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SUFFICIENCY CONDITIONS FOR THE EXISTENCE OF TRANSVERSALS

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1. Introduction. A *transversal* of a family of non-empty sets $\mathscr{F} = \langle F_\nu : \nu \in I \rangle$ is a 1-1 map

$$\varphi : I \to S(\mathscr{F}) = \bigcup_{\nu \in I} F_\nu$$

such that $\varphi(\nu) \in F_\nu$ $(\nu \in I)$. A number of problems in combinatorial mathematics reduce to the question of whether or not a certain family of sets has a transversal. An up-to-date account of this theory is to be found in the book by Mirsky [**9**]. The best known result of this kind is the following theorem.

THEOREM. *If $\mathscr{F} = \langle F_\nu : \nu \in I \rangle$ is either a finite family or an arbitrary family of finite sets, then $\mathscr{F}$ has a transversal if and only if*

$$(1.1) \qquad \left| \bigcup_{\nu \in J} F_\nu \right| \geqslant |J|$$

holds for all finite sets $J \subset I$.

This was proved for finite $\mathscr{F}$ by P. Hall [**7**] (and in an equivalent graph theoretical formulation by J. König [**8**]) and for an arbitrary family of finite sets by M. Hall [**6**]. We shall refer to (1.1) as Hall's condition. If $\mathscr{F}$ is an infinite family with infinite sets, then the problem of finding necessary and sufficient conditions for the existence of a transversal assumes a different complexity and remains unsolved. Rado and Jung [**12**] observed that if $\mathscr{F}$ has just one infinite member, say F_{ν_0}, then there is a transversal if and only if (1.1) holds and

$$F_{\nu_0} \not\subset \bigcup_{J \in \mathscr{C}} \bigcup_{\nu \in J} F_\nu$$

where $\mathscr{C}$ is the set of critical subsets of I, i.e., $J \in \mathscr{C}$ if and only if J is a finite subset of I for which equality holds in (1.1). Brualdi and Scrimger [**3**] and Folkman [**5**] considered the more general problem of a family containing an arbitrary finite number of infinite sets. More recently, Nash-Williams [**10**] conjectured a condition which is both necessary and sufficient for an arbitrary *countable* family of sets to have a transversal, and this was proved by Damerell and Milner [**4**]. The conditions given by these authors are not so easily stated and the reader is referred to the original papers.

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That there can be no entirely elementary set of conditions which are necessary and sufficient for an arbitrary family of sets to have a transversal may perhaps be illustrated by considering the two families

$$\mathscr{F}_1 = \langle \alpha + 1 : \omega \leqq \alpha < \omega_1 \rangle \quad \text{and} \quad \mathscr{F}_2 = \langle \alpha : \omega \leqq \alpha < \omega_1 \rangle.$$

Here ω denotes the first infinite ordinal, ω_1 the first uncountable ordinal and an ordinal $\alpha = \{\beta : \beta < \alpha\}$ is regarded as the set of all smaller ordinals. Clearly $\mathscr{F}_1$ has a transversal since $\alpha \in \alpha + 1$. However, $\mathscr{F}_2$ has no transversal. For, if $\varphi(\alpha) \in \alpha$ ($\omega \leqq \alpha < \omega_1$), then by a theorem of Alexandroff and Urysohn [**1**] on regressive functions, there is some $\gamma < \omega_1$ such that $\varphi(\alpha) = \gamma$ for uncountably many $\alpha < \omega_1$. The family $\mathscr{F}_2$ gives a partial answer to [**9**, Problem 3, p. 220].) It is difficult to imagine any criterion involving inequalities between cardinals of sets which will be delicate enough to distinguish between the families $\mathscr{F}_1$ and $\mathscr{F}_2$.

In view of the difficulty just mentioned it seems of interest therefore to have conditions which, though not necessary, are at least sufficient to ensure the existence of a transversal in a family having infinite members. In this connection Professor L. Mirsky asked if the following condition (which is a kind of dual of the finiteness condition in M. Hall's theorem) is sufficient for the existence of a transversal: *each member of* $\mathscr{F}$ *is infinite and each element* $x \in S(\mathscr{F})$ *belongs to only a finite number of sets* $F \in \mathscr{F}$.

If $\mathscr{F} = \langle F_\nu : \nu \in I \rangle$ is a family, we write $F \in \mathscr{F}$ if $F = F_\nu$ for some $\nu \in I$. The cardinality of the family is $|\mathscr{F}| = |I|$. For any set A, put $\mathscr{F}(A) = \langle F_\nu : \nu \in I,\ A \cap F_\nu \neq \emptyset \rangle$ and write $\mathscr{F}(x)$ instead of $\mathscr{F}(\{x\})$. Mirsky's question is answered affirmatively by the following theorem.

THEOREM 1. *If the family of nonempty sets* $\mathscr{F}$ *satisfies*

(1.2) $|F| \geqq |\mathscr{F}(x)|$ *for all* $F \in \mathscr{F}$ *and* $x \in S(\mathscr{F})$,

then $\mathscr{F}$ *has a transversal.*

Dr. C. J. Knight conjectured that the following, more local type of condition, is also sufficient for a transversal. We write $\mathscr{F} \in \mathscr{K}$ if and only if the members of $\mathscr{F}$ are nonempty and

(1.3) $|F| \geqq |\mathscr{F}(F)| \quad (F \in \mathscr{F})$.

The main result proved in this paper settles Knight's conjecture.

THEOREM 2. *If* $\mathscr{F} \in \mathscr{K}$, *then* $\mathscr{F}$ *has a transversal.*

A common weakening of the conditions (1.2) and (1.3) is the condition

(1.4) $|F| \geqq |\mathscr{F}(x)| \quad (x \in S(\mathscr{F}),\ F \in \mathscr{F}(x)$ i.e., $x \in F \in \mathscr{F})$.

We write $\mathscr{F} \in \mathscr{L}$ if the members of $\mathscr{F}$ are nonempty and (1.4) is satisfied. Thus a strengthening of both Theorems 1 and 2 is

THEOREM 3. *If* $\mathcal{F} \in \mathcal{L}$, *then* $\mathcal{F}$ *has a transversal.*

Suppose $\mathcal{F} = \langle F_\nu : \nu \in I \rangle \in \mathcal{L}$. Let J be a finite set, $J \subset I$, and let $\mathcal{F}'$ be the sub-family $\langle F_\nu : \nu \in J \rangle$. For $p \in \{1, 2, \ldots, |J|\}$, put

$$n_p = |\{\nu \in J : |F_\nu| = p\}|, \quad m_p = |\{x \in S(\mathcal{F}') : |\mathcal{F}'(x)| = p\}|.$$

Considering the number of pairs (x, F) with $x \in F \in \mathcal{F}'$, $|F| \leqq p$, we obtain by (1.4) the inequality

$$n_1 + 2n_2 + \ldots + pn_p \leqq m_1 + 2m_2 + \ldots + pm_p \quad (1 \leqq p \leqq |J|).$$

It follows that

$$n_1 + n_2 + \ldots + n_p \leqq m_1 + m_2 + \ldots + m_p \quad (1 \leqq p \leqq |J|),$$

and hence (1.1) holds. It follows from this that $\mathcal{L}$ is a sufficient condition for a family of finite sets to have a transversal. The conditions $\mathcal{L}$ and $\mathcal{K}$ are easily seen to be equivalent if all the members of $\mathcal{F}$ are infinite sets and therefore, $\mathcal{L}$ is also sufficient (by Theorem 2) for a family of infinite sets to have a transversal. In an early version of this paper we left Theorem 3 as an open question since we could not prove the special case

(1.5) *if* $\mathcal{F}$ *is a countable family of countable sets and* $\mathcal{F} \in \mathcal{L}$, *then* $\mathcal{F}$ *has a transversal.*

In fact, (1.5) and Theorem 2 implies the complete result stated as Theorem 3 (see § 6). Shelah [**13**] has since proved (1.5) and a simpler proof of this result is given in [**2**]. In § 7 we prove an even stronger result (Theorem 4).

Theorem 3 has an interesting formulation in terms of bipartite graphs. A bipartite graph is a triple $\Gamma = \langle X, \Delta, Y \rangle$ with vertex set $X \cup Y$ (X, Y disjoint sets) and edge set $\Delta \subset \{\{x, y\} : x \in X, y \in Y\}$. Let $v(z) = |\{u \in X \cup Y : \{u, z\} \in \Delta\}|$ $(z \in X \cup Y)$ be the valency function of Γ. Then Theorem 3 is equivalent to the following statement: *If* $\Gamma = \langle X, \Delta, Y \rangle$ *is a bipartite graph such that* $v(x) > 0$ *for* $x \in X$ *and* $v(x) \geqq v(y)$ *whenever* $x \in X$, $y \in Y$ *and* $\{x, y\} \in \Delta$, *then there is a matching from* X *into* Y, *i.e. there is a* 1-1 *function* $\varphi : X \rightarrow Y$ *such that* $\{x, \varphi(x)\} \in \Delta$ $(x \in X)$.

2. Notation. Capital letters denote sets and the cardinal power of A is $|A|$. Small Latin and Greek letters denote ordinal numbers unless stated otherwise. As usual, an ordinal α is the set $\{\beta : \beta < \alpha\}$ of all smaller ordinals. A cardinal number is an initial ordinal, i.e., α is a cardinal if $\beta < \alpha \Rightarrow |\beta| < |\alpha|$. The letters κ, λ, μ always denote infinite cardinals. κ^+ is the successor cardinal of κ.

$\mathcal{F}$ will always denote the family of non-empty sets $\langle F_\nu : \nu \in I \rangle$ with index set I. We write $|\mathcal{F}| = |I|$ and $S(\mathcal{F}) = \bigcup_{\nu \in I} F_\nu$. We shall abuse the usual terminology of sets by applying it to families of sets, but this should not lead to any confusion. Thus, we write $A \in \mathcal{F}$ if $A = F_\nu$ for some $\nu \in I$. We write

$A, B \in \mathscr{F}, A \neq B$ to mean that A, B are *different* members of $\mathscr{F}$, i.e., $A = F_\mu$, $B = F_\nu$ and $\mu \neq \nu$ (even though we may have $A = B$ in the usual set theoretical sense). $\mathscr{F}' = \langle F_\nu : \nu \in I' \rangle$ is a *subfamily* of $\mathscr{F}$, and we write $\mathscr{F}' \subset \mathscr{F}$, if $I' \subset I$; in this case we also write $\mathscr{F} - \mathscr{F}' = \langle F_\nu : \nu \in I - I' \rangle$. We write $\mathscr{F}'' \subset\subset \mathscr{F}$ if $\mathscr{F}'' = \langle G_\nu : \nu \in I \rangle$ and $G_\nu \subset F_\nu (\nu \in I)$. The family $\mathscr{F}' = \langle F_\nu : \nu \in I' \rangle$ is *disjoint* from $\mathscr{F}$ if $I \cap I' = \emptyset$; it is *strongly disjoint* from $\mathscr{F}$ if it is disjoint and in addition $S(\mathscr{F}) \cap S(\mathscr{F}') = \emptyset$. If $\mathscr{F}, \mathscr{F}'$ are disjoint, then $\mathscr{F} \cup \mathscr{F}' = \langle F_\nu : \nu \in I \cup I' \rangle$.

A *transversal* of $\mathscr{F}$ is an 1-1 function $\varphi : I \rightarrow S(\mathscr{F})$ such that $\varphi(\nu) \in F_\nu$ $(\nu \in I)$. Let Trans $(\mathscr{F})$ be the set of all transversals of $\mathscr{F}$. If $\varphi \in$ Trans $(\mathscr{F})$, $\psi \in$ Trans $(\mathscr{F}')$, then range $(\varphi) = \{\varphi(\nu) : \nu \in I\}$ and φ, ψ are said to be *disjoint* if range $(\varphi) \cap$ range $(\psi) = \emptyset$. Thus, if $\mathscr{F}, \mathscr{F}'$ are disjoint families and φ, φ' are disjoint transversals of $\mathscr{F}$ and $\mathscr{F}'$ respectively, then $\varphi \cup \varphi' \in$ Trans $(\mathscr{F} \cup \mathscr{F}')$.

For $A \subset S(\mathscr{F})$, let $\mathscr{F}(A)$ denote the subfamily of $\mathscr{F}$,

$$\mathscr{F}(A) = \langle F_\nu : \nu \in I, A \cap F_\nu \neq \emptyset \rangle.$$

In particular, for a singleton we write $\mathscr{F}(x)$ instead of $\mathscr{F}(\{x\})$. $\mathscr{F}$ has property $\mathscr{K}$, $\mathscr{F} \in \mathscr{K}$, if and only if

(2.1) $|F| \geqq |\mathscr{F}(F)| \quad (F \in \mathscr{F})$,

and $\mathscr{F} \in \mathscr{L}$ if and only if

$$|F| \geqq |\mathscr{F}(x)| \quad (x \in S(\mathscr{F}), F \in \mathscr{F}(x) \text{ i.e., } x \in F \in \mathscr{F}).$$

If λ is an infinite cardinal we write

$$\mathscr{F}^\lambda = \langle F_\nu : \nu \in I, |F_\nu| = \lambda \rangle.$$

$\mathscr{F}^{<\lambda}$, $\mathscr{F}^{\leqq\lambda}$, $\mathscr{F}^{>\lambda}$, $\mathscr{F}^{\geqq\lambda}$ are similarly defined. For $x \in S(\mathscr{F})$ put

$$\rho_{\mathscr{F}}(x) = \inf \{|F| : F \in \mathscr{F}(x)\}.$$

Thus $\mathscr{F} \in \mathscr{L}$ if and only if $\rho_{\mathscr{F}}(x) \geqq |\mathscr{F}(x)|$ $(x \in S(\mathscr{F}))$. We usually write $S = S(\mathscr{F})$, and then

$$S^\lambda = \{x \in S : \rho_{\mathscr{F}}(x) = \lambda\}.$$

$S^{<\lambda}$, $S^{\leqq\lambda}$ are similarly defined.

A λ-*component* of $\mathscr{F}$ is a minimal non-empty subfamily $\mathscr{H} \subset \mathscr{F}^{\leqq\lambda}$ such that

$$A \in \mathscr{H}, B \in \mathscr{F}^{\leqq\lambda}(A) \Rightarrow B \in \mathscr{H}.$$

Let $\mathscr{F}^{\leqq\lambda} = \langle F_\nu : \nu \in I_\lambda \rangle$. Consider the graph $\mathscr{G}_\lambda$ on the index set I_λ in which $\{\rho, \sigma\}$ is an edge if and only if $\rho, \sigma \in I_\lambda$, $\rho \neq \sigma$ and $F_\rho \cap F_\sigma \neq \emptyset$. Then $\mathscr{H} = \langle F_\nu : \nu \in J \rangle$ is a λ-component of $\mathscr{F}$ exactly when J is the vertex set of a connected component of the graph $\mathscr{G}_\lambda$. Two different λ-components of $\mathscr{F}$ are strongly disjoint subfamilies of $\mathscr{F}$. A *large* λ-*component* of $\mathscr{F}$ is a minimal non-empty subfamily $\mathscr{H} \subset \mathscr{F}$ such that

$$A \in \mathscr{H}, A \cap B \cap S^{\leqq\lambda} \neq \emptyset \Rightarrow B \in \mathscr{H}.$$

Thus every set $A \in \mathscr{F}$ is a member of a large λ-component of $\mathscr{F}$; two large λ-components are disjoint subfamilies of $\mathscr{F}$ but they are not in general strongly disjoint.

If $\mathscr{F} \in \mathscr{L}$, then for any $\lambda \geqq \omega$, the valency of a vertex ν in the graph $\mathscr{G}_\lambda$ described above is at most λ and hence the vertex set of a connected component has cardinality at most λ, i.e. if $\mathscr{H}$ is a λ-component of $\mathscr{F}$, then $|\mathscr{H}| \leqq \lambda$.

Suppose $\mathscr{F}$ is a family of sets such that (2.1) holds and

(2.2) $\quad |A \cap S^{\leqq\lambda}| \leqq \lambda \quad$ for $A \in \mathscr{F}$.

Now (2.1) implies that each element $x \in S^{\leqq\lambda}$ is a member of at most λ different sets of the family $\mathscr{F}$. Therefore, by (2.2), there are at most $\lambda^2 = \lambda$ different sets $B \in \mathscr{F}$ such that $A \cap B \cap S^{\leqq\lambda} \neq \emptyset$. This implies that every large λ-component of $\mathscr{F}$ also has cardinality at most λ.

The cofinality of the cardinal λ, is the least cardinal $\mu = \text{cf}\ (\lambda)$ such that λ can be expressed as the union of μ subsets each of cardinal less than λ. λ is *regular* if cf $(\lambda) = \lambda$ and *singular* if cf $(\lambda) < \lambda$.

A set of ordinals $C \subset \lambda$ is *stationary* in λ if for every regressive function $f : C \rightarrow \lambda$ (i.e., $f(\gamma) < \gamma$ for $\gamma \in C - \{0\}$), there is γ_0 such that

$$|\{\gamma \in C : f(\gamma) = \gamma_0\}| = \lambda.$$

We use the well-known result (e.g. [**11**]) that if $\lambda > \omega$ is regular then the set $C = \{\gamma < \lambda : \gamma \text{ is a limit ordinal}\}$ is stationary in λ. A set $C \subset \lambda$ is *cofinal* in λ if for every $x \in \lambda$ there is $y \in C$ such that $x \leqq y$.

3. Elementary lemmas and proof of Theorem 1. We need the following well-known fact.

LEMMA 1. *If* $|\mathscr{F}| \leqq \lambda \leqq |F|$ $(F \in \mathscr{F})$, *then there are sets* $g(F) \subset F$ $(F \in \mathscr{F})$ *such that* $|g(F)| = \lambda$ *and* $g(F_1) \cap g(F_2) = \emptyset$ *for* $F_1, F_2 \in \mathscr{F}$ *and* $F_1 \neq F_2$.

Proof. We may assume that $\mathscr{F} = \langle F_\nu : \nu < \alpha \rangle, \alpha \leqq \lambda$. Let $\langle \nu_\rho : \rho < \lambda \rangle$ be any sequence of ordinals such that $\nu_\rho < \alpha (\rho < \lambda)$ and $|\{\rho < \lambda : \nu_\rho = \nu\}| = \lambda$ $(\nu < \alpha)$. Now by transfinite induction we can choose elements $x_\rho \in F_{\nu_\rho} - \{x_\sigma : \sigma < \rho\}$ and the lemma holds with $g(F_\nu) = \{x_\rho : \rho < \lambda$ and $\nu_\rho = \nu\}$ $(\nu < \alpha)$.

Since a family of non-empty pairwise disjoint sets obviously has a transversal, we have the following corollary.

COROLLARY 1. *If* $|F| \geqq \lambda \geqq |\mathscr{F}|$ $(F \in \mathscr{F})$, *then* Trans $(\mathscr{F}) \neq \emptyset$.

LEMMA 2. *If* $\mathscr{F} \in \mathscr{K}$ *and* $|\mathscr{F}| \leqq \aleph_0$, *then* Trans $(\mathscr{F}) \neq \emptyset$.

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Remark. The condition $\mathscr{F} \in \mathscr{K}$ can be replaced by the weaker hypothesis $\mathscr{F} \in \mathscr{L}$, but the proof is much more difficult in this case (see [**1**; **13**]).

Proof of Lemma 2. We may assume that $\mathscr{F} = \langle F_i : i < \tau \rangle$, where $\tau \leqq \omega$. Let $n < \tau$ and suppose that elements $\varphi(i) \in F_i$ have been chosen for $i < n$. Since $F_n \in \mathscr{F}(F_n)$ and $\mathscr{F} \in \mathscr{K}$, we have that

$$|\{i < n : F_i \in \mathscr{F}(F_n)\}| < |F_n|$$

and hence there is $\varphi(n) \in F_n - \{\varphi(i) : i < n\}$. This defines a transversal φ of $\mathscr{F}$ by induction.

LEMMA. 3. *Let* $\mathscr{F} \in \mathscr{K}$. *If either* (i) $|F| \leqq \aleph_0$ *for all* $F \in \mathscr{F}$ *or* (ii) $|F| = \lambda$ *for all* $F \in \mathscr{F}$, *then* Trans $(\mathscr{F}) \neq \emptyset$.

Proof. If (i) holds put $\mu = \omega$; if (ii) holds put $\mu = \lambda$. Then $\mathscr{F}$ is the union of its μ-components $\mathscr{G}_i$ $(i \in J)$ which are pairwise strongly disjoint. Since $|\mathscr{G}_i| \leqq \mu$ and $\mathscr{G}_i \in \mathscr{K}$ it follows, from Lemma 2 in the case $\mu = \omega$ and from Corollary 1 in the case $\mu > \omega$, that Trans $(\mathscr{G}_i) \neq \emptyset$. Lemma 3 follows since the $\mathscr{G}_i$ are strongly disjoint.

Proof of Theorem 1. The hypothesis implies that there is a cardinal number m such that $|F| \geqq m \geqq |\mathscr{F}(x)|$ for all $F \in \mathscr{F}$ and $x \in S(\mathscr{F})$. Let F' be any subset of F of power m $(F \in \mathscr{F})$. Then it will be enough to show that the family $\mathscr{F}' = \langle F' : F \in \mathscr{F} \rangle \subset \subset \mathscr{F}$ has a transversal. If m is finite then Trans $(\mathscr{F}') \neq \emptyset$ by Hall's theorem. If m is infinite, then for $F' \in \mathscr{F}'$ and $x \in F'$ we have

$$|\mathscr{F}'(x)| \leqq |\mathscr{F}(x)| \leqq m = |F'|$$

i.e., $\mathscr{F}' \in \mathscr{K}$. Therefore, since the members of $\mathscr{F}'$ all have the same cardinality, it follows from Lemma 3 (ii) that Trans $(\mathscr{F}') \neq \emptyset$.

4. A strengthening of $\mathscr{K}$. It will be convenient to consider the following strengthening of condition $\mathscr{K}$. We write $\mathscr{F} \in \mathscr{K}^+$ if and only if the following three conditions are satisfied:

(i) $\mathscr{F} \in \mathscr{K}$,
(ii) $A \in \mathscr{F}^{>\mu} \Rightarrow |A \cap S^{\leqq\mu}| < \mu$,
(iii) $\lambda > \omega$, $A \in \mathscr{F}^{\lambda}$, $A \cap S^{<\lambda} \neq \emptyset \Rightarrow A \subset S^{<\lambda}$.

It follows from (ii) and (iii) that if $A \in \mathscr{F}^{\lambda}$ and $A \cap S^{<\lambda} \neq \emptyset$, then λ is a limit cardinal.

LEMMA 4. *Let* $\mathscr{F} \in \mathscr{K}^+$, $A \in \mathscr{F}^{\lambda}$, $A \cap S^{<\lambda} \neq \emptyset$. *Then* cf $(\lambda) = \omega$.

Proof. The hypothesis implies that λ is a limit cardinal. Suppose that $\text{cf}(\lambda) = \kappa > \omega$. Let $\langle \lambda_\alpha : \alpha < \kappa \rangle$ be a closed increasing sequence of ordinals with $\lambda = \lim_{\alpha<\kappa}\lambda_\alpha$. By (ii), for each limit ordinal $\alpha < \kappa$ there is an ordinal $f(\alpha) < \alpha$ such that

$$|A \cap S^{\leqq\lambda_\alpha}| \leqq \lambda_{f(\alpha)}.$$

The set of limit ordinals $\alpha < \kappa$ is a stationary subset of κ. Hence there is $\beta < \kappa$ such that $f(\alpha) = \beta$ on some cofinal set $U \subset \kappa$. Since U is cofinal in κ, it follows that

$$|A \cap S^{\leqq \lambda_\alpha}| \leqq \lambda_\beta \quad \text{for all } \alpha < \kappa.$$

By (iii), and the fact that the sets $S^{\leqq \lambda_\alpha}$ increase with α, we have

$$A \subset S^{<\lambda} = \bigcup_{\alpha<\kappa} S^{\leqslant \lambda_\alpha}.$$

This gives the contradiction $|A| \leqq \lambda_\beta^+ < \lambda$.

Before stating the next lemma, we remind the reader that $\mathscr{K} \subset \mathscr{L}$.

LEMMA 5. *Let* $\mathscr{J} \in \{\mathscr{K}, \mathscr{L}\}$, $\mathscr{F} \in \mathscr{J}$. *Then there is* $\mathscr{F}_1 \subset\subset \mathscr{F}$ *so that*
(i) $\mathscr{F}_1^{\leqq \omega} \in \mathscr{J}$,
(ii) $\mathscr{F}_1^{>\omega} \in \mathscr{K}^+$.
(iii) $\mathscr{F}_1^{\leqq \omega}$ *and* $\mathscr{F}_1^{>\omega}$ *are strongly disjoint.*

Proof. We shall define sets $g(F) \subset F$ for $F \in \mathscr{F}$ by induction on the cardinality of F. For $F \in \mathscr{F}^{\leqq \omega}$ put $g(F) = F$. Now let $\lambda > \omega$ and assume that $g(F)$ is defined for $F \in \mathscr{F}^{<\lambda}$. Let $A \in \mathscr{F}^{\lambda}$. Then we define $g(A)$ as follows.

For $\omega \leqq \mu < \lambda$, put $A(\mu) = \{x \in A : x \in g(B)$ for some $B \in \mathscr{F}^{\leqq \mu}\}$, and for $\mu \geqq \lambda$ put $A(\mu) = A$. Then $A(\mu) \subset A(\kappa)$ for $\mu \leqq \kappa$. Put

$$C(\mu) = A(\mu) - \bigcup_{\omega \leqslant \kappa < \mu} A(\kappa).$$

Since $|A(\lambda)| = \lambda$, there is a smallest cardinal, say λ_0, such that $\omega \leqq \lambda_0 \leqq \lambda$ and $|A(\lambda_0)| \geqq \lambda_0$.

Case 1. If $|C(\lambda_0)| \geqq \lambda_0$, let $g(A)$ be any λ_0-subset of $C(\lambda_0)$.

Case 2. If $|C(\lambda_0)| < \lambda_0$, put

$$g(A) = \bigcup_{\omega < \kappa < \lambda_0} A(\kappa) - A(\omega).$$

Notice that if Case 2 holds, then $\lambda_0 > \omega$ (since $C(\omega) = A(\omega)$) and so $|A(\omega)| < \omega$ and hence $|g(A)| = \lambda_0$. Thus, in either case, $|g(A)| = \lambda_0$ and

(4.1) $\quad g(A) \subset A(\lambda_0)$.

The family $\mathscr{F}_1 = \langle g(A) : A \in \mathscr{F} \rangle$ has the required properties.

To prove this we first show that

(4.2) $\quad A \in \mathscr{F}, x \in S_1 \cap A, \rho_1(x) \leqq \mu \Rightarrow x \in A(\mu)$,

where $S_1 = S(\mathscr{F}_1)$ and $\rho_1 = \rho_{\mathscr{F}_1}$. From the hypothesis that $\rho_1(x) \leqq \mu$, it follows that there is some $F \in \mathscr{F}$ such that $x \in g(F)$ and $|g(F)| \leqq \mu$.

(i)′ If $|F| \leqq \mu$, then $x \in A(\mu)$ by the definition of $A(\mu)$.

(ii)′ If $|F| > \mu$, then $g(F) \subset F(\mu)$ by (4.1) and hence there is $B \in \mathscr{F}^{\leqq \mu}$ such that $x \in g(B)$. This again implies that $x \in A(\mu)$, and (4.2) follows. We now verify that $\mathscr{F}_1$ has the required properties. Let $C \in \mathscr{F}_1^{\leqq\omega}$, $x \in C$. There is $A \in \mathscr{F}$ such that $C = g(A)$ and $x \in A$. If $|A| \leqq \omega$, then $C = A$ and we have

(a) $|C| = |A| \geqq |\mathscr{F}(A)| \geqq |\mathscr{F}_1(C)|$ if $\mathscr{J} = \mathscr{K}$ and

(b) $|C| = |A| \geqq |\mathscr{F}(x)| \geqq |\mathscr{F}_1(x)|$ if $\mathscr{J} = \mathscr{L}$. Suppose $|A| > \omega$. Then $|C| = \omega$ and $C \subset A(\omega)$. Hence there is $B \in \mathscr{F}^{\leqq\omega}$ such that $x \in g(B) = B$. Then, since $\mathscr{K} \subset \mathscr{L}$, $\omega = |C| \geqq |B| \geqq |\mathscr{F}(x)| \geqq |\mathscr{F}_1(x)|$ and also

$$|\mathscr{F}_1(C)| = \left|\bigcup_{x\in C} \mathscr{F}_1(x)\right| \leqslant \omega = |C|.$$

This proves (i).

Let $\lambda > \omega$, $C \in \mathscr{F}_1^{\lambda}$, $x \in C$. There is $A \in \mathscr{F}$ such that $C = g(A) \subset A(\lambda) \subset S^{\leqq\lambda}$. Thus $\rho_{\mathscr{F}}(x) \leqq \lambda$ and so x is a member of at most λ sets $B \in \mathscr{F}$ and hence at most λ sets $g(B) \in \mathscr{F}_1$. It follows that $|\mathscr{F}_1(C)| \leqq \lambda^2 = |C|$ and hence $\mathscr{F}_1^{>\omega} \in \mathscr{K}$.

Now suppose $C \in \mathscr{F}_1^{>\mu}$. There is $\lambda > \mu$ such that $C = g(A)$, $A \in \mathscr{F}^{\lambda}$. Since $|C| > \mu$, it follows from the definition of $g(A)$ that $|A(\mu)| < \mu$. Therefore, by (4.2),

$$|C \cap S_1^{\leqq\mu}| \leqq |A(\mu)| < \mu.$$

Now let $\lambda > \omega$, $C \in \mathscr{F}_1^{\lambda}$, $C \cap S_1^{<\lambda} \neq \emptyset$. There is $A \in \mathscr{F}^{\kappa}$ such that $C = g(A)$ and $\kappa \geqq \lambda$. Now $C \subset A(\lambda)$ and from the definition of $g(A)$, either

(a) $g(A) \cap A(\mu) = \emptyset$ for $\omega \leqq \mu < \lambda$ or

(b) $g(A) \subset \bigcup_{\omega<\mu<\lambda} A(\mu)$.

Now (a) is false by (4.2) and the assumption that $C \cap S_1^{<\lambda} \neq \emptyset$. So (b) holds. But if $x \in A(\mu) \cap S_1$, then $\rho_1(x) \leqq \mu$ by the definition of $A(\mu)$. Hence $g(A) \subset S_1^{<\lambda}$. This proves (ii).

Finally, suppose $C \in \mathscr{F}_1^{>\omega}$. Then $C = g(A)$ for some $A \in \mathscr{F}^{>\omega}$ and from the definition of $g(A)$, we have $C \cap A(\omega) = \emptyset$. Therefore, by (4.2), $\rho_1(x) > \omega$ for all $x \in C$. This proves that $\mathscr{F}_1^{\leqq\omega}$ and $\mathscr{F}_1^{>\omega}$ and strongly disjoint.

5. Proof of Theorem 2. We shall prove the result by induction on

$$\mu(\mathscr{F}) = \inf\{\mu : |F| \leqq \mu \text{ for all } F \in \mathscr{F}\}.$$

By Lemma 3 (i) the theorem is true if $\mu(\mathscr{F}) = \omega$. Now assume that $\lambda > \omega$ and that

(5.1) $\quad \mathscr{F}' \in \mathscr{K},\ \mu(\mathscr{F}') < \lambda \Rightarrow \text{Trans}\ (\mathscr{F}') \neq \emptyset.$

Let $\mathscr{F} \in \mathscr{K}$, $\mu(\mathscr{F}) = \lambda$. We have to prove that Trans $(\mathscr{F}) \neq \emptyset$. Since $\mathscr{F}_1 \subset\subset \mathscr{F}$ and Trans $(\mathscr{F}_1) \neq \emptyset \Rightarrow$ Trans $(\mathscr{F}) \neq \emptyset$, we may assume by Lemma 5 that $\mathscr{F}_1 \in \mathscr{K}^+$ (and that $\mathscr{F}_1^{\leqq\omega} = \emptyset$, but we do not use this fact). We shall consider separately the three cases (1) λ a successor cardinal, (2) λ a regular limit cardinal, (3) λ a singular limit cardinal.

Case 1. $\lambda = \mu^+$: Since $\mathscr{F} \in \mathscr{K}^+$, it follows from Lemma 4, that $\mathscr{F}^{\lambda}$ and $\mathscr{F}^{<\lambda}$ are strongly disjoint families (since cf $(\lambda) > \omega$). Now $\mathscr{F}^{<\lambda} = \mathscr{F}^{\leqq\mu}$ has

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a transversal by (5.1) and $\mathscr{F}^{\lambda}$ has a transversal by Lemma 3 (ii). Hence $\mathscr{F} = \mathscr{F}^{<\lambda} \cup \mathscr{F}^{\lambda}$ also has a transversal.

Case 2. *λ a regular limit cardinal:* By Lemma 4, since cf $(\lambda) > \omega$, the families $\mathscr{F}^{\lambda}$ and $\mathscr{F}^{<\lambda}$ are strongly disjoint. Now $\mathscr{F}^{\lambda}$ has a transversal by Lemma 3 (ii) and so is enough to show that $\mathscr{F}^{<\lambda}$ has a transversal.

Let $A \in \mathscr{F}^{<\lambda}$. Put $X_0 = A, X_{n+1} = \bigcup \{B \in \mathscr{F}^{<\lambda} : B \cap X_n \neq \emptyset\}$ $(n < \omega)$, $X = \bigcup_{n<\omega} X_n$. Then, by induction on n, we have $|X_n| < \lambda$ $(n < \omega)$ and hence $|X| < \lambda$. Hence the λ-component of $\mathscr{F}^{<\lambda}$ containing A, $\mathscr{G}(A) = \langle B \in \mathscr{F}^{<\lambda} : B \cap X \neq \emptyset \rangle$, has cardinality $< \lambda$. Since λ is weakly inaccessible, it follows that $\mu(\mathscr{G}(A)) < \lambda$ and hence $\mathscr{G}(A)$ has a transversal by (5.1). Since $\mathscr{F}^{<\lambda}$ is the union of all its λ-components which are pairwise strongly disjoint, it follows that Trans $(\mathscr{F}^{<\lambda}) \neq \emptyset$.

Case 3. cf $(\lambda) = \kappa < \lambda$: Let $\langle \lambda_\alpha : \alpha \leqq \kappa \rangle$ be a continuous increasing sequence of cardinals,

$$\kappa < \lambda_0 < \lambda_1 < \ldots < \lambda_\kappa = \lambda = \lim_{\alpha<\kappa} \lambda_\alpha.$$

Denote by $\mathscr{C}_\alpha$ the set of all the large λ_α-components of $\mathscr{F}$, and let $\mathscr{C} = \bigcup_{\alpha\leqq\kappa} \mathscr{C}_\alpha$. If $\mathscr{G} \in \mathscr{C}_\alpha$, then $|\mathscr{G}| \leqq \lambda_\alpha$ and we may write

$$\mathscr{G}^{\lambda_\alpha} = \langle G_\nu : \nu < \xi(\mathscr{G}) \rangle$$

where $\xi(\mathscr{G})$ is some initial ordinal $\leqq \lambda_\alpha$. For any ordinal β put

$$\mathscr{G}(\beta) = \begin{cases} \langle G_\nu : \nu < \beta \rangle, & \text{if } \beta \leqq \xi(\mathscr{G}), \\ \mathscr{G}^{\lambda_\alpha}, & \text{if } \beta > \xi(\mathscr{G}). \end{cases}$$

If $\mathscr{G}, \mathscr{G}' \in \mathscr{C}$, $\mathscr{G} \neq \mathscr{G}'$ and β, β' are ordinals, then

(5.2) $\quad \mathscr{G}(\beta) \cap \mathscr{G}'(\beta') = \emptyset.$

For, there are $\alpha, \alpha' \leqq \kappa$ such that $\mathscr{G} \in \mathscr{C}_\alpha$, $\mathscr{G}' \in \mathscr{C}_{\alpha'}$. If $\alpha = \alpha'$ then $\mathscr{G}$ and $\mathscr{G}'$ are disjoint since a set $F \in \mathscr{F}^{\lambda_\alpha}$ is a member of exactly one large λ_α-component; if $\alpha \neq \alpha'$ then $\mathscr{G}^{\lambda_\alpha}$ and $\mathscr{G}'^{\lambda_{\alpha'}}$ are disjoint since members of these families have cardinalities λ_α and $\lambda_{\alpha'}$ respectively.

For $\alpha \leqq \kappa$ put

$$\mathscr{F}_\alpha^* = \bigcup_{\mathscr{G}\in\mathscr{C}} \bigcup_{\gamma<\alpha} \mathscr{G}(\lambda_\gamma), \quad \mathscr{F}_\alpha^{**} = \mathscr{F}^{<\lambda_\alpha} \cup \mathscr{F}_\alpha^*.$$

It is easy to see that

(5.3) $\quad \mathscr{F}_{\alpha_0}^{**} = \bigcup_{\alpha<\alpha_0} \mathscr{F}_\alpha^{**}$

if α_0 is a limit ordinal. For, if $A \in \mathscr{F}^{\lambda_{\alpha_0}}$, then there is a large λ_{α_0}-component $\mathscr{G} \in \mathscr{C}$ and $\gamma < \alpha_0$ so that $A \in \mathscr{G}(\lambda_\gamma)$ and hence $A \in \mathscr{F}_{\gamma+1}^{**}$. We also remark that (put $\mathscr{F}_{\kappa+1}^* = \mathscr{F}_\kappa^*$)

(5.4) $\quad |\langle B \in \mathscr{F}_{\alpha+1}^* : A \cap B \neq \emptyset \rangle| \leqq \lambda_\alpha \quad (\alpha \leqq \kappa, A \in \mathscr{F}).$

For, to each $\rho \leqq \kappa$ there is at most one large λ_ρ-component containing A, and

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if $|B| = \lambda_\rho$ and A, B are members of different large λ_ρ-components then $A \cap B = \emptyset$. Thus $|\langle B \in \mathscr{F}_{\alpha+1}^* : A \cap B \neq \emptyset\rangle| \leqq \kappa \cdot \lambda_\alpha = \lambda_\alpha$.

We are going to define functions φ_α for $\alpha \leqq \kappa$ by transfinite induction so that

(i) *φ_α is a transversal of $\mathscr{F}_\alpha^{**}$, and*

(ii) *φ_α is an extension of φ_γ for $\gamma < \alpha$.*

Then φ_κ will be a transversal of $\mathscr{F} = \mathscr{F}_\kappa^{**}$ as required.

Let $\alpha_0 \leqq \kappa$ and assume that φ_α has already been defined for $\alpha < \alpha_0$ so that (i) and (ii) hold. If $\alpha_0 = 0$, then $\mathscr{F}_{\alpha_0}^{**} = \mathscr{F}^{\leqq\lambda_0}$ has a transversal φ_0 by (5.1). If α_0 is a limit ordinal, then $\varphi_{\alpha_0} = \bigcup \varphi_\alpha$ is a transversal of $\mathscr{F}_{\alpha_0}^{**}$ of the required kind by (5.3) and (ii). It only remains to define φ_{α_0} in the case when α_0 is a successor ordinal, say $\alpha_0 = \alpha + 1$.

First we show that

(5.5) $|A \cap \text{range } (\varphi_\alpha)| \leqq \lambda_\alpha \quad (A \in \mathscr{F}).$

We may assume $A \in \mathscr{F}^{>\lambda_\alpha}$. Then $|A \cap S^{\leqq\lambda_\alpha}| < \lambda_\alpha$ and each element $x \in A \cap S^{\leqq\lambda_\alpha}$ is a member of at most λ_α different sets $B \in \mathscr{F}$. Therefore,

$$|\langle B \in \mathscr{F}^{\leqq\lambda_\alpha} : A \cap B \neq \emptyset\rangle| \leqq \lambda_\alpha.$$

This and (5.4) proves (5.5).

Put

$$\mathscr{F}_1 = \mathscr{F}^{\leqq\lambda_{\alpha+1}} - \mathscr{F}_\alpha^{**}, \quad \mathscr{F}_2 = \mathscr{F}_{\alpha+1}^* - (\mathscr{F}_\alpha^{**} \cup \mathscr{F}_1).$$

Then $\mathscr{F}_{\alpha+1}^{**}$ is the disjoint union of $\mathscr{F}_\alpha^{**}$, $\mathscr{F}_1$ and $\mathscr{F}_2$. The members of $\mathscr{F}_1$ all have cardinality $\lambda_{\alpha+1}$ and so, by (5.5) and Lemma 3 (ii), there is a transversal ψ_1 of $\mathscr{F}_1$ which is disjoint from φ_α. We shall extend $\varphi_\alpha' = \varphi_\alpha \cup \psi_1$ to a transversal of $\mathscr{F}_{\alpha+1}^{**}$ by selecting suitable elements from each set $F \in \mathscr{F}_2$. We do this component by component.

Let $\mathscr{C} = \{\mathscr{G}_\sigma : \sigma < \tau\}$. Let $\sigma < \tau$ and suppose we have already defined a transversal χ, say, of $\mathscr{G}_\sigma^* = \bigcup_{\rho<\sigma}\mathscr{G}_\rho(\lambda_\alpha) - \mathscr{F}_\alpha^{**} \cup \mathscr{F}_1$ which is disjoint from φ_α'. If

$$A \in \mathscr{G}' = \mathscr{G}_\sigma(\lambda_\alpha) - (\mathscr{G}_\sigma^* \cup \mathscr{F}_\alpha^{**} \cup \mathscr{F}_1),$$

then $|A| > \lambda_{\alpha+1}$. Therefore, $|A \cap S^{\leqq\lambda_{\alpha+1}}| < \lambda_{\alpha+1}$ and so

$$|A \cap \text{range } \psi_1| \leqq \lambda_\alpha.$$

Also, by (5.4) we have

$$|A \cap \text{range } (\chi)| \leqq \lambda_\alpha.$$

These two inequalities together with (5.5) show that

(5.6) $|A \cap \text{range } (\varphi_\alpha \cup \psi_1 \cup \chi)| < \lambda_{\alpha+1} \quad (A \in \mathscr{G}').$

Since $|\mathscr{G}'| \leqq \lambda_\alpha < |A|$ $(A \in \mathscr{G}')$, it follows from (5.6) and Corollary 1 that $\mathscr{G}'$ has a transversal χ' disjoint from $\varphi_\alpha \cup \psi_1 \cup \chi$. It follows, by transfinite induction on $\sigma < \tau$, that $\mathscr{F}_2$ has a transversal ψ_2 disjoint from $\varphi_\alpha \cup \psi_1$. Then

$\varphi_{\alpha+1} = \varphi_\alpha \cup \psi_1 \cup \psi_2$ is a transversal of $\mathscr{F}_{\alpha+1}^{**}$ which extends φ_α. This completes the proof of Theorem 2.

6. Proof of Theorem 3. We assume the special case of this theorem (proved in [**13**; **2**]):

(6.1) *if* $\mathscr{F}'$ *is a countable family of countable sets, then*

$$\mathscr{F}' \in \mathscr{L} \Rightarrow \text{Trans}\ (\mathscr{F}') \neq \emptyset.$$

Now let $\mathscr{F}$ be an arbitrary family satisfying condition $\mathscr{L}$. By Lemma 5 there is $\mathscr{F}_1 \subset \subset \mathscr{F}$ such that $\mathscr{F}_1^{\leqq\omega}$ and $\mathscr{F}_1^{>\omega}$ are strongly disjoint, $\mathscr{F}_1^{\leqq\omega} \in \mathscr{L}$ and $\mathscr{F}_1^{>\omega} \in \mathscr{K}^+$.

The ω-components of $\mathscr{F}_1^{\leqq\omega}$ are countable and strongly disjoint and every such component has a transversal by (6.1). $\mathscr{F}_1^{>\omega}$ has a transversal by Theorem 2. Therefore $\mathscr{F}_1$, and hence $\mathscr{F}$, has a transversal.

7. A generalization. We shall now prove a generalization of Theorem 3 using a different idea. A family $\mathscr{F}$ has property $\mathscr{P}$ if and only if the following three conditions are satisfied:

$\mathscr{P}_1$. $\mathscr{F}^{<\omega} \in \mathscr{L}$;
$\mathscr{P}_2$. $|\mathscr{F}^\lambda(x)| \leqq \lambda$ *for* $x \in S(\mathscr{F})$ *and* $\lambda \geqq \omega$;
$\mathscr{P}_3$. *If* λ *is inaccessible and* $x \in S(\mathscr{F})$, *then* $\{\mu < \lambda : \mathscr{F}^\mu(x) \neq \emptyset\}$ *is a non-stationary subset of* λ.

It is clear that if $\mathscr{F} \in \mathscr{L}$, then $\mathscr{F} \in \mathscr{P}$ (if λ inaccessible, $x \in S(\mathscr{F})$ and $\mathscr{F}^\kappa(x) \neq \emptyset$, then $|\{\mu < \lambda : \mathscr{F}^\mu(x) \neq \emptyset\}| \leqq \kappa$). It is also easy to verify that

(7.1) *if* $\mathscr{F} \in \mathscr{P}$ *and* $g(F) \subset F$, $|g(F)| = |F| (F \in \mathscr{F})$, *then*
$\mathscr{F}_1 = \langle g(F) : F \in \mathscr{F} \rangle \in \mathscr{P}$.

THEOREM 4. $\mathscr{F} \in \mathscr{P} \Rightarrow \text{Trans}\ (\mathscr{F}) \neq \emptyset$.

Proof of Theorem 4. For each infinite cardinal μ, the μ-components of $\mathscr{F}$ are pairwise strongly disjoint. Every such component has cardinality $\leqq \mu$ and so, by Lemma 1, the μ-sets of a μ-component can be replaced by subsets of power μ which are pairwise disjoint. By (7.1) the family thus obtained still enjoys property $\mathscr{P}$. So we may assume without loss of generality that

(7.2) if $\lambda \geqq \omega$ and $A, B \in \mathscr{F}^\lambda$, $A \neq B$, then $A \cap B = \emptyset$.

As in the proof of Theorem 2, we shall prove the theorem by transfinite induction on $\mu(\mathscr{F})$. If $\mu(\mathscr{F}) = \omega$, then $\mathscr{F} \in \mathscr{L}$ and $\text{Trans}\ (\mathscr{F}) \neq \emptyset$ by Theorem 3. Suppose $\mu(\mathscr{F}) = \lambda > \omega$.

Case 1. $\lambda = \kappa^+$: By (7.2) the members of $\mathscr{F}$ having power κ^+ are pairwise disjoint. Therefore, if we replace every such set by a subset of power ω, the resulting family $\mathscr{F}_1$, say, has property $\mathscr{P}$ and $\mu(\mathscr{F}_1) = \kappa$. Thus $\text{Trans}\ (\mathscr{F}_1) \neq \emptyset$ by the induction hypothesis and hence $\text{Trans}\ (\mathscr{F}) \neq \emptyset$.

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Case 2. $\lambda = \mu(\mathscr{F})$ *is singular*: Let cf $(\lambda) = \kappa < \lambda$, and let $\langle \lambda_\rho : \rho \leqq \kappa \rangle$ be a closed increasing sequence of ordinals

$$\kappa < \lambda_0 < \ldots < \lambda_\kappa = \lambda = \lim_{\rho < \kappa} \lambda_\rho.$$

Form a new family $\mathscr{F}_1$ from $\mathscr{F}$ by replacing each set $A \in \bigcup_{\rho \leqq \kappa} \mathscr{F}^{\lambda_\rho}$ by a subset $g(A) \subset A$ of power κ. Any element $x \in S(\mathscr{F})$ belongs to at most κ new sets of power κ and so $\mathscr{F}_1 \in \mathscr{P}$. We may as well assume that $\mathscr{F} = \mathscr{F}_1$, i.e.

(7.3) $\bigcup_{\rho \leqslant \kappa} \mathscr{F}^{\lambda_\rho} = \emptyset.$

If $A \in \mathscr{F}^{\leqq \lambda_\rho}$, let $\mathscr{G}_\rho(A)$ be the unique λ_ρ-component of $\mathscr{F}$ which contains A; if $A \in \mathscr{F}^{> \lambda_\rho}$, let $\mathscr{G}_\rho(A) = \emptyset$, the empty family. Then

$$\mathscr{G}_\rho(A) \subset \mathscr{G}_\sigma(A) \quad \text{for } \rho < \sigma \leqq \kappa.$$

Also, by (7.3),

$$\mathscr{G}_\sigma(A) = \bigcup_{\rho < \sigma} \mathscr{G}_\rho(A) \quad \textit{if } \sigma \textit{ is a limit ordinal} \leqslant \kappa.$$

Put $S_\rho(A) = S(\mathscr{G}_\rho(A)) (\rho \leqq \kappa)$. Since $|\mathscr{G}_\rho(A)| \leqq \lambda_\rho$ and $|B| < \lambda_\rho$ $(B \in \mathscr{G}_\rho(A))$, it follows that $|S_\rho(A)| \leqq \lambda_\rho$ $(\rho < \kappa)$. For $B \in \mathscr{F} - \mathscr{G}_\rho(A)$ we have that either (i) $|B| \leqq \lambda_\rho$ and $B \cap S_\rho(A) = \emptyset$ or (ii) $|B| > \lambda_\rho$. Therefore, by (7.1),

$$\mathscr{G}_\rho^*(A) = \langle B - S_\rho(A) : B \in \mathscr{G}_{\rho+1}(A) - \mathscr{G}_\rho(A) \rangle \in \mathscr{P}$$

for $\rho < \kappa$. Now $\mathscr{G}_0(A)$ has a transversal and so does $\mathscr{G}_\rho^*(A)$ $(\rho < \kappa)$ since $\mu(\mathscr{G}_\rho^*(A)) \leqq \lambda_{\rho+1} < \lambda$. Therefore, since the families $\mathscr{G}_0(A)$, $\mathscr{G}_\rho^*(A)$ $(\rho < \kappa)$ are pairwise strongly disjoint, the family

$$\mathscr{G}'(A) = \mathscr{G}_0(A) \cup \bigcup_{\rho < \kappa} \mathscr{G}_\rho^*(A)$$

has a transversal. This clearly implies that the λ-component, $\mathscr{G}_\kappa(A)$, containing A also has a transversal. This holds for any $A \in \mathscr{F}$ and so $\mathscr{F}$ has a transversal since the λ-components of $\mathscr{F}$ are strongly disjoint.

Case 3. λ *is weakly inaccessible*: Since the λ-components of $\mathscr{F}$ are strongly disjoint, we may assume that $\mathscr{F}$ has but a single λ-component. Then $|\mathscr{F}| \leqq \lambda$ and $|S(\mathscr{F})| \leqq \lambda$ and so we can assume further that $\mathscr{F}$ is a family of subsets of λ. (As usual, an ordinal is the set of all smaller ordinals.) Now by (7.2) the members of $\mathscr{F}$ which have power λ are pairwise disjoint and, if we replace these by subsets of power ω, the resulting family still has property $\mathscr{P}$. Thus we can assume that

(7.4) $|A| < \lambda \quad (A \in \mathscr{F}).$

By $\mathscr{P}_3$, for each $x \in \lambda$ there is a function $f_x : \lambda \to \lambda$ such that

$$f_x(\alpha) \leqq \alpha \quad (\alpha < \lambda),$$

$$x \leqq f_x(|A|) < |A| \quad (A \in \mathscr{F}(x), |A| > x),$$

(7.5) $|\{\alpha < \lambda : f_x(\alpha) = \gamma\}| < \lambda \quad (\gamma < \lambda).$

We now define a function $g : \lambda \to \lambda$ by putting

$$g(\alpha) = \sup (\alpha \cup \{y \in \lambda : (\exists x < \alpha)(\exists A \in \mathscr{F}(x))(y \in A \text{ and } f_x(|A|) < \alpha)\}).$$

(If C is a set of ordinals then $\sup C$ is the smallest ordinal β such that $\beta > \gamma$ for all $\gamma \in C$.) We immediately have from the definition of g, (7.5) and $\mathscr{P}_2$, that

(7.6) $\alpha \leqq g(\alpha) \leqq g(\beta) < \lambda$ for $\alpha < \beta < \lambda$.

If α is a limit ordinal such that $g(\gamma) < \alpha$ for all $\gamma < \alpha$, then $g(\alpha) = \alpha$. Put

$$C = \{0\} \cup \{\alpha < \lambda : \alpha \text{ a limit ordinal, } g(\alpha) = \alpha\}.$$

Now C is a cofinal subset of λ. For if $\gamma < \lambda$, put $\alpha_0 = \gamma$, $\alpha_{n+1} = g(\alpha_n + 1)$ $(n < \omega)$. Then $\gamma < \alpha = \lim_{n<\omega}\alpha_n$ and $\alpha \in C$. Therefore, we may write

$$C = \{\beta_\nu : \nu < \lambda\},$$

where $0 = \beta_0 < \beta_1 < \ldots < \lambda = \lim_{\nu<\lambda}\beta_\nu$ and β_ν is a limit ordinal satisfying $g(\beta_\nu) = \beta_\nu$ $(\nu < \lambda)$.

We will prove that, for $A \in \mathscr{F}$ there is $\nu = \nu(A) < \lambda$ such that

(7.7) $|A \cap [\beta_\nu, \beta_{\nu+1})| = |A|$.

Let x be the first element of A. If $|A| \leqq x$, then $f_x(|A|) \leqq x$ and hence $A \subset [x, g(x + 1))$. Now there is $\nu < \lambda$ such that $x \in [\beta_\nu, \beta_{\nu+1})$. Then $g(x + 1) \leqq \beta_{\nu+1} = g(\beta_{\nu+1})$ and (7.7) holds. Now suppose that $|A| > x$. There is $\nu < \lambda$ such that $f_x(|A|) \in [\beta_\nu, \beta_{\nu+1})$. Hence, there is γ such that

$$x \leqq f_x(|A|) < \gamma < \beta_{\nu+1}.$$

Then $A \subset [x, g(\gamma))$. Since $g(\gamma) < \beta_{\nu+1}$ and $\beta_\nu \leqq f_x(|A|) < |A|$, we again obtain (7.7).

By (7.7) and (7.1) we can replace each set $A \in \mathscr{F}$ by the subset $g(A) = A \cap [\beta_\nu, \beta_{\nu+1})$ to obtain a family $\mathscr{F}_1$ also with property $\mathscr{P}$. For $A \in \mathscr{F}$, if $\mathscr{G}(A)$ is the λ-component of $\mathscr{F}_1$ containing $g(A)$, then $\mathscr{G}(A)$ is a family of subsets of $[\beta_{\nu(A)}, \beta_{\nu(A)+1})$. Thus $\mu(\mathscr{G}(A)) < \lambda$ and so $\mathscr{G}(A)$ has a transversal. Since different λ-components of $\mathscr{F}_1$ are strongly disjoint, it follows that $\mathscr{F}_1$ (and hence $\mathscr{F}$) has a transversal.

References

1. Alexandroff and Urysohn, *Memoire sur les espaces topologiques compacts*, Verh. Nederl. Akad. Wentensch. Sect. I, 14, Nr. 1, S1 (1929).

2. B. Bollobas and E. C. Milner, *A theorem in transversal theory*, Bull. London Math. Soc. *5* (1973), 267–270.

3. R. A. Brualdi and E. B. Scrimger, *Exchange systems, matchings and transversals*, J. Combinatorial Theory *5* (1968), 244–257.

4. R. M. Damerell and E. C. Milner, *Necessary and sufficient conditions for transversals of countable set systems* (to appear in J. Combinatorial Theory).

5. J. Folkman, *Transversals of infinite families with finitely many infinite members*, RAND Corporation Memorandum RM-5676-PR, 1968; J. Combinatorial Theory *9* (1970), 200–220.

6. Marshall Hall, Jr., *Distinct representatives of subsets*, Bull. Amer. Math. Soc. *54* (1948), 922–926.

7. P. Hall, *On representatives of subsets*, J. London Math. Soc. *10* (1935), 26–30.

8. J. König, *Graphok es matrixok*, Mat. Lapok *38* (1931), 116–119.

9. L. Mirsky, *Transversal theory* (Academic Press, New York, 1971).

10. C. St. J. A. Nash Williams, *Proceedings of conference in combinatorics and graph theory*, Oxford, 1972.

11. W. Neumer, *Verallgemeinerung eines Satzes von Alexandroff and Urysohn*, Math. Z. *54* (1951), 254–261.

12. R. Rado, *Note on the transfinite case of Hall's theorem on representatives*, J. London Math. Soc. *42* (1967), 321–324.

13. S. Shelah, *A substitute for Hall's theorem for families with infinite sets*, J. Combinatorial Theory (A) *16* (1974), 199–208.

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