach space; this fails for $p < 1$. For fixed $p$, whether or not an analytic function is in $L^p_a(G, w\,dA)$ depends upon the growth rate

Received by the editors May 29, 1984.
AMS Subject Classification (1980): 30H05, 32A30, 46E10.
The author was supported in part by the National Science Foundation.




https://doi.org/10.4153/CMB-1985-029-1 Published online by Cambridge University Press

of the function near the boundary of $G$, so in this context the distinction between $p < 1$ and $p \geq 1$ seems unnatural. Thus it is worthwhile to do the small amount of extra work necessary to allow $p$ to be any positive number. I would like to thank Joel Shapiro for supplying the reference which shows that the Fredholm alternative is valid even where $p < 1$.

This paper studies questions of when $gL_a^p(G, w\, dA)$ is contained in $L_a^s(G, w\, dA)$ for $p < s$; for comparison we give some references to the case where $p \geq s$. For $p = s$, it is well known that $gL_a^p(G, w\, dA)$ is contained in $L_a^p(G, w\, dA)$ if and only if $g$ is a bounded analytic function on $G$; see for example Lemma 11 of [2]. Information concerning what happens when $p > s$ can be found in [1] and [4].

The following proposition is presented for purposes of motivation and comparison; it deals with measurable functions, while Theorem 4 deals with analytic functions.

PROPOSITION 1. *Suppose* $0 < p < s \leq \infty$ *and* $g$ *is a complex valued function defined on* $G$ *such that*

$$gL^p(G, w\, dA) \subset L^s(G, w\, dA).$$

*Then* $g = 0$ *almost everywhere on* $G$.

PROOF. Clearly $g$ is measurable. If the conclusion is false, then there is a positive number $t$ such that the set $G_t$ defined by

$$G_t = \{z \in G : |g(z)| > t\}$$

has positive measure. Now

$$(g|G_t)L^p(G_t, w\, dA) \subset L^s(G_t, w\, dA),$$

and so

$$L^p(G_t, w\, dA) \subset L^s(G_t, w\, dA).$$

However, by Theorem 1 of [6], this is impossible, and thus we are done.

For $0 < p < \infty$ and $g \in L^p(G, w\, dA)$, define $\|g\|_p$ by

$$\|g\|_p = \left(\int_G |g|^p\, w\, dA\right)^{1/p}.$$

For $f, g \in L^p(G, w\, dA)$, the distance $d(f, g)$ from $f$ to $g$ is defined to be $\|f - g\|_p$ for $1 \leq p < \infty$ and $\|f - g\|_p^p$ for $0 < p < 1$. As is well known, $d$ defines a metric on $L^p(G, w\, dA)$ which makes $L^p(G, w\, dA)$ (and thus $L_a^p(G, w\, dA)$) into a topological vector space. The following lemma shows that the functions in each bounded subset of $L_a^p(G, w\, dA)$ are uniformly bounded on each compact subset of $G$.

LEMMA 2. *Let* $0 < p < \infty$, *and let* $K$ *be a compact subset of* $G$. *Then there is a constant* $c < \infty$ *such that*

$$|f(z)| \leq c\|f\|_p \text{ for all } f \text{ in } L_a^p(G, w\, dA).$$

PROOF. Temporarily fix $z \in K$, and let $0 < r < \infty$ be such that the closed ball $V$ of radius $r$ centered at $z$ lies in $G$. Let $b$ denote the supremum of $w^{-1/p}$ over $V$. If $f$ is analytic on $G$, then $|f|^p$ is subharmonic, so (see [9], Proposition 1.5.4 and equation (2) on page 20)

$$\begin{aligned}|f(z)| &\leq r^{-N/p}\left(\int_V |f|^p \, dA\right)^{1/p}\\ &\leq br^{-N/p}\left(\int_V |f|^p \, w \, dA\right)^{1/p}\\ &\leq br^{-N/p}\,\|f\|_p.\end{aligned}$$

Since $K$ is compact, we can choose an $r$ for each $z$ in $K$ so that $br^{-N/p}$ remains bounded, giving the desired result.

A linear map $M$ from a topological vector space $X$ to a topological vector space $Y$ is called compact if there is an open set $E$ in $X$ containing 0 such that the closure of $M(E)$ in $Y$ is compact. Compact operators will play a crucial role in the proof of Theorem 4.

For $f$ and $h$ analytic functions on $G$, define the multiplication operator $M_h$ by $M_h(f) = hf$. It will be clear from the context which spaces we intend to be the domain and range of $M_h$.

Suppose that $s$, $t$, and $p$ are positive numbers such that $(1/s) + (1/t) = (1/p)$. Let $f \in L^s(G, w\, dA)$ and $h \in L^t(G, w\, dA)$. Then a slight generalization of Hölder's inequality (see [5], pages 84–85) shows that $hf \in L^p(G, w\, dA)$ and

$$\|hf\|_p \leq \|h\|_t \|f\|_s.$$

This inequality shows that $M_h$ is a continuous map from $L^s(G, w\, dA)$ to $L^p(G, w\, dA)$. The following lemma shows that far more is true if $h$ is analytic and we restrict the domain to the Bergman space $L^s_a(G, w\, dA)$.

LEMMA 3. *Let $0 < p < s \leq \infty$, and let $0 < t < \infty$ be such that $(1/s) + (1/t) = (1/p)$, and let $h$ be a function in $L^t_a(G, w\, dA)$. Then*

$$M_h : L^s_a(G, w\, dA) \to L^p_a(G, w\, dA)$$

*is compact.*

PROOF. We will see that the image under $M_h$ of the unit ball of $L^s_a(G, w\, dA)$ has a compact closure in $L^p_a(G, w\, dA)$. To do this, let $\{f_n\}$ be a sequence in the unit ball of $L^s_a(G, w\, dA)$. We need to show that $\{hf_n\}$ has a subsequence which is convergent in $L^p_a(G, w\, dA)$.

By Lemma 2, $\{f_n\}$ is uniformly bounded on each compact subset of $G$, and so $\{f_n\}$ is a normal family. Thus there is an analytic function $f$ defined on $G$ and a subsequence of $\{f_n\}$ (for convenience, replace $\{f_n\}$ with the subsequence) such that $f_n$ converges to $f$ uniformly on each compact subset of $G$. In the case where $s = \infty$, we clearly have $f \in L^s_a(G, w\, dA)$. For the case where $s < \infty$, Fatou's Lemma shows that

$$\int_G |f|^s\, w\, dA = \int_G \lim |f_n|^s\, w\, dA \leq \text{lim-inf} \int_G |f_n|^s\, w\, dA \leq 1,$$

so $f \in L^s_a(G, w\, dA)$ in either case.

Now let $\epsilon$ be a positive number. Let $K$ be a compact subset of $G$ such that

$$\int_{G \sim K} |h|^t\, w\, dA < \epsilon.$$

Since $f_n$ tends to $f$ uniformly on $K$, there is a positive integer $M$ such that

$$\int_K |h(f_n - f)|^p\, w\, dA < \epsilon \text{ for all } n > M.$$

Let $n > M$. Then

$$\begin{aligned}\int_G |hf_n - hf|^p\, w\, dA &= \int_K |h(f_n - f)|^p\, w\, dA + \int_{G \sim K} |hf_n - hf|^p\, w\, dA \\ &< \epsilon + \left(\int_{G \sim K} |h|^t\, w\, dA\right)^{p/t} \|f_n - f\|_s^p \\ &\leq \epsilon + \epsilon^{p/t}\|f_n - f\|_s^p.\end{aligned}$$

Thus we see that $hf_n$ converges to $hf$ in $L^p_a(G, w\, dA)$, and so we are done.

We are now ready to prove the main result of this paper.

THEOREM 4. *Let* $0 < p < s \leq \infty$, *and let g be an analytic function defined on G such that*

$$gL^p_a(G, w\, dA) \subset L^s_a(G, w\, dA).$$

*Then* $g = 0$.

PROOF. First we need to verify that $L^p_a(G, w\, dA)$ is a complete metric space. So let $\{f_n\}$ be a Cauchy sequence in $L^p_a(G, w\, dA)$. From Lemma 2, we see that for each compact set $K \subset G$, the sequence $\{f_n|K\}$ is a Cauchy sequence in the space of continuous functions on $K$. Thus there is an analytic function $f$ defined on $G$ such that $f_n$ converges to $f$ uniformly on each compact subset of $G$. As in the proof of Lemma 3, Fatou's Lemma implies that $f \in L^p_a(G, w\, dA)$. Now for $\epsilon > 0$, let $M$ be such that

$$\int_G |f_m - f_n|^p\, w\, dA < \epsilon \text{ for all } n, m > M.$$

If $n > M$, then another application of Fatou's Lemma shows that

$$\int_G |f - f_n|^p\, w\, dA \leq \text{lim-inf}_m \int_G |f_m - f_n|^p\, w\, dA \leq \epsilon.$$

Thus $f_n$ converges to $f$ in $L^p_a(G, w\, dA)$, and so $L^p_a(G, w\, dA)$ is complete.

Consider the multiplication operator

$$M_g : L_a^p(G, w\, dA) \to L_a^s(G, w\, dA).$$

Since we know that $L_a^p(G, w\, dA)$ and $L_a^s(G, w\, dA)$ are complete, the Closed Graph Theorem (see [8], Theorem 2.15, for a version that applies when $p < 1$) can be used to show that $M_g$ is continuous. To do this, suppose $f_n$ converges to $f$ in $L_a^p(G, w\, dA)$ and $gf_n$ converges to $v$ in $L_a^s(G, w\, dA)$. By Lemma 2, $f_n$ converges pointwise to $f$ on $G$, and similarly $gf_n$ converges pointwise to $v$ on $G$. Thus $v = gf$, and so $M_g$ is continuous.

Let $0 < t < \infty$ be such that $(1/s) + (1/t) = (1/p)$. Let $h$ be a nonzero function in $L_a^t(G, w\, dA)$. By Lemma 3, the multiplication operator

$$M_h : L_a^s(G, w\, dA) \to L_a^p(G, w\, dA)$$

is compact. Let $E$ be an open subset of $L_a^s(G, w\, dA)$ containing 0 such that $M_h(E)$ has compact closure in $L_a^p(G, w\, dA)$. Let $F = M_g^{-1}(E)$. Since $M_g$ is continuous, $F$ is an open subset of $L_a^p(G, w\, dA)$ containing 0. Also, $(M_h M_g)(F)$ is contained in $M_h(E)$, and so

$$M_{gh} = M_h M_g : L_a^p(G, w\, dA) \to L_a^p(G, w\, dA)$$

is a compact operator.

Let $f$ be a nonzero function in $L_a^p(G, w\, dA)$. Suppose the conclusion of the theorem is false, so $g$ is also nonzero. Fix a point $z$ in $G$ such that $g$, $f$, and $h$ are all nonzero at $z$. Every function in the range of the multiplication operator

$$M_{g(z)h(z)-gh} : L_a^p(G, w\, dA) \to L_a^p(G, w\, dA)$$

is zero at $z$. In particular, $f$ is not in the range of this operator, so $M_{g(z)h(z)-gh}$ is not onto. Since $M_{gh}$ is a compact operator, $M_{g(z)h(z)-gh}$ is a nonzero scalar times a compact perturbation of the identity operator. But $M_{g(z)h(z)-gh}$ is not onto, so the Fredholm alternative (which holds even if $p < 1$; see [10], Theorem 1) implies that $M_{g(z)h(z)-gh}$ is not injective. However, every nonzero multiplication operator is clearly injective on $L_a^p(G, w\, dA)$, so $g(z)h(z) - gh$ must be identically zero.

So $gh$ is a constant function for each $h \in L_a^t(G, w\, dA)$. Since the dimension of $L_a^t(G, w\, dA)$ is greater than 1, this can happen only if $g = 0$. Thus we have completed the proof.

In addition to its use in the proof of Theorem 4, Lemma 3 has another interesting application. The classical Hardy spaces $H^p$ of the unit disk in the complex plane have the property that if $p \neq s$, then there is an infinite dimensional subspace $X$ of $H^p \cap H^s$ which is closed in both $H^p$ and $H^s$. For example, take $X$ to be the set of functions in $H^1$ whose Taylor coefficients vanish outside a fixed lacunary sequence (see [7], page 203, for a clean statement of the theorem needed for this example). The following theorem shows that in this respect the Bergman spaces behave differently from the Hardy spaces.

https://doi.org/10.4153/CMB-1985-029-1 Published online by Cambridge University Press

THEOREM 5. *Suppose that* $\int_G w\,\mathrm{d}A < \infty$. *Let* $0 < p < s \le \infty$, *and let* $X$ *be a subspace of* $L^p_a(G, w\,\mathrm{d}A) \cap L^s_a(G, w\,\mathrm{d}A)$ *which is closed in both* $L^p_a(G, w\,\mathrm{d}A)$ *and* $L^s_a(G, w\,\mathrm{d}A)$. *Then* $X$ *is finite dimensional.*

PROOF. Let $X_p$ (respectively, $X_s$) denote $X$ with the topology it inherits as a subspace of $L^p_a(G, w\,\mathrm{d}A)$ (respectively, $L^s_a(G, w\,\mathrm{d}A)$). Applying Lemma 3 with $h = 1$ shows that the inclusion of $L^s_a(G, w\,\mathrm{d}A)$ into $L^p_a(G, w\,\mathrm{d}A)$ is a compact operator. Thus there is an open subset $E \subset L^s_a(G, w\,\mathrm{d}A)$ with $0 \in E$ such that the closure of $E$ in $L^p_a(G, w\,\mathrm{d}A)$ is compact. Thus the closure of $E \cap X_p$ is compact in $X_p$.

As in the proof of Theorem 4, the Closed Graph Theorem implies that the identity map from $X_p$ to $X_s$ is continuous. Thus $E \cap X_p$ is open in $X_p$. Thus $X_p$ is a locally compact topological vector space, and so by Theorem 1.22 of [8], we can conclude that $X$ is finite dimensional, as desired.

## REFERENCES

1. Rohan Attele, *Analytic multipliers of Bergman spaces*, to appear in Michigan Math. J.
2. P. L. Duren, B. W. Romberg, and A. L. Shields, *Linear functionals on* $H^p$ *spaces with* $0 < p < 1$, J. reine angew. Math. **238** (1969), pp. 32–60.
3. William W. Hastings, *A Carleson measure theorem for Bergman spaces*, Proc. AMS **52** (1975), pp. 237–241.
4. V. L. Oleinik and B. S. Pavlov, *Embedding theorems for weighted classes of harmonic and analytic functions*, J. Soviet Math. **2** (1974), pp. 135–142; a translation of Zap. Nauch. Sem. LOMI Steklov **22** (1971), pp. 94–102.
5. Michael Reed and Barry Simon, *Methods of Modern Mathematical Physics*, Vol. 1, Functional Analysis, Academic Press, New York, 1972.
6. Juan L. Romero, *When is* $L^p(u)$ *contained in* $L^q(u)$? Amer. Math. Monthly **90** (1983), pp. 203–206.
7. Walter Rudin, *Trigonometric series with gaps*, J. Math. Mech. **9** (1960), pp. 203–227.
8. Walter Rudin, *Functional Analysis*, McGraw-Hill Book Co., New York, 1973.
9. Walter Rudin, *Function Theory on the Unit Ball of* $\mathbb{C}^n$, Springer-Verlag, New York, 1980.
10. J. H. Williamson, *Compact linear operators in linear topological spaces*, J. LMS **29** (1954), pp. 149–156.

DEPARTMENT OF MATHEMATICS
MICHIGAN STATE UNIVERSITY
EAST LANSING, MI 48824 USA