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A NOTE ON SPECIAL INVOLUTIONS

W. D. MUNN

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Abstract

The algebra consisting of those linear transformations of a complex inner product space that have a formal adjoint is shown to possess a special involution. Two earlier results concerning special involutions are then generalized.

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For a given complex inner product space V, the set of all linear transformations of V that have a formal adjoint constitutes a complex algebra and is shown to possess an involution that is special in the sense of Easdown and Munn (Theorem 1). It follows that the standard involution on a C^*-algebra is special (Corollary 1) – a result first noted by Hofmann – and that hermitian conjugation is a special involution on the algebra of all $I \times I$ row-finite and column-finite complex matrices, where I is an arbitrary nonempty set (Corollary 2). By combining Corollary 1 with a result of Barnes, it is proved that the natural involution on the l^1-algebra of an inverse semigroup is special (Theorem 2).

Let * be an involution on a semigroup S (that is, a permutation of S such that, for all a, b in S, $(ab)^* = b^*a^*$ and $a^{**} = a$). We say that * is *special* if and only if, for each nonempty finite subset T of S,

$$(\exists t \in T)(\forall u, v \in T) \quad t^*t = u^*v \Rightarrow u = v.$$

This definition is clearly equivalent to the one given originally in [2]. An involution [a special involution] on a complex algebra R is a mapping $^* : R \to R$ that is an automorphism of $(R, +)$ and an involution [a special involution] on $(R, \cdot)$, with the further property that, for all $a \in R$ and all $\lambda \in \mathbb{C}$ (the complex field), $(\lambda a)^* = \bar{\lambda} a^*$, where $\bar{\lambda}$ denotes the complex conjugate of λ. Examples of special involutions include hermitian conjugation on the algebra of all $n \times n$ complex matrices [2] and the natural involution on the complex semigroup algebra of an inverse semigroup [3]. Both of these are generalized here. By a *star subalgebra* of a complex algebra R with an involution * we mean a subalgebra S of R such that $a^* \in S$ for all $a \in S$. Observe that if * is special and S is a star subalgebra of R then * induces a special involution on S.

In the theorem below we examine a certain subalgebra of the algebra of all linear transformations of a complex inner product space, namely the subalgebra consisting of all elements that possess a 'formal adjoint'.

THEOREM 1. *Let V be a complex vector space that admits an inner product $\langle \mid \rangle$, let $L(V)$ denote the algebra of all linear transformations of V and let*

$$A(V) := \{a \in L(V) : (\exists b \in L(V))(\forall x, y \in V)\ \langle ax|y\rangle = \langle x|by\rangle\}.$$

Then

(i) *$A(V)$ is a subalgebra of $L(V)$,*
(ii) *to each $a \in A(V)$ there corresponds a unique $a^* \in A(V)$ such that, for all $x, y \in V$, $\langle ax|y\rangle = \langle x|a^*y\rangle$,*
(iii) *the mapping $^* : A(V) \to A(V)$, $a \mapsto a^*$ is a special involution.*

PROOF. (i) This is routine.

(ii) Let $a \in A(V)$. Suppose that $b, c \in L(V)$ are such that, for all $x, y \in V$, $\langle ax|y\rangle = \langle x|by\rangle = \langle x|cy\rangle$. Then, for all $y \in V$, $\langle (b-c)y|(b-c)y\rangle = 0$ and so $(b-c)y = 0$. Thus $b = c$. This establishes the existence of a unique $a^* \in L(V)$ such that, for all $x, y \in V$, $\langle ax|y\rangle = \langle x|a^*y\rangle$. Then, for all $x, y \in V$,

$$\langle a^*x|y\rangle = \overline{\langle y|a^*x\rangle} = \overline{\langle ay|x\rangle} = \langle x|ay\rangle$$

and so $a^* \in A(V)$. (This argument shows also that $a^{**} = a$.)

(iii) It is easily checked that * is an involution on $A(V)$: we must prove that it is special.

Let T be a nonempty finite subset of $A(V)$. Write

$$U := \{a - b : a, b \in T\}.$$

We show first that there exists a linear functional χ on $A(V)$ such that

(1) $(\forall a \in A(V))\quad \chi(a^*a)$ is real and nonnegative,
(2) $(\forall u \in U)\quad \chi(u^*u) = 0$ implies $u = 0$.

If $U = \{0\}$ we take χ to be the zero mapping. Suppose, therefore, that $U \neq \{0\}$. Let $u_1, u_2, \ldots, u_n$ be the nonzero elements of U. For each $r \in \{1, 2, \ldots, n\}$, choose

$x_r \in V$ such that $u_r x_r \neq 0$. We define $\chi : A(V) \to \mathbb{C}$ by

$$(\forall a \in A(V)) \quad \chi(a) := \sum_{i=1}^{n} \langle ax_i | x_i \rangle.$$

It is clear that χ is linear. To establish (1), we simply note that, for any $a \in A(V)$,

$$\chi(a^*a) = \sum_{i=1}^{n} \langle a^*ax_i | x_i \rangle = \sum_{i=1}^{n} \langle ax_i | ax_i \rangle;$$

further, (2) holds, since, for $r \in \{1, 2, \dots, n\}$,

$$\chi(u_r^* u_r) = \sum_{i=1}^{n} \langle u_r x_i | u_r x_i \rangle \geq \langle u_r x_r | u_r x_r \rangle > 0.$$

Choose $t \in T$ such that $\chi(t^*t) = \max\{\chi(a^*a) : a \in T\}$. Suppose that $t^*t = a^*b$, where $a, b \in T$. We complete the proof by showing that $a = b$. Since $t^*t = (a^*b)^* = b^*a$, we have that $(a-b)^*(a-b) = a^*a + b^*b - 2t^*t$. Hence, by (1) and the choice of t,

$$0 \leq \chi((a-b)^*(a-b)) = \chi(a^*a) + \chi(b^*b) - 2\chi(t^*t) \leq 0$$

and so $\chi((a-b)^*(a-b)) = 0$. But $a - b \in U$. Hence, by (2), $a = b$. □

COROLLARY 1 (Hofmann). *The standard involution on a C^*-algebra is special.*

PROOF. It is sufficient to consider the case of the C^*-algebra $B(V)$ of all bounded linear operators on a complex Hilbert space V. Clearly $B(V)$ is a star subalgebra of $A(V)$ and the standard involution on $B(V)$ is the restriction of the involution * on $A(V)$. Since * is special, the result follows. □

Let I be a nonempty set. An $I \times I$ complex matrix $[\alpha_{ij}]$ is said to be *row-finite* if and only if, for all $i \in I$, the set $\{j \in I : \alpha_{ij} \neq 0\}$ is finite (possibly empty). Similarly, $[\alpha_{ij}]$ is *column-finite* if and only if, for all $j \in I$, $\{i \in I : \alpha_{ij} \neq 0\}$ is finite. The set $\mathbb{C}_I$ of all $I \times I$ complex matrices that are both row-finite and column-finite is a complex algebra under the usual matrix operations and is closed under hermitian conjugation.

COROLLARY 2. *Let I be a nonempty set. Then hermitian conjugation is a special involution on $\mathbb{C}_I$.*

PROOF. Let V denote the complex vector space consisting of all $I \times \{1\}$ 'column' vectors with at most finitely many nonzero entries. Then the mapping $\theta : \mathbb{C}_I \to L(V)$ defined by $\theta(a)x = ax$ $(x \in V)$, where ax is the usual matrix product, is an injective homomorphism. Moreover, V admits an inner product $\langle\ |\ \rangle$ defined by $\langle x|y\rangle = \Sigma_i \xi_i \bar{\eta}_i$, where ξ_i and η_i denote the ith components of x and y respectively; and it is easily seen that, for all $a \in \mathbb{C}_I$ and all $x, y \in V$, $\langle ax|y\rangle = \langle x|a^\dagger y\rangle$, where $a^\dagger$ denotes the hermitian conjugate of a. Thus, for all $a \in \mathbb{C}_I$, $\theta(a) \in A(V)$ and $(\theta(a))^* = \theta(a^\dagger)$. But, by the theorem, * is special. Hence, since $\operatorname{im}\theta$ is a star subalgebra of $A(V)$ and θ is injective, it follows that hermitian conjugation is a special involution on $\mathbb{C}_I$. □

Observe that if I is infinite then $\operatorname{im}\theta$ above contains unbounded linear operators on V.

Each of these corollaries generalizes the result, due to Lavers [2, Example 4], that, for any positive integer n, hermitian conjugation is a special involution on the algebra of all $n \times n$ complex matrices.

A further application of Theorem 1 arises in the context of certain Banach algebras. Let S be a semigroup. We denote by $l^1(S)$ the Banach algebra consisting of all functions $a : S \to \mathbb{C}$ of countable support such that $\Sigma_{x\in S}|a(x)| < \infty$, where addition and scalar multiplication are the usual pointwise operations, multiplication is convolution, and the norm $\|\ \|$ is defined by

$$(\forall a \in l^1(S)) \quad \|a\| := \sum_{x\in S} |a(x)|.$$

Now suppose that S is an inverse semigroup; thus, to each $x \in S$ there corresponds a unique element $x^{-1} \in S$ (the 'inverse' of x) such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. It is well known that inversion $(x \mapsto x^{-1})$ is an involution on S. As is readily checked, inversion on S induces an involution † on $l^1(S)$ by the rule that

$$(\forall x \in S) \quad a^\dagger(x) := \overline{a(x^{-1})}.$$

We now combine Corollary 1 with a result of Barnes [1] to show that † is special.

THEOREM 2 (Crabb). *Let S be an inverse semigroup. Then the involution on $l^1(S)$ induced by inversion on S is special.*

PROOF. As above, let † denote the involution on $l^1(S)$ induced by inversion on S. By [1, Theorem 2.3], there exists a Hilbert space V and a (continuous) injective homomorphism $\theta : l^1(S) \to B(V)$, the algebra of all bounded linear operators on V, such that, for all $a \in l^1(S)$, $(\theta(a))^* = \theta(a^\dagger)$, where * denotes the standard involution on $B(V)$. But, by Corollary 1, * is special. Hence, since $\operatorname{im}\theta$ is a star subalgebra of $B(V)$ and θ is injective, it follows that † is special. □

This extends [3, Theorem 5.1].

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Department of Mathematics
University of Glasgow
Glasgow G12 8QW
Scotland
U.K.
e-mail: wdm@maths.gla.ac.uk