Now $\alpha^2 + \beta^2 + \gamma^2 = \Sigma^2 - 2 > 0$, so $8\Sigma^2 - 3 > 13 > 0$, so (since $\Sigma > 0$) $8\Sigma^2 + 8\sqrt{3}\Sigma - 3 > 0$, so $\Sigma > \sqrt{3}$.

The reader may wish to repeat the argument using $A \geq H$.

Finally, note the corollaries

$$\sin A \sin B \sin C < \frac{3\sqrt{3}}{8} \quad \text{and} \quad \cos A \cos B \cos C < \frac{1}{8}.$$ 

References


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**90.78 Watt quadrilaterals and the 2005 IMO**

In [1], we defined a *Watt quadrilateral* as a quadrilateral with a pair of opposite sides of equal length. There, we proved a number of results concerning Watt quadrilaterals, several concerning circumcircles of various triangles.

The following problem was set (using slightly different notation) at the International Mathematical Olympiad of July 2005.

**Problem**

Let $ABCD$ be a convex Watt quadrilateral with $BC$ and $AD$ equal but not parallel. Consider two interior points, $K_i$ in $BC$ and $J_i$ in $DA$, such that $BK_i = DJ_i$. Let $AC$ and $BD$ meet at $U$. Let $AC$ and $J_iK_i$ meet at $V_i$, and $BD$ and $J_iK_i$ meet at $W_i$. Show that the circumcircles of triangle $UV_iW_i$, as $J_i$ and $K_i$ vary, have a common point other than $U$.

Before proceeding, we remark that we have found it convenient, in our Maple code and elsewhere, to regard $t$ as a parameter, and to have

$$\frac{|DJ_i|}{|DA|} = t = \frac{|BK_i|}{|BC|},$$

The $J$ and $K$ of [1] (e.g. Property 6) correspond to $t = \frac{1}{2}$ in $J_i$ and $K_i$. See also Figure 1. Other notation we use from [1] includes:

- $F$ and $G$ denote the midpoints of $AC$ and of $BD$ respectively.
- $Q'$ is defined as the point of intersection of the perpendicular bisectors of $AC$ and $BD$.

The set of points $\{Q', F, U, G\}$ is concyclic, and $UQ'$ is a diameter.
Various solutions of the IMO problem are now available on the web. The function of this note is to call attention to [1], some results of which readers might wish to use in constructing their own solutions to the IMO problem.

Here are various results:

• For any $t$, the point $Q'$ is on the circumcircle of $UV_iW_i$. (There are many ways to organize the proofs. One algebraic approach involves using the fact that, if the cross ratio of four complex numbers is real, the four numbers are concyclic. To prove the result about $Q'$, this is applied to the complex numbers, $z_u, z_v, z_q', z_w$, the notation being that of §4.1 of [1].)

• The point $Q'$ is the second point of intersection of the circumcircles of $AUD$ and of $BUC$. Triangles $AQ'D$ and $CQ'B$ are congruent.

• The circumcentres of the triangles $UV_iW_i$ are collinear, and this line of circumcentres is the perpendicular bisector of $UQ'$, the radical axis of circles $UV_iQ'W_i$ and circle $UFQ'G$.

• $Q'$ is on the perpendicular bisector of $J_iK_i$.

• The midpoint of $J_iK_i$ is on the line joining the midpoint $F$ of $AC$ to the midpoint $G$ of $BD$.

• Let $J$ be the midpoint of $AD$ and $K$ be the midpoint of $BC$. Then the midpoint of $V_iW_i$ lies on the line $JK$.

It was reasonable for the IMO, for competition purposes, to specify $ABCD$ as a convex quadrilateral, and $K_i$ and $J_i$ interior points on $BC$ and $DA$ respectively. However the Watt quadrilateral is a linkage [1] and the problem property is more general. It persists even if $ABCD$ is not convex and/or $J_i$ and $K_i$ are not interior points. The reader is invited to explore these options, perhaps using approaches outlined in [1].
90.79 Two proofs of a Pythagorean-like theorem

In [1] there are several proofs of the following theorem: if $C$ is a point on the base of isosceles $\triangle ABD$ such that $BA = BD = c$, $BC = a$, $AC = b$ and $DC = d$, then $c^2 = a^2 + bd$. (See Figure 1).

**Figure 1**

In [1] and republished in [2] $\angle ACB$ was obtuse, but [3] showed that the result also holds when $\angle ACB$ is acute. If the angle is a right angle, then the formula reduces to the Pythagorean theorem. Darvasi used Heron's formula to derive the result and Hoehn used the intersecting chords theorem, the law of cosines, Pythagoras theorem, and Stewart's theorem to provide his four proofs.

In this note we first prove the result by using Ptolemy's theorem. This theorem is stated in [4] as follows: in a cyclic quadrilateral, the product of the diagonals is equal to the sum of the products of the two pairs of opposite sides.

**First proof**

For this proof we construct lines through points $B$ and $C$ parallel to $DA$ and $DB$, respectively, to form parallelogram $BECD$ (see Figures 2(i) and 2(ii)). Also draw $EA$. Using properties of isosceles $\triangle BDA$, parallel lines, and parallelogram $BECD$, we obtain $\angle EBA \cong \angle BAD \cong \angle BDA \cong \angle BEC \cong \angle ECA$ and $CE = DB = AB$. Therefore $\angle BEC \cong \angle EBA$ and $\triangle BAC \cong \triangle ECA$ by the SAS congruence theorem for triangles.