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ON A THEOREM OF BOVDI

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1. Introduction. If p is a prime, we call an element $x \neq 1$ of a group G a generalized p-element if, for every $n \geqq 1$, there exists $r \geqq 0$ such that $x^{p^r} \in G_n$, where G_n is the nth term of the lower central series of G. Bovdi [**1**] proved that if G is a finitely generated group having a generalized p-element, and if $\bigcap_n \Delta^n(Z(G) = 0$ where $\Delta(Z(G))$ is the augmentation ideal, then G is residually a finite p-group.

We recall that if R is a ring, then the nth dimension subgroup of G over R, denoted by $D_n(R(G))$, is defined to be $\{g \mid g - 1 \in \Delta^n(R(G))\}$. In this note, we show that if G is finitely generated, then $\bigcap_n D_n(\mathbf{Z}_p^\wedge(G)) = 1 \Leftrightarrow \bigcap_n \Delta^n(\mathbf{Z}_p^\wedge(G)) = 0 \Leftrightarrow G$ is residually a finite p-group. Here $\mathbf{Z}_p^\wedge$ is the ring of p-adic integers. As a preliminary result, we obtain the structure of $D_n(R(G))$ in terms of $D_n(\mathbf{Z}(G))$, where R is a commutative ring with unity such that $(R, +)$ is torsion free.

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2. Dimension subgroups. We first state a lemma which was proved by Sandling [**7**, Corollary 6.5].

LEMMA 1. *If R is a commutative ring with* 1 *containing* $\mathbf{Q}$, *the field of rational numbers, then* $D_n(R(G)) = D_n(\mathbf{Q}(G))$.

THEOREM 2. (cf. [**7**, Theorem 6.1]). *Let R be a commutative ring with* 1 *such that $(R, +)$ is torsion free and let G be a group. Let $\pi = \{q \mid q$ is a prime natural number with $qR = R\}$. Then $D_n(R(G)) = T_\pi(G \bmod D_n(\mathbf{Z}(G)))$, the torsion π-subgroup of G modulo $D_n(\mathbf{Z}(G))$.*

Proof. If π consists of all primes, then Lemma 1 says that $D_n(R(G)) = D_n(\mathbf{Q}(G))$ and it is well known [**2**] that $D_n(\mathbf{Q}(G)) = T(G \bmod D_n(\mathbf{Z}(G)))$, the torsion subgroup of G modulo $D_n(\mathbf{Z}(G))$. Therefore, we assume that there is a prime not in π.

Let $x \in T_\pi$ $(G \bmod D_n(\mathbf{Z}(G)))$. Then $x^q - 1 \in \Delta^n(\mathbf{Z}(G))$ for some π-number q. Hence $q(x-1) + \ldots + (x-1)^q \in \Delta^n(\mathbf{Z}(G)) \subseteq \Delta_n(R(G))$, since $(R, +)$ is torsion free. Since q is invertible in R, we get $x - 1 \in \Delta^n(R(G))$, hence $x \in D_n(R(G))$.

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Now assume that $x \in D_n(R(G))$, i.e., that $x - 1 \in \Delta^n(R(G))$. We recall that a map $f: G \to \mathbf{Q}/\mathbf{Z}$ is called a $\mathbf{Z}$-polynomial map of degree $\leqq n - 1$ [**5**] if the $\mathbf{Z}$-linear extension of f to $\mathbf{Z}(G)$ vanishes on $\Delta^n(\mathbf{Z}(G))$. We will also denote this extension by f. Now let $S^{-1}R$ be the ring of fractions constructed from R using the multiplicative system $S = \{x | x \text{ is not a zero-divisor in } R\}$.

We note that since $(R, +)$ is torsion free, we have a copy of $\mathbf{Z}$ in S. Form the R-module $S^{-1}R/R$ and define $\alpha: \mathbf{Q} \to S^{-1}R/R$ by $\alpha(q) = q + R$. This makes sense since we have a copy of $\mathbf{Q}$ in $S^{-1}R$. Then $\operatorname{Ker} \alpha = \{q | q \in R\} = \{a/b \mid b \text{ is a } \pi\text{-number}\}$, where a/b is considered in lowest terms. Hence, $\mathbf{Z} \subseteq \operatorname{Ker} \alpha$, and we have a $\mathbf{Z}$-homomorphism $\bar{\alpha}: \mathbf{Q}/\mathbf{Z} \to S^{-1}R/R$ with $\operatorname{Ker} \bar{\alpha} = \{a/b + \mathbf{Z} | b \text{ is a } \pi\text{-number}\}$.

Now let $f: G \to \mathbf{Q}/\mathbf{Z}$ be any polynomial map of degree $\leqq n - 1$ and let $y = \sum r_i(x_{i_1} - 1) \ldots (x_{i_n} - 1) \in \Delta^n(R(G))$. Thinking of $\bar{\alpha} \circ f$ extended R-linearly, we see that $(\bar{\alpha} \circ f)y = 0$, since $\bar{\alpha}$ is a group homomorphism and f vanishes on $\Delta^n(\mathbf{Z}(G))$. Hence, $\bar{\alpha} \circ f$ vanishes on $\Delta^n(R(G))$ and, in particular, $f(x - 1) \in \operatorname{Ker} \bar{\alpha}$ and has denominator a π-number.

Now, I claim that there exists a π-number k_1 with $k_1(x - 1) \in \Delta^n(\mathbf{Z}(G))$. Assume that this is not true, and consider the subgroup $\langle x - 1 + \Delta^n(\mathbf{Z}(G))\rangle$ of the abelian group $\Delta(\mathbf{Z}(G))/\Delta^n(\mathbf{Z}(G))$. Since $\mathbf{Q}/\mathbf{Z}$ contains elements of all orders, we could then construct a homomorphism $\rho: \langle x - 1 + \Delta^n(\mathbf{Z}(G))\rangle \to \mathbf{Q}/\mathbf{Z}$ with $\rho(x - 1 + \Delta^n(\mathbf{Z}(G)))$ having denominator not a π-number (here we use the fact that not all primes are in π). Since $\mathbf{Q}/\mathbf{Z}$ is divisible, we can extend ρ to $\bar{\rho}: \Delta(\mathbf{Z}(G))/\Delta^n(\mathbf{Z}(G) \to \mathbf{Q}/\mathbf{Z}$. Now define $f: G \to \mathbf{Q}/\mathbf{Z}$ by $f(g) = \bar{\rho}(g - 1 + \Delta^n(\mathbf{Z}(G)))$. It is clear that f is a $\mathbf{Z}$-polynomial map of degree $\leqq \mathrm{n} - 1$ with $f(x - 1) = f(x)$ having denominator not a π-number. This contradicts the conclusion of the previous paragraph.

Hence, there exists a π-number k_1 with $k_1(x - 1) \in \Delta^n(\mathbf{Z}(G))$. When $n = 1$, the theorem holds trivially since $D_1(\mathbf{Z}(G)) = G$, so assume that $n > 1$. In that case, we have

$$x^{k_1} - 1 = k_1(x - 1) + \ldots + (x - 1)^{k_1} \in \Delta^2(\mathbf{Z}(G)).$$

If $n = 2$, the theorem is proved. If not, repeat the argument with x^{k_1}, which is in $D_n(R(G))$, and obtain a π-number k_2 with $k_2(x^{k_1} - 1) \in \Delta^n(\mathbf{Z}(G))$.

Therefore, $x^{k_1k_2} - 1 = k_2(x^{k_1} - 1) + \ldots + (x^{k_1} - 1)^{k_2} \in \Delta^{\min(4,n)}(\mathbf{Z}(G))$. Continuing this argument, we obtain a π-number $l = k_1k_2 \ldots k_t$ with $x^l - 1 \in \Delta^n(\mathbf{Z}(G))$. Hence $x \in T_\pi(G \bmod D_n(\mathbf{Z}(G)))$ as required.

In the case where R is the ring of p-adic integers, we obtain the following:

COROLLARY. *Let* $K = \{p | p \text{ is prime, } x^p \in D_n(\mathbf{Z}(G)) \text{ for some } x \in G - D_n(\mathbf{Z}(G))\}$. *Then* $\bigcap_{p \in K} D_n(\mathbf{Z}_p^\wedge(G)) = D_n(\mathbf{Z}(G))$.

As far as the corresponding problem for powers of the augmentation ideal is concerned, we have the following:

PROPOSITION 3. *Let* $x \in \Delta(\mathbf{Z}(G))$ *satisfy* $x \in \Delta^n(\mathbf{Z}_p^\wedge(G))$ *for all* p. *Then* $x \in \Delta^n(\mathbf{Z}(G))$.

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Proof. Let π_p be the natural map: $\mathbf{Q}/\mathbf{Z} \to \mathbf{Q}_p{}^\wedge/\mathbf{Z}_p{}^\wedge$, where $\mathbf{Q}_p{}^\wedge$ is the field of p-adic numbers. Let $f: G \to \mathbf{Q}/\mathbf{Z}$ be any polynomial map of degree $\leqq n-1$. Then $(\pi_p \circ f)(x) = 0$ for all p as before, since $x \in \Delta^n(\mathbf{Z}_p{}^\wedge(G)$ for all p. As before, we are thinking of $\pi_p \circ f$ extended $\mathbf{Z}_p{}^\wedge$-linearly. However, $\operatorname{Ker} \pi_p = \{a/b + \mathbf{Z} \mid b$ is not divisible by $p\}$ and $f(x) \in \operatorname{Ker} \pi_p$ for all p. Hence, $f(x) = 0$ in $\mathbf{Q}/\mathbf{Z}$. This says, as in the proof of Theorem 2, that $x \in \Delta^n(\mathbf{Z}(G))$, since if $x \notin \Delta^n(\mathbf{Z}(G))$, then there exists $\rho: \Delta\mathbf{Z}(G))/\Delta^n(\mathbf{Z}(G)) \to \mathbf{Q}/\mathbf{Z}$ with $\rho(x + \Delta^n(\mathbf{Z}(G))) \neq 0$. This follows from the divisibility of $\mathbf{Q}/\mathbf{Z}$. However, if we define $f: G \to \mathbf{Q}/\mathbf{Z}$ by $f(g) = \rho(g - 1 + \Delta^n(\mathbf{Z}(G)))$, we obtain a polynomial map with $f(x) \neq 0$, which is a contradiction. Hence $x \in \Delta^n(\mathbf{Z}(G))$.

The last section of this proof is essentially in [**6**].

Note. When G/G_n is torsion of finite exponent e, it is not difficult to see that if $x \in \Delta(\mathbf{Z}(G))$ satisfies $x \in \Delta^n(\mathbf{Z}_p{}^\wedge(G))$ for all $p|e$, then $x \in \Delta^n(\mathbf{Z}(G))$.

3. Main theorem.

We require some preliminary lemmas.

LEMMA 4 [**7**]. *Let G be finitely generated, nilpotent and π-torsion free, where π is a collection of primes, and where not every prime is in π. Let π' be the set of primes not in π, Then G is residually a finite π'-group.*

LEMMA 5 [**3**]. *Let G be a nilpotent p-group of finite exponent. Let R be a commutative ring with 1 satisfying $\bigcap_n p^n R = 0$. Then $\bigcap_n \Delta^n(R(G)) = 0$.*

LEMMA 6. *Let G be residually a nilpotent p-group of finite exponent and let R be as in Lemma 5. Then $\bigcap_n \Delta^n(R(G)) = 0$.*

Proof. This is essentially found in [**4**]. Let $x \in \bigcap_n \Delta^n(R(G))$. Say $x = \sum a_i g_i$ with $g_1 = 1$. Since the class of nilpotent p-groups of finite exponent is closed under subgroups and direct products, we can find $H \lhd G$ such that G/H is a nilpotent p-group of finite exponent and such that $g_i g_j{}^{-1}$ is not in H for all $i \neq j$. By projecting to $R(G/H)$, we see that $\bar{x} \in \bigcap_n \Delta^n(R(G/H)) = 0$, by Lemma 5. However, by the choice of H, this implies that $x = 0$. Hence, $\bigcap_n \Delta^n(R(G)) = 0$ as required.

THEOREM 7. *Let G be finitely generated. Then the following conditions are equivalent:*

(i) $\bigcap_n \Delta^n(\mathbf{Z}_p{}^\wedge(G)) = 0$;

(ii) $\bigcap_n D_n(\mathbf{Z}_p{}^\wedge(G)) = 1$;

(iii) *G is residually a finite p-group.*

Proof. (i) $\Rightarrow$ (ii) is immediate. Now we assume (ii) and prove (iii). By Theorem 2, $D_n(\mathbf{Z}_p{}^\wedge(G)) = T_\pi$ ($G \bmod D_n(\mathbf{Z}(G))$), where $\pi = \{q | q$ is prime, $q \neq p\}$. Hence, $G/D_n(\mathbf{Z}_p{}^\wedge(G))$ is π-torsion free. By Lemma 4, $G/D_n(\mathbf{Z}_p{}^\wedge(G))$ is residually a finite p-group. Since $\bigcap_n D_n(\mathbf{Z}_p{}^\wedge(G)) = 1$, G is residually a finite p-group. Hence, (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i) is a special case of Lemma 6. This completes the proof.

We also observe the following:

PROPOSITION 8. *Let G have a generalized p-element. Then*

$$\bigcap_n \Delta^n(\mathbf{Z}(G)) = 0 \Leftrightarrow \bigcap_n \Delta^n(\mathbf{Z}_p^\wedge(G)) = 0.$$

Proof. It has been shown by Mital [4] that if G has a generalized p-element, then $\bigcap_n \Delta^n(\mathbf{Z}(G)) = 0 \Rightarrow G$ is residually a nilpotent p-group of finite exponent. By Lemma 6, $\bigcap_n \Delta^n(\mathbf{Z}_p^\wedge(G)) = 0$. The other direction is trivial.

4. Lie dimension subgroups. In [7], Sandling introduced the concept of lie dimension subgroups of G. Given $a, b \in R(G)$, define $(a, b) = ab - ba$. Given subsets A, B of $R(G)$, define $(A, B) = \{(a, b) \mid a \in A, b \in B\}$. Then the lie powers $\Delta^{(n)}$ of the augmentation ideal $\Delta(R(G))$ are defined inductively:

(i) $\Delta^{(1)} = \Delta$

(ii) $\Delta^{(n)} = (\Delta^{(n-1)}, \Delta)R(G)$.

Define the nth lie dimension subgroup $D_{(n)}(R(G))$ to be $\{g \mid g - 1 \in \Delta^{(n)}(R(G))\}$. Then it is proved in [7] that $\{D_{(n)}(R(G))\}$ form a descending central series and that $G_n \leqq D_{(n)}(R(G)) \leqq D_n(R(G))$. Using similar arguments to those used in the proofs of Theorem 2 and Proposition 3, and using some results of [7], we can obtain:

THEOREM 2′. *Let R be a commutative ring with* 1 *such that* $(R, +)$ *is torsion-free and let* $\pi = \{q \mid q$ *is prime and* $qR = R\}$. *Then* $D_{(n)}(R(G)) = G_2 \cap T_\pi$ (G mod $D_{(n)}(\mathbf{Z}(G))$), *for* $n \geqq 2$.

PROPOSITION 3′. *Let* $x \in \Delta(\mathbf{Z}(G))$ *satisfy* $x \in \Delta^{(n)}(\mathbf{Z}_p^\wedge(G))$ *for all* p. *Then* $x \in \Delta^{(n)}(\mathbf{Z}(G))$.

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