

# An Intertwining Result for $p$ -adic Groups

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*Abstract.* For a reductive  $p$ -adic group  $G$ , we compute the supports of the Hecke algebras for the  $K$ -types for  $G$  lying in a certain frequently-occurring class. When  $G$  is classical, we compute the intertwining between any two such  $K$ -types.

## 0 Introduction

Let  $F$  denote a nonarchimedean local field of residual characteristic  $p$ ,  $\mathbf{G}$  a connected reductive algebraic group defined over  $F$ , and  $G = \mathbf{G}(F)$  the group of  $F$ -points of  $\mathbf{G}$ .

Howe proposed and developed (in the case of  $G = \mathrm{GL}(N)$ ) the idea of studying the admissible (complex) representations of  $G$  by examining their restriction to appropriate compact open subgroups. (Such restrictions had already played a role in the work of Mautner [12], Shalika [23], and Tanaka [27] on groups of type  $A_1$ .) Later, Howe and Moy [9, 10, 11] obtained much detailed information under various tameness hypotheses. In particular, if  $p \geq N$ , their work provides a classification of the irreducible representations of  $\mathrm{GL}(N)$ . There has been recent progress on two fronts.

Let  $J$  be a compact open subgroup of  $G$  and  $(\rho, W)$  a smooth irreducible representation of  $J$ . Write  $\mathfrak{R}$  for the category of smooth representations of  $G$  and  $\mathfrak{R}_\rho$  for the full subcategory consisting of all smooth representations that are generated by their  $\rho$ -isotypic vectors. If  $\mathfrak{R}_\rho$  is closed under subquotients, then Bushnell and Kutzko [3] call  $\rho$  a *type*. This occurs if and only if  $\mathfrak{R}_\rho$  is a finite sum of components of the Bernstein decomposition of  $G$ . When  $\rho$  is a type, there is an equivalence of categories between  $\mathfrak{R}_\rho$  and the category of left modules over  $\mathcal{H}(G, \rho)$ . Here  $\mathcal{H}(G, \rho)$  denotes the convolution algebra of  $\rho^\vee$ -spherical functions  $\Psi: G \rightarrow \mathrm{End}_{\mathbb{C}}(W^\vee)$ . The program of constructing a type for a given component of the Bernstein decomposition and identifying the resulting Hecke algebra may be viewed as a precise form of the philosophy of classifying representations in terms of compact-open data. Bushnell and Kutzko [2, 4] have completed this program for  $G = \mathrm{GL}(N)$ .

Moy and Prasad [17, 18] define a class of “unrefined minimal  $K$ -types”, which has the property (among others) that every admissible representation must contain one. These  $K$ -types (“fundamental strata” in the language of Bushnell-Kutzko) provide an essential first step in the work of Howe-Moy and Bushnell-Kutzko on  $\mathrm{GL}(N)$ . They arise roughly as follows. Let  $\mathcal{B} = \mathcal{B}(\mathbf{G}, F)$  be the Bruhat-Tits building of  $G$ , and let  $\mathfrak{g}$  and  $\mathfrak{g}^*$  be the Lie algebra of  $G$  and its dual, respectively. To each point  $x$  in  $\mathcal{B}$ , one can associate a parahoric subgroup  $G_x = G_{x,0}$  of  $G$ , together with a filtration  $\{G_{x,r}\}$  of normal subgroups of  $G_x$ , indexed by positive real numbers. Similarly, one has filtrations  $\{\mathfrak{g}_{x,r}\}$  and  $\{\mathfrak{g}_{x,r}^*\}$  of lattices in  $\mathfrak{g}$  and  $\mathfrak{g}^*$ , respectively, indexed by real numbers. Write  $G_{x,r^+} = \bigcup_{s>r} G_{x,s}$  and define  $\mathfrak{g}_{x,r^+}$  and  $\mathfrak{g}_{x,r^+}^*$  similarly. If  $r > 0$ , then the quotient  $G_{x,r}/G_{x,r^+}$  is abelian, and (except in some

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Received by the editors September 18, 1998; revised January 6, 2000.

AMS subject classification: Primary: 22E50; secondary: 20G05.

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cases not considered here) isomorphic to  $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$ . Its Pontryagin dual is isomorphic to  $\mathfrak{g}_{x,-r}^*/\mathfrak{g}_{x,(-r)^+}^*$ . (This last isomorphism is non-canonical, but in a way that turns out not to be relevant for us.)

Consider a pair  $(G_{x,r}, \rho)$ , where  $\rho$  is an irreducible representation of  $G_{x,r}/G_{x,r^+}$ . Such a pair is an *unrefined minimal  $K$ -type of (positive) depth  $r$*  if  $r > 0$  and  $\rho$  corresponds to a coset in  $\mathfrak{g}^*$  containing no nilpotent elements. One can also define unrefined minimal  $K$ -types of depth zero, using the fact that  $G_{x,0}/G_{x,0^+}$  is a finite reductive group.

One expects that every unrefined minimal  $K$ -type  $(G_{x,r}, \rho)$  is a type in the sense of Bushnell-Kutzko (for a “large” sum of components of the Bernstein decomposition when  $r > 0$ ). For  $r = 0$ , this is known [13, 18]. Whenever this is so, we have an equivalence of categories between  $\mathfrak{R}_\rho$  and  $\mathcal{H}(G, \rho)\text{-Mod}$ , as noted above. But even if  $\rho$  is not a type, we still have a correspondence between irreducible smooth representations of  $G$  containing  $\rho$  and simple  $\mathcal{H}(G, \rho)$ -modules (see [5] or [2, 4.2.3 and 4.2.4]).

To determine the structure of  $\mathcal{H}$ , one typically first determines its *support*, which is defined to be the set

$$\text{supp } \mathcal{H} = \{g \in G \mid f(g) \neq 0 \text{ for some } f \in \mathcal{H}\}.$$

By an elementary argument (see the discussion preceding [7, Lemma 2.2], or [2, 4.1.1]), the support of  $\mathcal{H}$  is equal to the *intertwining* of the pair  $(K, \rho)$ , which is defined to be the set

$$\mathcal{J}_G(\rho) = \{g \in G \mid \text{Hom}_{K \cap \mathfrak{g}Kg^{-1}}(\rho, {}^g\rho) \neq 0\}.$$

Here,  ${}^g\rho$  is the representation of  $gKg^{-1}$  given by  ${}^g\rho(gkg^{-1}) = \rho(k)$  for all  $k \in K$ .

The purpose of this paper is to compute  $\mathcal{J}_G(\rho)$  for all  $\rho$  in a certain frequently-occurring class. We summarize our results below.

Recall that, for  $r > 0$ , characters of  $G_{x,r}/G_{x,r^+}$  correspond to certain cosets in  $\mathfrak{g}^*$ . The same is true (see [1, Section 1.7]) for characters of all subgroups of groups of the form  $G_{x,s}/G_{x,2s}$ . Suppose  $\rho$  is such a character, and  $\Upsilon \subset \mathfrak{g}^*$  is the corresponding coset. Then

$$\mathcal{J}_G(\rho) = \{g \in G \mid \text{Ad}^*(g)\Upsilon \cap \Upsilon \neq \emptyset\}.$$

This is proved in [1, Section 1.8], using an imitation of a proof of a more specialized result in [11]. Thus, for  $\rho$  arising in this way, the computation of  $\mathcal{J}_G(\rho)$  boils down to an understanding of the coadjoint action of  $G$  on  $\mathfrak{g}^*$ .

Under certain mild conditions, one can establish a  $G$ -equivariant isomorphism  $\mathfrak{g} \rightarrow \mathfrak{g}^*$ . One can then attach to each  $\rho$  (as above) a coset  $\Upsilon$  in  $\mathfrak{g}$ . This coset is particularly easy to compute in case our isomorphism takes each  $\mathfrak{g}_{x,r}$  to  $\mathfrak{g}_{x,r}^*$ . We include the details in Section 4. Thus,

$$\mathcal{J}_G(\rho) = \{g \in G \mid \text{Ad}(g)\Upsilon \cap \Upsilon \neq \emptyset\}.$$

We will also denote this set by  $\mathcal{J}_G(\Upsilon)$ .

Suppose  $(G_{x,r}, \rho)$  is an unrefined minimal  $K$ -type (with  $r > 0$ ) and let  $\Upsilon \in \mathfrak{g}_{x,-r}/\mathfrak{g}_{x,(-r)^+}$  be the corresponding coset (in  $\mathfrak{g}$ , not  $\mathfrak{g}^*$ ). One can often pick an element  $a \in \Upsilon$  having certain good properties (see Proposition 5.4). When this is the case, one can find a subgroup  $J_+ \subset G_{x,(r/2)^+}$  such that  $\rho$  extends in a trivial way to a character  $\bar{\rho}$  of  $J_+$ , and such that every representation of  $G$  that contains  $\rho$  must contain  $\bar{\rho}$ . (This is all handled in Section 6.) Thus,

for the purpose of studying representations of  $G$ , it is enough to consider  $\mathcal{H}(G, \bar{\rho})$  instead of  $\mathcal{H}(G, \rho)$ .

In this paper, we show that, under mild hypotheses on  $p$ ,

$$J_G(\bar{\rho}) = J_{C_G(a)}J,$$

where  $J$  is a certain group slightly larger than  $J_+$ , and  $C_G(a)$  is the stabilizer of  $a$  in  $G$ . The precise result is stated in Section 7. (We note that for  $K$ -types of depth zero, the structure of the Hecke algebra has been described by Morris [15].)

Suppose  $(K_1, \rho_1)$  and  $(K_2, \rho_2)$  are two  $K$ -types. Using similar methods, one can sometimes compute the intertwining set

$$J_G(\rho_1, \rho_2) = \{g \in G \mid \text{Hom}_{K \cap gKg^{-1}}(\rho_1, \rho_2) \neq 0\}.$$

This is equal to the support of

$$\mathcal{H}(G, \rho_1, \rho_2) = \left\{ f: G \rightarrow \text{Hom}(\rho_2^\vee, \rho_1^\vee) \mid \begin{array}{l} f(k_1 g k_2) = \rho_1^\vee(k_1) f(g) \rho_2^\vee(k_2) \text{ for all } k_i \in K_i \\ \text{and } g \in G, \text{ and } f \text{ is compactly supported.} \end{array} \right\}$$

We pursue the matter in Section 9.

This work is a combined generalization of methods developed for use in [1] and [21]. Yu's construction of supercuspidal representations [28] relies on a similar intertwining result, which he obtains in a way that does not require one to identify  $\mathfrak{g}^*$  with  $\mathfrak{g}$ .

### 1 Notation, Conventions, and Some Primes to Avoid

Let  $\mathbb{Z}$  and  $\mathbb{R}$  denote the sets of integers and real numbers, respectively.

Let  $F$  be a nonarchimedean local field of residual characteristic  $p$ ,  $\mathcal{O}$  its ring of integers, and  $\mathcal{P}$  the prime ideal in  $\mathcal{O}$ . Let  $\bar{F}$  be an algebraic closure of  $F$ . For any extension  $E$  of  $F$ , let  $\mathcal{P}_E$  denote the prime ideal in the ring of integers  $\mathcal{O}_E$  of  $E$ .

Let  $\mathbf{G}$  be a connected reductive group defined over  $F$ , let  $\mathbf{Z}$  be the connected component of the center of  $\mathbf{G}$ , and let  $\mathbf{G}'$  be the derived group of  $\mathbf{G}$ . Let  $\mathfrak{g}$ ,  $\mathfrak{z}$ , and  $\mathfrak{g}'$  denote the Lie algebras of  $G = \mathbf{G}(F)$ ,  $Z = \mathbf{Z}(F)$ , and  $G' = \mathbf{G}'(F)$ , respectively.

There are several restrictions that we will need to impose on the characteristic and residual characteristic of  $F$  at various points in our argument. In order to describe them, we must establish some more notation, some of which comes from [24].

For the rest of this section, fix a maximal torus  $\mathbf{T} \subset \mathbf{G}$ . There are associated lattices  $\mathbf{X} = \mathbf{X}^*(\mathbf{T}) = \text{Hom}(\mathbf{T}, \text{GL}_1)$  and  $\mathbf{X}^\vee = \mathbf{X}_*(\mathbf{T}) = \text{Hom}(\text{GL}_1, \mathbf{T})$  of characters and cocharacters, respectively, of  $\mathbf{T}$ , and these lattices contain a root system  $\Phi = \Phi(\mathbf{G}, \mathbf{T})$  and a coroot system  $\Phi^\vee$ . There is a natural, nondegenerate pairing

$$\langle \cdot, \cdot \rangle: \mathbf{X} \otimes \mathbf{X}^\vee \rightarrow \mathbb{Z}$$

given by  $(\beta \circ \alpha)(x) = x^{\langle \alpha, \beta \rangle}$  for all  $x \in \text{GL}_1$ . This extends to a real-valued pairing between  $\mathbf{X} \otimes \mathbb{R}$  and  $\mathbf{X}^\vee \otimes \mathbb{R}$ . Let

$$\begin{aligned} P^\vee &= \{v \in \mathbb{Z}\Phi^\vee \otimes \mathbb{R} \mid \langle v, \alpha \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}, \\ \mathbf{X}_0 &= \{\alpha \in \mathbf{X} \mid \langle \alpha, y \rangle = 0 \text{ for all } y \in \Phi^\vee\}, \\ \mathbf{X}_0^\vee &= \{y \in \mathbf{X}^\vee \mid \langle \alpha, y \rangle = 0 \text{ for all } \alpha \in \Phi\}. \end{aligned}$$

$\Phi$	$z(\Phi)$	$\text{bad}(\Phi)$	$c(\Phi)$
$A_n$	$n + 1$	$\{\}$	
$B_n, C_n$	2	$\{2\}$	
$D_n$	4	$\{2\}$	$(c(D_4) \leq 31)$
$E_6$	3	$\{2, 3\}$	$\leq 113$
$E_7$	2	$\{2, 3\}$	$\leq 373$
$E_8$	1	$\{2, 3, 5\}$	$\leq 1291$
$F_4$	1	$\{2, 3\}$	$\leq 61$
$G_2$	1	$\{2, 3\}$	7

Table: Some data associated to root systems

Then  $\mathbb{Z}\Phi^\vee \subset P^\vee$ , and  $P^\vee/\mathbb{Z}\Phi^\vee$  is a finite group depending only on  $\Phi$ . Let  $z(\Phi)$  denote the order of this group. This can be interpreted as the order of the center of the simply connected cover  $\mathbf{G}'_{\text{sc}}$  of  $\mathbf{G}'$ . For  $\Phi$  simple, the values of  $z(\Phi)$  are given in the table. In general, it is clear that  $z(\Phi) = \prod z(\Phi_i)$ , where the product is taken over the irreducible factors  $\Phi_i$  of  $\Phi$ .

Let  $\widetilde{\mathbb{Z}\Phi^\vee}$  be the largest subgroup of  $\mathbf{X}^\vee$  that contains  $\mathbb{Z}\Phi^\vee$  with finite index.

Then the center of  $\mathbf{G}'$  has order  $|P^\vee/\widetilde{\mathbb{Z}\Phi^\vee}|$ , which we denote by  $z(\mathbf{G})$ , and  $|\pi_1(\mathbf{G}')| = |\widetilde{\mathbb{Z}\Phi^\vee}/\mathbb{Z}\Phi^\vee|$ . Define

$$k(\mathbf{G}) = |\mathbf{X}^\vee/(\mathbf{X}_0^\vee + \widetilde{\mathbb{Z}\Phi^\vee})|.$$

Then  $k(\mathbf{G})$  may be interpreted as the order of the kernel of the canonical isogeny  $\mathbf{Z} \times \mathbf{G}' \rightarrow \mathbf{G}$ . Note that  $k(\mathbf{G})$  divides  $z(\mathbf{G})$ , which in turn divides  $z(\Phi)$ . Moreover,  $k(\mathbf{G}) \cdot |\pi_1(\mathbf{G}')|$  is the order of the kernel of the canonical isogeny  $\mathbf{Z} \times \mathbf{G}'_{\text{sc}} \rightarrow \mathbf{G}$ .

We use an observation of Springer and Steinberg [25, Section I.4.3]. A subset  $\Phi_1 \subset \Phi$  is called a closed subsystem if  $\mathbb{Z}\Phi_1 \cap \Phi = \Phi_1$  and if whenever  $\alpha$  and  $\beta$  are in  $\Phi_1$ , and  $s_\alpha$  is the reflection corresponding to  $\alpha$ , then  $s_\alpha(\beta) \in \Phi_1$ . Say that a prime  $q$  is “bad” for  $\Phi$  if  $\mathbb{Z}\Phi/\mathbb{Z}\Phi_1$  has  $q$ -torsion for some closed subsystem  $\Phi_1$  of  $\Phi$ . Let  $\text{bad}(\mathbf{G}) = \text{bad}(\Phi)$  denote the set of bad primes for  $\Phi$ . The sets  $\text{bad}(\Phi)$  for the irreducible root systems are given in the table. More generally, a prime is bad for  $\Phi$  if it is bad for some irreducible factor of  $\Phi$ .

Here is one role that bad primes will play for us. For any semisimple  $a \in \mathfrak{g}$ , let

$$C_{\mathbf{G}}(a) = \{g \in \mathbf{G} \mid \text{Ad}(g)a = a\},$$

$$C_G(a) = \{g \in G \mid \text{Ad}(g)a = a\},$$

$$C_{\mathfrak{g}}(a) = \{b \in \mathfrak{g} \mid \text{ad}(b)a = 0\}.$$

Then  $C_{\mathfrak{g}}(a)$  is the Lie algebra of  $C_G(a)$ , and  $C_G(a)$  is the group of  $F$ -points of  $C_{\mathbf{G}}(a)$ . Suppose for the moment that the characteristic of  $F$  is not a bad prime for  $\mathbf{G}$ . This implies, for example, that the connected component of  $C_G(a)$  is an  $E$ -Levi subgroup of  $\mathbf{G}$ , for some finite extension  $E/F$ . If we also assume that the characteristic does not divide  $|\pi_1(\mathbf{G}')|$ , then the characteristic is not a “torsion” prime for  $\mathbf{G}$  in the sense of [26]. Therefore, by [26, Theorem 3.14], the group  $C_G(a)$  is connected, and thus is an  $E$ -Levi subgroup of  $\mathbf{G}$ .

Finally, we let  $c(\mathbf{G}) = c(\Phi)$  be the smallest integer such that for any prime  $q > c(\mathbf{G})$ , we are guaranteed that  $\mathbf{X}/\mathbf{Y}$  has no  $q$ -torsion for any  $\mathbf{Y} \subset \mathbf{X}$  which has a set of  $\mathbb{Z}$ -generators each of which is either a root or a sum of two distinct roots. We will use one argument where we have to assume that  $p > c(\mathbf{G})$ . This argument is necessary only for exceptional and triality groups, so we have only estimated  $c$  for the associated root systems, usually obtaining a crude upper bound. However, we stress that the condition  $p > c(\mathbf{G})$  is not itself a hypothesis of our main theorem; its purpose is to guarantee that the other hypotheses will often be satisfied.

## 2 Some Results Concerning the Moy-Prasad Filtrations

Let  $\mathbf{T}$  be an  $F$ -torus in  $\mathbf{G}$  and  $\mathfrak{t}$  the Lie algebra of  $T = \mathbf{T}(F)$ . Then  $T$  and  $\mathfrak{t}$  have natural filtrations, defined as follows. Let  $E/F$  be a minimal splitting field for  $\mathbf{T}$  and  $\nu_E$  its normalized valuation. For any  $r \in \mathbb{R}$ , let

$$\mathfrak{t}_r = \{H \in \mathfrak{t} \mid \nu_E(d\chi(H)) \geq r \text{ for all } \chi \in \mathbf{X}^*(\mathbf{T})\}.$$

For any  $r > 0$ , let

$$T_r = \{t \in T \mid \nu_E(\chi(t) - 1) \geq r \text{ for all } \chi \in \mathbf{X}^*(\mathbf{T})\}.$$

Following [17], one can associate to any point  $x$  in the building  $\mathcal{B}$  of  $G$  a parahoric subgroup  $G_x = G_{x,0}$  of  $G$ , a filtration  $\{G_{x,r}\}_{r \geq 0}$  of  $G_x$ , and a filtration  $\{\mathfrak{g}_{x,r}\}_{r \in \mathbb{R}}$  of the Lie algebra  $\mathfrak{g}$  of  $G$ . Moreover, these filtrations are compatible with the given filtrations of  $T$  and  $\mathfrak{t}$  if  $x$  lies in the apartment of  $\mathbf{T}(E)$  in  $\mathcal{B}(\mathbf{G}, E)$ .

It will be convenient for us to normalize the filtrations as in [19] rather than as in [18]. The normalizations differ in the following way. Let  $K$  be a maximal unramified extension of  $F$ , and let  $L$  be the minimal extension of  $K$  such that  $\mathbf{G}$  is  $L$ -split. Let  $\ell = [L : K]$ . Moy and Prasad normalize the filtrations with respect to a valuation normalized for  $L$ , while Pan and Yu use a valuation normalized for  $F$ . In particular, what we call  $\mathfrak{g}_{x,r}$  and  $G_{x,r}$  would be called  $\mathfrak{g}_{x,r\ell}$  and  $G_{x,r\ell}$  (respectively) according to the Moy-Prasad definition. Note that these normalizations are the same if  $\mathbf{G}$  splits over an unramified extension.

Suppose  $E/F$  is a finite Galois extension, and  $\mathbf{M}$  is an  $E$ -Levi subgroup of  $\mathbf{G}$ . Let  $M = \mathbf{M}(F)$  and  $\mathfrak{m} = \text{Lie}(M)$ . Since every maximal  $E$ -split torus in  $\mathbf{M}$  is also a maximal  $E$ -split torus in  $\mathbf{G}$ , one can embed  $\mathcal{B}(\mathbf{M}, E)$  in  $\mathcal{B}(\mathbf{G}, E)$ , and the set of such embeddings is an affine space. The group  $\text{Gal}(E/F)$  acts on this space, and must have a fixed point, which is a Galois-equivariant embedding. All such embeddings have the same image.

If  $E/F$  is tame, then  $\mathcal{B}(\mathbf{G}, F)$  is the  $\text{Gal}(E/F)$  fixed point set in  $\mathcal{B}(\mathbf{G}, E)$  (see [22, Proposition 5.1.1]). The image of  $\mathcal{B}(\mathbf{M}, F)$  lies in  $\mathcal{B}(\mathbf{G}, F)$ . Given  $x$  lying in this image, we can use this embedding to identify  $x$  with an element of  $\mathcal{B}(\mathbf{M}, F)$ . The groups and lattices  $M_x, M_{x,r}, \mathfrak{m}_{x,r}$  are independent of the choice of embedding.

We list a few basic properties of the filtrations. (For a proof of Proposition 2.1, see [1]. For Proposition 2.2 and Lemma 2.3, see [1, 17, 20].)

**Proposition 2.1** *Let  $F^\sharp/F$  be a finite extension of ramification degree  $e$ , and let  $\mathbf{G}^\sharp = \mathbf{G} \times_F F^\sharp$  be  $\mathbf{G}$  regarded as an  $F^\sharp$ -group. Define  $G^\sharp, \mathfrak{g}^\sharp$ , etc., accordingly. Then for all  $r \in \mathbb{R}$  and for all*

$x \in \mathcal{B}(\mathbf{G}, F)$ ,

$$(G_{x,r}^\sharp) \cap G_x = G_{x,r/e} \quad (r > 0),$$

$$(\mathfrak{g}_{x,r}^\sharp) \cap \mathfrak{g} = \mathfrak{g}_{x,r/e}.$$

Note that  $\mathfrak{g}^\sharp$  is just  $\mathfrak{g} \otimes F^\sharp$ .

**Proposition 2.2** *The Moy-Prasad filtrations have the following properties:*

- (a) For any  $g \in G$ , let  $gx$  be the image of  $x$  under the action of  $G$  on  $\mathcal{B}(\mathbf{G}, F)$ . Then  $\text{Int}(g)G_{x,r} = G_{gx,r}$  and  $\text{Ad}(g)\mathfrak{g}_{x,r} = \mathfrak{g}_{gx,r}$ .
- (b)  $[G_{x,r}, G_{x,s}] \subset G_{x,r+s}$  and  $[\mathfrak{g}_{x,r}, \mathfrak{g}_{x,s}] \subset \mathfrak{g}_{x,r+s}$ .
- (c) If  $\mathfrak{m}$  is the Lie algebra of  $M$ , and  $x \in \mathcal{B}(\mathbf{M}, F) \cap \mathcal{B}(\mathbf{G}, F)$ , then  $\mathfrak{g}_{x,r} \cap \mathfrak{m} = \mathfrak{m}_{x,r}$  and  $G_{x,r} \cap M = M_{x,r}$ .

Moy and Prasad also define filtration lattices  $\{\mathfrak{g}_{x,r}^*\}$  in the dual  $\mathfrak{g}^*$  of  $\mathfrak{g}$  by

$$\mathfrak{g}_{x,r}^* = \{\chi \in \mathfrak{g}^* \mid \chi(\mathfrak{g}_{x,(-r)^+}) \subset \mathcal{P}\}.$$

These lattices satisfy statements analogous to those in Proposition 2.1 and Proposition 2.2 (except for Proposition 2.2(b), which makes no sense in this context).

As a notational convenience, we write  $G_{x,r^+} = \bigcup_{s>r} G_{x,s}$ ,  $\mathfrak{g}_{x,r^+} = \bigcup_{s>r} \mathfrak{g}_{x,s}$ , and  $\mathfrak{g}_{x,r^+}^* = \bigcup_{s>r} \mathfrak{g}_{x,s}^*$ .

Suppose that  $\mathbf{G}$  contains a maximal torus that splits over some tamely ramified extension. Then for all  $x \in \mathcal{B}(\mathbf{G}, F)$  and for all  $0 < s \leq t \leq 2s$ , there is an isomorphism  $\mathfrak{g}_{x,s}/\mathfrak{g}_{x,t} \rightarrow G_{x,s}/G_{x,t}$ . One can define (in a non-canonical way) a filtration-preserving homeomorphism  $\varphi_x: \mathfrak{g}_{x,0^+} \rightarrow G_{x,0^+}$  that is compatible with each of these isomorphisms and that also has other desirable properties. For example (see [1] for a construction and proofs):

**Lemma 2.3** *If  $X \in \mathfrak{g}_{x,r}$  and  $Y \in \mathfrak{g}_{x,s}$  with  $r, s > 0$ , then*

$$\text{Ad}(\varphi_x(X))Y \equiv Y + [X, Y] \pmod{\mathfrak{g}_{x,2r+s}}.$$

**Lemma 2.4** *Let  $f: \mathbf{G}^{(1)} \rightarrow \mathbf{G}^{(2)}$  be a central isogeny with kernel of order  $k$ . Then  $df: \mathfrak{g}^{(1)} \rightarrow \mathfrak{g}^{(2)}$  is an isomorphism if and only if the characteristic of  $F$  is zero or prime to  $k$ . Let  $x$  be a point in the building  $\mathcal{B}(\mathbf{G}^{(1)}, F)$ , which we identify with  $\mathcal{B}(\mathbf{G}^{(2)}, F)$ , and let  $r \in \mathbb{R}$ . If  $p$  is prime to  $k$ , then  $df(\mathfrak{g}_{x,r}^{(1)}) = \mathfrak{g}_{x,r}^{(2)}$ .*

**Proof** The first statement follows from [24, Section 2.6]. From Proposition 2.1, we may assume that the  $\mathbf{G}^{(i)}$  are  $F$ -split. Let  $\mathfrak{t}^{(i)}$  be the Lie algebra of a maximal split torus in  $\mathbf{G}^{(i)}(F)$ , chosen so that  $df(\mathfrak{t}^{(1)}) = \mathfrak{t}^{(2)}$ . The result then follows from the way in which the Moy-Prasad filtrations of the  $\mathfrak{t}^{(i)}$  are defined. ■

### 3 Decomposition of the Lie Algebra

**Proposition 3.1** *If the characteristic of  $F$  does not divide  $k(\mathbf{G})$ , then  $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ .*

**Proof** This follows from Lemma 2.4, but here is a direct proof. It is enough to consider the case where  $\mathbf{G}$  is  $F$ -split. Let  $T = \mathbf{T}(F)$  be a maximal  $F$ -split torus in  $G = \mathbf{G}(F)$ , let  $\mathfrak{t}$  be its Lie algebra, and let  $\Phi$  be the root system for  $\mathbf{G}$  with respect to  $\mathbf{T}$ . Then  $\mathfrak{g}$  and  $\mathfrak{g}'$  have root space decompositions given by

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{g}' = \mathfrak{t}' \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}'_\alpha,$$

where  $\mathfrak{t}' = \mathfrak{t} \cap \mathfrak{g}'$ , and  $\mathfrak{g}_\alpha$  and  $\mathfrak{g}'_\alpha$  are the  $\alpha$ -eigenspaces in  $\mathfrak{g}$  and  $\mathfrak{g}'$  coming from the adjoint action of  $T$ . Then  $\mathfrak{g}_\alpha = \mathfrak{g}'_\alpha$ , so it is enough to show that  $\mathfrak{t} = \mathfrak{z} \oplus \mathfrak{t}'$ .

Let  $\widetilde{\mathbb{Z}\Phi}^\vee$  and  $\mathbf{X}_0^\vee$  be as in Section 1. Then  $\mathbf{X}_0^\vee \otimes F$  and  $\widetilde{\mathbb{Z}\Phi}^\vee \otimes F$  may be naturally identified with  $\mathfrak{z}$  and  $\mathfrak{t}'$ , respectively. Our hypothesis implies

$$\mathfrak{t} = \mathbf{X}_*(\mathbf{T}) \otimes F = (\mathbf{X}_0^\vee \otimes F) \oplus (\widetilde{\mathbb{Z}\Phi}^\vee \otimes F) = \mathfrak{z} \oplus \mathfrak{t}'. \quad \blacksquare$$

**Proposition 3.2** *If  $p$  does not divide  $k(\mathbf{G})$ , then  $\mathfrak{g}_{x,r} = \mathfrak{z}_r \oplus \mathfrak{g}'_{x,r}$  for all  $x \in \mathcal{B}(\mathbf{G}, F)$  and all  $r \in \mathbb{R}$ .*

**Proof** This follows from Lemma 2.4. \blacksquare

### 4 Adjoint vs. Coadjoint Representations

When  $F$  has characteristic zero, one may use an extension of the Killing form to identify  $\mathfrak{g}$  and  $\mathfrak{g}^*$ . However, this identification need not respect Moy-Prasad filtrations, and need not exist at all if  $F$  has positive characteristic.

**Proposition 4.1** *Suppose that either of the following conditions is true:*

1.  $\mathbf{G}$  is a form of  $\mathrm{GL}_n$ ;
2. the absolute Dynkin diagram of  $\mathbf{G}$  has no bonds of order  $p$ , and  $p$  does not divide  $2k(\mathbf{G})|\pi_1(\mathbf{G}')|$ .

*Then there exists an  $F$ -valued, nondegenerate,  $G$ -invariant, symmetric, bilinear form  $B$  on  $\mathfrak{g}$  such that, under the associated identification of  $\mathfrak{g}$  with  $\mathfrak{g}^*$ , each  $\mathfrak{g}_{x,r}$  is identified with  $\mathfrak{g}_{x,r}^*$ .*

**Proof** We start by constructing  $B$  under the assumption that  $\mathbf{G}$  is  $F$ -split. If  $\mathbf{G}$  is a general linear group, then identify  $\mathfrak{g}$  with a matrix algebra in the usual way, and take  $B$  to be the trace form.

Now suppose instead that the second hypothesis holds. Continue to assume that  $\mathbf{G}$  splits over  $F$ . Then without loss of generality, we may assume that  $\mathbf{G}$  is defined and split over  $\mathbb{Z}$ . Fix a maximal  $F$ -split  $F$ -torus  $\mathbf{T} \subset \mathbf{G}$ , let  $\Phi$  be the corresponding root system, and let

$\Delta \subset \Phi$  be a system of simple roots. Proposition 3.1 implies that  $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$ . Fix an  $\mathcal{O}$ -basis  $v_1, \dots, v_r$  of  $\mathfrak{z}_0$ .

Define an  $F$ -valued bilinear form  $(\ , \ )$  on  $\mathfrak{z}$  by  $(v_i, v_j) = \delta_{ij}$ . We take  $B = B' \oplus (\ , \ )$  where  $B'$  is defined as follows.

Let  $(\mathbf{T}, \{X_\alpha\})$  be a Chevalley splitting for  $\mathbf{G}$ , defined over  $\mathbb{Z}$ . That is,  $X_\alpha = dx_\alpha(1)$  where the  $\alpha$ -th root group homomorphism  $x_\alpha$  is defined over  $\mathbb{Z}$ . Let  $\mathfrak{g}'_{\mathbb{Z}}$  be the  $\mathbb{Z}$ -module generated by a  $\mathbb{Z}$ -basis for  $\Phi^\vee$  and  $\{X_\alpha\}_{\alpha \in \Phi}$ . Then we may write  $\mathfrak{g}'_{\mathbb{Z}} = \mathbb{Z}\Phi^\vee + \sum_{\alpha \in \Phi} \mathbb{Z}X_\alpha$ . Suppose  $\Phi$  has irreducible factors  $\Phi_1, \dots, \Phi_r$ . Then

$$\mathfrak{g}'_{\mathbb{Z}} = \sum_i \mathfrak{g}'_{\mathbb{Z}}(i)$$

where

$$\mathfrak{g}'_{\mathbb{Z}}(i) = \mathbb{Z}\Phi_i^\vee + \sum_{\alpha \in \Phi_i} \mathbb{Z}X_\alpha.$$

Let  $B_i: \mathfrak{g}'_{\mathbb{Z}}(i) \times \mathfrak{g}'_{\mathbb{Z}}(i) \rightarrow \mathbb{Z}$  be the restriction of the Killing form of  $\mathfrak{g}'_{\mathbb{Z}} \otimes \mathbb{C}$  to  $\mathfrak{g}'_{\mathbb{Z}}(i)$ .

We define  $B'_i$  to equal  $\ell_i^{-1}B_i$  where  $\ell_i$  is the minimum value of  $B_i(\alpha^\vee, \alpha^\vee)$  as  $\alpha^\vee$  ranges through  $\Phi_i^\vee$ .

Let  $B' = B'_1 \oplus \dots \oplus B'_r$ . Note that  $B'$  takes values in  $\mathbb{Z}[S]$ , where  $S$  consists of  $1/2$  and all reciprocals of integers which occur as ratios of squares of root lengths for roots in some  $\Phi_i^\vee$  ( $i = 1, \dots, r$ ). Indeed  $B'(\alpha^\vee, \beta^\vee) = 0$  if  $\alpha^\vee \in \Phi_i^\vee, \beta^\vee \in \Phi_j^\vee$  and  $i \neq j$ . If  $\alpha^\vee, \beta^\vee \in \Phi_i^\vee$  and  $\ell_i = B_i(\gamma^\vee, \gamma^\vee)$  then

$$\begin{aligned} B'(\alpha^\vee, \beta^\vee) &= \ell_i^{-1}B_i(\alpha^\vee, \beta^\vee) \\ &= \frac{B_i(\beta^\vee, \beta^\vee) B_i(\alpha^\vee, \beta^\vee)}{B_i(\gamma^\vee, \gamma^\vee) B_i(\beta^\vee, \beta^\vee)} \\ &= \frac{1}{2}r_i \langle \beta, \alpha^\vee \rangle \end{aligned}$$

where  $r_i$  is a ratio of squares of root lengths in  $\Phi_i^\vee$ , so that  $B'(\alpha^\vee, \beta^\vee) \in \mathbb{Z}[S]$ . Further,  $B'(X_\alpha, X_\beta) = 0$  unless  $\alpha + \beta = 0$ , and

$$2B'(X_\alpha, X_{-\alpha}) = B'(\alpha^\vee, \alpha^\vee)$$

and hence  $B'(X_\alpha, X_{-\alpha})$  belongs to  $\mathbb{Z}[S]$ .

It is clear that  $B'$  is nondegenerate, and can be extended to a nondegenerate form on  $\tilde{\mathfrak{g}}'_{\mathbb{Z}} = \widetilde{\mathbb{Z}\Phi^\vee} + \sum \mathbb{Z}X_\alpha$  with values in  $\mathbb{Z}[S, |\pi_1(\mathbf{G}')|^{-1}]$ . On tensoring with  $F$ , we obtain a nondegenerate  $F$ -bilinear form on  $\tilde{\mathfrak{g}}'_{\mathbb{Z}} \otimes F = \mathfrak{g}'$ , which we also denote by  $B'$ . As above we define  $B$  to equal  $B' \oplus (\ , \ )$ . It is easy to check (e.g. by standard formulas for the adjoint action of  $G$  on  $\mathfrak{g}$ ) that  $B$  is  $G$ -invariant.

Now drop the assumption that  $\mathbf{G}$  is  $F$ -split. Then  $\mathbf{G}$  is  $E$ -split for some Galois extension  $E$  of  $F$ . Fix an maximal  $E$ -split torus  $\mathbf{T} \subset \mathbf{G}$ . As appropriate, define or choose  $\Phi, \Delta, \{X_\alpha\}$ , and  $\tilde{\mathfrak{g}}'_{\mathbb{Z}}$  as above. Then the reasoning above gives us a nondegenerate  $E$ -bilinear form  $B = B' \oplus (\ , \ )$  on  $\mathfrak{g}_E = \mathfrak{g} \otimes E$ . It is clear that  $(\ , \ )$  is  $\text{Gal}(E/F)$ -invariant if  $(\ , \ )$  is defined with



respect to an  $\mathcal{O}$ -basis for  $\mathfrak{z}_0$ , as before. We show  $\sigma(B'(X, Y)) = B'(\sigma(X), \sigma(Y))$  for  $X, Y \in \mathfrak{g}'_E$  and  $\sigma \in \text{Gal}(E/F)$ . It will follow that  $B$  defines a nondegenerate  $G$ -invariant  $F$ -bilinear form on  $\mathfrak{g} = \mathfrak{g}'_E^{\text{Gal}(E/F)}$ . From our assumptions on  $p$ ,  $\{\alpha^\vee\}_{\alpha \in \Delta} \cup \{X_\alpha\}_{\alpha \in \Phi}$  determines an  $E$ -basis for  $\tilde{\mathfrak{g}}'_Z \otimes E = \mathfrak{g}'_E$ , so we may restrict our attention to  $X$  and  $Y$  lying in this basis. Let  $H_\alpha$  denote the basis element corresponding to  $\alpha^\vee$ . Then  $H_\alpha = [X_\alpha, X_{-\alpha}] = d\alpha^\vee(1)$ .

Fix  $\sigma \in \text{Gal}(E/F)$ . We have  $\sigma X_\alpha = \epsilon_\alpha X_{\sigma(\alpha)}$  where  $\epsilon_\alpha = \pm 1$  (since  $x_\alpha$  is defined over  $\mathbb{Z}$ ,  $\epsilon_i$  must be a unit in  $\mathbb{Z}$ ). Since  $d\alpha^\vee$  is defined over  $\mathbb{Z}$ ,  $\sigma(H_\alpha) = H_{\sigma(\alpha)}$ . Since  $\sigma(-\alpha) = -\sigma(\alpha)$ , we see that  $\epsilon_{-\alpha} = \epsilon_\alpha$ . Using

$$2B'(X_\alpha, X_{-\alpha}) = B'(H_\alpha, H_\alpha) = \ell_i^{-1} \sum_{\beta \in \Phi} \langle \beta, \alpha^\vee \rangle$$

(where  $\alpha \in \Phi_i$ ), we then obtain  $B'(X_\alpha, X_{-\alpha}) = B'(\sigma(X_\alpha), \sigma(X_{-\alpha}))$  (since the canonical pairing between characters and cocharacters is  $\text{Gal}(E/F)$ -invariant). It follows that the form  $B'$  defined on  $\tilde{\mathfrak{g}}'_Z$  is  $\text{Gal}(E/F)$ -invariant and hence that  $\sigma(B(X, Y)) = B(\sigma(X), \sigma(Y))$  for  $X, Y \in \mathfrak{g}_E$ .

Now that we have constructed  $B$ , we have an associated isomorphism  $\Psi_B: \mathfrak{g} \rightarrow \mathfrak{g}^*$  given by  $(\Psi_B(X))(Y) = B(X, Y)$ . It remains to show that  $\Psi_B(\mathfrak{g}_{x,r}) = \mathfrak{g}_{x,r}^*$ . From Proposition 2.1, it is enough to show this in the case where  $\mathbf{G}$  is  $F$ -split.

It is clear from the construction of  $B$  that  $B(X_\alpha, X_{-\alpha}) \in \mathcal{O}^\times$  for any  $\alpha \in \Phi$ . Also, note that  $\{H_\alpha\}$  is an  $\mathcal{O}$ -basis for  $\widetilde{\mathbb{Z}\Phi}^\vee \otimes \mathcal{O}$ ; and for all  $\alpha, \beta \in \Phi$ ,  $B(H_\alpha, H_\beta) \in \mathcal{O}$  and is a unit if  $\alpha = \beta$ . It follows easily from these two observations and the definitions that  $\Psi_B(\mathfrak{g}_{x,r}) = \mathfrak{g}_{x,r}^*$ . ■

**Remark 4.2** If  $\mathbf{G}$  is the almost-direct product  $(\prod \mathbf{G}_i)/\mathbf{C}$  where each  $\mathbf{G}_i$  satisfies some hypothesis of Proposition 4.1, then it is clear from the proof that we can construct a bilinear form  $B$  having the desired properties if  $p$  does not divide  $|\mathbf{C}|$ .

### 5 The $K$ -Types Under Consideration

A given representation of  $G$  will contain many unrefined minimal  $K$ -types, some of which will have better properties than others. The problem of determining in general which properties are most useful is still open. But for our purposes, it is convenient to restrict to unrefined minimal  $K$ -types that, after field extension, look like they are contained in a principal series representation.

**Definition 5.1** Say that a set  $\Upsilon \subset \mathfrak{g}$  is *good* if  $\Upsilon \in \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$  for some  $x \in \mathcal{B}$  and some  $r \in \mathbb{R}$ , and there is some maximal  $F$ -torus  $\mathbf{T} \subset \mathbf{G}$  such that  $\text{Lie}(\mathbf{T}(F))$  intersects  $\Upsilon$ ,  $\mathbf{T}$  splits over a tamely ramified extension  $E$  of  $F$ , and  $x$  belongs to the apartment of  $\mathbf{T}(E)$  in  $\mathcal{B}(\mathbf{G}, E)$ .

For example, in the work of Howe [8], Corwin [6], Moy [16], Roche [21], and (apparently) Morris [14] on (respectively)  $\text{GL}_n$ , division algebras,  $\text{GSP}_4$ , the principal series, and classical groups, always in the tame case, the positive depth representations that arise all contain characters parametrized by good cosets.

It is clear from the definition that any semisimple  $a$  that splits over a tame extension lies in some good coset. From [1, Lemma 1.9.5], any good coset is free of nilpotent elements.

Call an unrefined minimal  $K$ -type of positive depth *good* if it corresponds to a good coset in  $\mathfrak{g}$ .

Suppose  $a \in \mathfrak{g}$  belongs to the Lie algebra  $\mathfrak{t}$  of some split maximal torus  $\mathbf{T}(F)$ . Then there is some  $r \in \mathbb{Z}$  such that  $a \in \mathfrak{t}_r \setminus \mathfrak{t}_{r+1}$ . Call  $a$  a  $\mathbf{T}$ -good element of depth  $r$  if for every root  $\alpha$  of  $\mathbf{G}$  with respect to  $\mathbf{T}$ ,  $d\alpha(a)$  either is zero or has valuation  $r$ . Here,  $d\alpha$  denotes the derivative of  $\alpha$ .

Suppose  $\mathbf{T}'$  is another split maximal torus with Lie algebra  $\mathfrak{t}'$ , and  $a \in \mathfrak{t} \cap \mathfrak{t}'$ . Then there is some  $g \in G$  such that  $\text{Ad}(g)a = a$  and  $\text{Ad}(g)\mathfrak{t} = \mathfrak{t}'$ . This implies that  $a$  is  $\mathbf{T}$ -good if and only if it is  $\mathbf{T}'$ -good, and its depth is independent of the choice of split torus.

Suppose that  $\mathbf{T}$  is an arbitrary maximal  $F$ -torus, split over some tame Galois extension  $E/F$ . Clearly,  $\text{Gal}(E/F)$  preserves the set of  $\mathbf{T}$ -good elements of  $\mathfrak{t} \otimes E$  of any given depth. This allows us to make the following definition.

**Definition 5.2** A semisimple element  $a \in \mathfrak{g}$  is *good* if for some (hence any) tamely ramified Cartan  $F$ -subgroup  $\mathbf{T}$  satisfying  $a \in \mathfrak{t}$ , where  $\mathfrak{t} = \text{Lie}(\mathbf{T}(F))$ , and some (hence any)  $E/F$  over which  $\mathbf{T}$  splits, for every root  $\alpha$  of  $\mathbf{G}$  with respect to  $\mathbf{T}$ ,  $d\alpha(a)$  either is zero or has  $E$ -normalized valuation  $r$ , where  $a \in (\mathfrak{t} \otimes E)_r \setminus (\mathfrak{t} \otimes E)_{r+1}$ . If this is also true whenever  $\alpha$  is a sum of two roots, then we will call  $a$  *very good*.

Roughly, good elements of a coset in  $\mathfrak{t}_r/\mathfrak{t}_{r+}$  are maximally singular among all elements of the coset. They play the same role as Howe's "standard" coset representatives [8]. More precisely, the definition is exactly what is necessary in order to make Lemma 6.2 work. Very good elements are only used in the proof of Lemma 5.12, where we establish the existence of good  $F$ -points while descending to  $F$  from a splitting field.

While good cosets will typically contain good elements, one can cook up some counterexamples. Therefore, it is useful to know under what circumstances one can guarantee the existence of good elements. That is, we are particularly interested in pairs  $(\mathbf{G}, \mathbf{T})$  such that the following statement holds for  $\mathfrak{t} = \text{Lie}(\mathbf{T}(F))$ :

(5.3) For every  $r \in \mathbb{R}$ , every coset in  $\mathfrak{t}_r/\mathfrak{t}_{r+}$  contains a good element.

This is not a severe restriction:

**Proposition 5.4** Let  $\mathbf{T}$  be a tamely ramified maximal  $F$ -torus in  $\mathbf{G}$ . Then Statement (5.3) holds for  $\mathbf{G}$  and  $\mathbf{T}$  under any of the following conditions:

1.  $\mathbf{G}$  is an inner form of  $\text{GL}_n$ ;
2.  $\mathbf{G}$  is a form of  $\text{GL}_n$ ,  $\text{Sp}_n$ , or  $\text{SO}_n$  (non-triality), and  $p$  is odd;
3.  $\mathbf{G}$  is a form of  $\text{SL}_n$  and  $\mathbf{T}$  has  $F$ -split rank zero;
4.  $\mathbf{G}$  has no factors of exceptional or triality type and  $p$  does not divide  $2 \cdot z(\Phi)$ , where  $\Phi$  is the root system of  $\mathbf{G}$  with respect to  $\mathbf{T}$ ;
5.  $\mathbf{T}$  is  $F$ -split,  $p \notin \text{bad}(\mathbf{G})$ , and  $p$  does not divide  $z(\mathbf{G})$ ;
6.  $p > c(\mathbf{G})$ .

This result will follow from a series of overlapping lemmata. Lemma 5.7 will handle the first three cases. The fourth case will then follow from the second, Lemma 5.8, and Lemma 5.9. The fifth case will follow from Lemma 5.9, Lemma 5.10, and the fact that  $k(\mathbf{G})$

divides  $z(\mathbf{G})$ . The last case will follow from the fifth and Lemma 5.12. Note that the last hypothesis trumps all of the previous hypotheses on  $p$ . Thus, when  $p > c(\mathbf{G})$ , we have a statement (and proof) that works uniformly for all  $\mathbf{G}$ .

**Remark 5.5** The fourth hypothesis of the proposition is too strong, particularly for some groups containing factors of type  $A_n$ . For any particular group, one can piece together Lemma 5.9, Lemma 5.8, and the first three parts of the proposition to get a sufficient, but possibly weaker, hypothesis on  $p$ .

Fix a uniformizer  $\varpi$  for  $F$ . For every tamely ramified extension  $E/F$ , choose a uniformizer  $\varpi_E$  for  $E$  such that  $\varpi_E^e = \varpi$ , where  $e$  is the ramification degree of  $E/F$ . We can do this in a coherent way, so that if  $E \supset E' \supset F$ , then  $\varpi_E^{e(E/E')} = \varpi_{E'}$ . For  $k \in \mathbb{Z}$ , let  $C_{E,k}$  be the set containing zero and all elements of  $E$  of the form  $\varepsilon \varpi_E^k$ , where  $\varepsilon$  is a root of unity of order prime to  $p$ .

If  $E/F$  is Galois, then  $\text{Gal}(E/F)$  preserves  $C_{E,k}$ . When  $p \neq 2$ , multiplication by  $-1$  also preserves  $C_{E,k}$ .

**Definition 5.6** Suppose  $G$  is a form of  $\text{GL}_n$ ,  $\text{SL}_n$ ,  $\text{Sp}_n$ , or  $\text{SO}_n$ . Choose a faithful representation of  $\mathfrak{g} \otimes \bar{F}$  in  $\mathfrak{gl}_n \otimes \bar{F}$ . Let  $\mathcal{G}$  denote the set of semisimple elements  $a$  of  $\mathfrak{g}$  such that for some tame extension  $E$  of  $F$ , and some  $k$ , all eigenvalues of  $a$  under this representation lie in  $C_{E,k}$ .

Note that the definition of  $\mathcal{G}$  does not depend on the choice of representation, and the set  $\mathcal{G}$  is invariant under conjugation by  $G$ . If  $\mathfrak{t}$  is any tamely ramified Cartan subalgebra of  $\mathfrak{g}$  then it is clear that each coset in  $\mathfrak{t}_r/\mathfrak{t}_{r^+}$  can contain at most one element of  $\mathcal{G}$ .

**Lemma 5.7** Suppose that  $\mathbf{G}$  and  $\mathbf{T}$  satisfy any of the first three hypotheses of Proposition 5.4. Then Statement (5.3) holds for  $\mathbf{G}$  and  $\mathbf{T}$ .

**Proof** In each of these cases, the Lie algebra  $\mathfrak{g} \otimes \bar{F}$  has a standard representation as a set of matrices, and we may concretely realize  $\mathfrak{t} \otimes \bar{F}$  as the subalgebra of diagonal matrices. Assume that  $\mathfrak{t}_r \neq \mathfrak{t}_{r^+}$ , since otherwise the only coset in  $\mathfrak{t}_r/\mathfrak{t}_{r^+}$  contains 0, which is good. Let  $E$  be the splitting field of  $\mathbf{T}$ , and  $e$  the ramification degree of  $E/F$ . Then  $er \in \mathbb{Z}$ , and given a coset  $\bar{a} = a + \mathfrak{t}_{r^+} \in \mathfrak{t}_r/\mathfrak{t}_{r^+}$ ,  $\bar{a} \subset \bar{a}_E = a + (\mathfrak{t} \otimes E)_{er+1} \in (\mathfrak{t} \otimes E)_{er}/(\mathfrak{t} \otimes E)_{er+1}$ . Consider the character of  $\mathfrak{t} \otimes \bar{F}$  that picks out a particular diagonal entry. The collection of such characters forms a  $\mathbb{Z}$ -basis for the weight lattice of  $\mathfrak{t} \otimes E$ , so the coset  $\bar{a}_E$  has a unique element  $a'$  lying in  $\mathcal{G}$ .

In the first and third cases, for every root  $\alpha$ ,  $d\alpha(a')$  is a difference of two elements of  $C_{E,er}$ . When  $p \neq 2$ , two times an element of  $C_{E,er}$  is also a difference of elements of  $C_{E,er}$ . Thus, in all three cases under consideration, for each  $\alpha$ ,  $d\alpha(a')$  is an element of  $C_{E,er}$  or a difference of two such elements. Therefore,  $d\alpha(a')$  either is zero or has  $E$ -normalized valuation  $er$  (and thus  $F$ -normalized valuation  $r$ ).

Let  $\gamma \in \text{Gal}(E/F)$ . In the first two cases, the Galois group permutes our chosen basis for the character lattice (up to factors of  $\pm 1$  in the second case), so  $\gamma(a')$  also has its entries lying in  $C_{E,er}$ . (In general, the Galois action need not have this property, even when  $\mathbf{G} =$

$SL_n$ . However, in the third case, the good elements we will construct for  $gl_n(E)$  happen to lie in  $sl_n(E)$ .) Since

$$\gamma(a') \equiv \gamma(a) = a \equiv a' \pmod{(\mathfrak{t} \otimes E)_{er+1}},$$

the uniqueness of  $a'$  implies that  $a'$  is fixed by the action of  $\text{Gal}(E/F)$ . Therefore,  $a'$  lies in  $\bar{a}$ . ■

**Lemma 5.8** *Let  $f: \mathbf{G}^{(1)} \rightarrow \mathbf{G}^{(2)}$  be a central isogeny,  $\mathbf{T}^{(i)} \subset \mathbf{G}^{(i)}$  a Cartan subgroup, and  $f(\mathbf{T}^{(1)}) = \mathbf{T}^{(2)}$ . Assume that  $p$  does not divide the order of the kernel of  $f$ . Then Statement (5.3) holds for  $\mathbf{G}^{(1)}$  and  $\mathbf{T}^{(1)}$  if and only if it holds for  $\mathbf{G}^{(2)}$  and  $\mathbf{T}^{(2)}$ .*

**Proof** From Lemma 2.4, the map  $df: \mathfrak{t}^{(1)} \rightarrow \mathfrak{t}^{(2)}$  is an isomorphism and preserves the filtration lattices. Therefore,  $df$  and its inverse send good elements to good elements. ■

**Lemma 5.9** *Let  $\mathbf{T} \subset \mathbf{G}$  be a Cartan subgroup, and  $\mathbf{T}' = \mathbf{G}' \cap \mathbf{T}$ . Suppose  $p$  does not divide  $c(\mathbf{G})$ . Then Statement (5.3) holds for  $\mathbf{G}$  and  $\mathbf{T}$  if and only if it holds for  $\mathbf{G}'$  and  $\mathbf{T}'$ .*

**Proof** The statement is trivial if  $\mathbf{G} = \mathbf{Z} \times \mathbf{G}'$ . The general case follows from Lemma 5.8. ■

**Lemma 5.10** *Suppose that  $\mathbf{G}$  is  $F$ -split and semisimple and let  $\mathfrak{t} \subset \mathfrak{g}$  be the Lie algebra of a split Cartan subgroup  $\mathbf{T}$ . Suppose that  $p$  does not divide  $z(\mathbf{G})$  and does not lie in  $\text{bad}(\mathbf{G})$ . Then Statement (5.3) holds for  $\mathbf{G}$  and  $\mathbf{T}$ .*

**Proof** Let  $\bar{a} \in \mathfrak{t}_i/\mathfrak{t}_{i+1}$ . It is enough to consider the case where  $i = 0$ . Let  $\Phi_{\bar{a}} = \{\alpha \in \Phi \mid d\alpha(a_1) \in \mathcal{P} \text{ for any (hence all) } a_1 \in \bar{a}\}$ . It is easy to check that  $\Phi_{\bar{a}}$  is a closed subsystem of  $\Phi$ .

Since  $p \notin \text{bad}(\mathbf{G})$ , the quotient  $(\mathbb{Z}\Phi \otimes \mathcal{O})/(\mathbb{Z}\Phi_{\bar{a}} \otimes \mathcal{O})$  is torsion free. In particular,  $\mathbb{Z}\Phi_{\bar{a}} \otimes \mathcal{O}$  has a complement in  $\mathbb{Z}\Phi \otimes \mathcal{O}$ . Hence any  $\mathbb{Z}$ -basis  $\alpha_1, \dots, \alpha_r$  of  $\Phi_{\bar{a}}$  extends to an  $\mathcal{O}$ -basis  $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s$  of  $\mathbb{Z}\Phi \otimes \mathcal{O}$ . For any such basis, consider the isomorphism  $\mathfrak{t}_0 \xrightarrow{\sim} \mathcal{O}^{r+s}$  given by

$$t \mapsto (d\alpha_1(t), \dots, d\alpha_r(t), d\beta_1(t), \dots, d\beta_s(t)).$$

Our first hypothesis on  $p$  is equivalent to  $\mathbf{X}/\mathbb{Z}\Phi$  having no  $p$ -torsion, so this isomorphism identifies  $\mathfrak{t}_1$  with a product of  $r + s$  copies of  $\mathcal{P}$ . For any  $a_1 \in \bar{a}$ , let  $a \in \mathfrak{t}_0$  be the element corresponding to  $(0, \dots, 0, d\beta_1(a_1), \dots, d\beta_s(a_1))$ . Then  $a \in \bar{a}$  is good. ■

**Lemma 5.11** *Suppose that  $\mathbf{G}$  is  $F$ -split and semisimple and let  $\mathfrak{t} \subset \mathfrak{g}$  be the Lie algebra of a split Cartan subgroup  $\mathbf{T}$ . Suppose that  $p > c(\mathbf{G})$ . Then Statement (5.3) holds for  $\mathbf{G}$  and  $\mathbf{T}$  with “good” replaced by “very good.”*

**Proof** Use the proof of Lemma 5.10 and the definition of  $c(\mathbf{G})$ . ■

**Lemma 5.12** *Suppose  $p > c(\mathbf{G})$ . Then Statement (5.3) holds for  $\mathbf{G}$  and  $\mathbf{T}$ .*

**Proof** Note that  $c(\mathbf{G})$  is always greater than the largest prime divisor of  $k(\mathbf{G})$ , so from Lemma 5.9, it is enough to consider the case where  $\mathbf{G}$  is semisimple. Let  $E$  be a tamely ramified splitting field of  $\mathbf{T}$ . Let  $e$  be the ramification degree of  $E/F$ . Suppose that a coset  $\bar{a} \in (\mathfrak{t} \otimes E)_{er}/(\mathfrak{t} \otimes E)_{er+1}$  contains an  $F$ -point  $a_0$ . Then it remains to show that  $\bar{a}$  contains a good  $F$ -point.

From Lemma 5.11,  $\bar{a}$  contains a very good  $E$ -point  $a$ . Let  $\gamma \in \text{Gal}(E/F)$ . Then  $\gamma a \equiv \gamma a_0 = a_0 \equiv a \pmod{\mathfrak{t}(E)_{er+1}}$ , so for all  $\alpha \in \Phi(\mathbf{G}(E), \mathbf{T}(E))$ ,  $d\alpha(a) \equiv d\alpha(\gamma a) = \gamma^{-1}d\alpha(a)$  modulo  $\mathcal{P}_E^{er+1}$ . The fact that  $a$  is very good then implies that  $d\alpha(a) = d\alpha(\gamma a)$ .

Since  $F$  does not have characteristic two and  $\mathbf{G}$  is semisimple, we may conclude that  $a = \gamma a$ . Thus,  $a$  is a good  $F$ -point in the coset  $\bar{a}$ . ■

### 6 Slightly Refined Minimal $K$ -Types

From now on, let  $r > 0$ , let  $\Upsilon \in \mathfrak{g}_{x,-r}/\mathfrak{g}_{x,(-r)^+}$  be a good coset, and let  $a \in \Upsilon$  be a good element such that  $x$  lies in the building of the centralizer of  $a$ . Thus, there is some  $F$ -torus  $\mathbf{T}$  and some tame extension  $E/F$  such that  $\mathbf{T}$  is  $E$ -split and  $x$  lies in the apartment of  $\mathbf{T}(E)$  in  $\mathcal{B}(\mathbf{G}, E)$ .

For any subset  $\mathfrak{s} \subset \mathfrak{g}$ , let  $\mathfrak{s}^\perp$  be the perpendicular of  $\mathfrak{s}$  with respect to the bilinear form  $B$  defined in Section 4. We need to see when the Moy-Prasad filtrations are compatible with the decomposition of  $\mathfrak{g}$  into perpendicular summands:

**Proposition 6.1** *Suppose  $\mathbf{M} \subset \mathbf{G}$  is an  $E$ -Levi subgroup for some finite extension  $E/F$ , and  $\mathfrak{m}$  is the Lie algebra of  $\mathbf{M}(F)$ . If  $x \in \mathcal{B}(\mathbf{M}, F) \cap \mathcal{B}(\mathbf{G}, F)$ , then for all  $r \in \mathbb{R}$ ,  $\mathfrak{g}_{x,r} = \mathfrak{m}_{x,r} \oplus (\mathfrak{m}^\perp \cap \mathfrak{g}_{x,r})$ .*

**Proof** This follows from Proposition 2.1. ■

Let  $\mathfrak{m} = \ker \text{ad}(a)$ . For each  $s \in \mathbb{R}$ , let  $\mathfrak{m}_{x,s}^\perp = \mathfrak{g}_{x,s} \cap \mathfrak{m}^\perp$ .

Fix an additive character  $\Lambda$  of  $F$  with conductor  $\mathcal{P}$ . Let

$$\begin{aligned} \mathfrak{J}_+ &= \mathfrak{m}_{x,r} \oplus \mathfrak{m}_{x,(r/2)^+}^\perp \\ \mathfrak{J} &= \mathfrak{m}_{x,r} \oplus \mathfrak{m}_{x,(r/2)}^\perp. \end{aligned}$$

Let  $J_+ = \varphi_x(\mathfrak{J}_+)$  and  $J = \varphi_x(\mathfrak{J})$ . These are groups. The element  $a$  corresponds by duality to a character  $\rho$  on  $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$ , defined by

$$\rho(X) = \Lambda(B(a, X)).$$

Note that  $\rho$  is trivial on  $\mathfrak{m}_{x,r}^\perp$ . Thus, we may extend  $\rho$  to a character  $\bar{\rho}$  of  $\mathfrak{J}_+$  by letting it be trivial on  $\mathfrak{m}_{x,(r/2)^+}^\perp$ .

**Lemma 6.2** *The map*

$$\text{ad}(a): \mathfrak{m}_{x,s}^\perp/\mathfrak{m}_{x,s^+}^\perp \rightarrow \mathfrak{m}_{x,-r+s}^\perp/\mathfrak{m}_{x,(-r+s)^+}^\perp$$

*is an isomorphism.*

**Proof** In view of Proposition 2.1, we may assume that  $a$  belongs to the Lie algebra of an  $F$ -split maximal torus, in which case the result follows directly from the fact that  $a$  is good. ■

**Lemma 6.3** *Let  $b \in \Upsilon = a + \mathfrak{g}_{x,(-r)^+}$ , and suppose that  $b \in \mathfrak{m} + \mathfrak{g}_{x,s}$  for some  $s > -r$ . Then  $b$  is conjugate under  $G_{x,s+r}$  to an element of  $a + \mathfrak{m}_{x,(-r)^+}$ .*

**Proof** (This proof, taken from [1], mimics arguments in [11, Lemma 2.4.3] and [16, Lemma 4.4].) We may assume that  $\mathfrak{m}_{x,s}^\perp \neq \mathfrak{m}_{x,s^+}^\perp$ . From Proposition 6.1, we may write  $b = b_1 + b^\perp$ , with  $b_1 \in a + \mathfrak{m}_{x,(-r)^+}$  and  $b^\perp \in \mathfrak{m}_{x,s}^\perp$ . From Lemma 6.2, there exists  $c^\perp \in \mathfrak{m}_{x,s+r}^\perp$  such that  $\text{ad}(b)(c^\perp) \equiv b^\perp \pmod{\mathfrak{g}_{x,s^+}}$ . Let  $g_0 = \varphi_x(c^\perp)$ . From Proposition 2.3,

$$\text{Ad}(g_0)(b) \equiv b - \text{ad}(b)(c^\perp) \equiv b_1 \pmod{\mathfrak{g}_{x,s^+}}.$$

Thus,

$$\text{Ad}(g_0)(b) \in b_1 + \mathfrak{g}_{x,s^+} \subset a + \mathfrak{m}_{x,(-r)^+} + \mathfrak{m}_{x,s_1}^\perp$$

for some  $s_1 > s$ . Continuing by induction, we obtain an increasing sequence  $s, s_1, s_2, \dots$  such that  $\bigcap_{i \in \mathbb{N}} \mathfrak{m}_{x,s_i}^\perp = \{0\}$ , and a collection  $g_i \in G_{x,s_i+r}$  such that for all  $t \in \mathbb{N}$ ,

$$\text{Ad}(g_t \cdots g_1 g_0)(b) \in a + \mathfrak{m}_{x,(-r)^+} + \mathfrak{m}_{x,s_t}^\perp.$$

Let  $g \in G_{x,s+r}$  be the (convergent) infinite product  $(\cdots g_1 g_0)$ . Then  $\text{Ad}(g)(b) \in a + \mathfrak{m}_{x,(-r)^+}$ . ■

**Lemma 6.4** *All extensions of  $\rho$  to  $\mathfrak{F}_+$  are  $G_x$ -conjugate.*

**Proof** Characters of  $\mathfrak{F}_+/\mathfrak{g}_{x,r^+}$  correspond (via the bilinear form  $B$  defined in Section 4 and the duality between  $\mathfrak{g}$  and  $\mathfrak{g}^*$ ) to cosets in  $\mathfrak{g}_{x,-r}/\mathfrak{F}_+^\bullet$ , where  $\mathfrak{F}_+^\bullet = \mathfrak{m}_{x,(-r)^+} \oplus \mathfrak{m}_{x,-r/2}^\perp$ . Let  $\rho'$  be an extension of  $\rho$  to  $\mathfrak{F}_+$ . Then  $\rho'$  is given by the formula

$$\rho'(b) = \Lambda(B(a + a_1, b)),$$

where  $a_1 \in \mathfrak{m}_{x,-r/2}^\perp$ . From Lemma 6.3, there is some  $g \in G_{x,r/2}$  such that  $\text{Ad}(g)(a + a_1) \equiv a \pmod{\mathfrak{m}_{x,(-r)^+}}$ . Thus, for all  $b \in \mathfrak{F}_+$ ,

$$\rho'(g^{-1}b) = \Lambda(B(a + a_1, g^{-1}b)) = \Lambda(B(a, b)) = \bar{\rho}(b),$$

where  $\bar{\rho}$  is our trivial extension of  $\rho$  to  $\mathfrak{F}_+$ . ■

Now regard  $\bar{\rho}$  as a character of  $J_+$  via the bijection between  $\mathfrak{F}_+$  and  $J_+$ .

**Corollary 6.5** *Any admissible representation of  $G$  that contains  $(G_{x,r}, \rho)$  also contains  $(J_+, \bar{\rho})$ .* ■

### 7 Main Theorem

Assume that the characteristic of  $F$  is not in  $\text{bad}(\mathbf{G})$  and does not divide  $|\pi_1(\mathbf{G}')|$ . Then, as remarked in Section 1, the centralizer in  $G$  of any semisimple element of  $\mathfrak{g}$  is an  $E$ -Levi subgroup of  $G$  for some extension  $E/F$ . As in Section 6, let  $\Upsilon \in \mathfrak{g}_{x,-r}/\mathfrak{g}_{x,(-r)^+}$  be a good coset, and let  $a \in \Upsilon$  be a good element such that  $x$  lies in the building of  $C_G(a)$ . Define  $J_+$ ,  $J$ ,  $\mathfrak{S}_+^\bullet$ , and  $\bar{\rho}$  as in Section 6. Suppose  $B$  is as in Proposition 4.1, and  $\bar{\rho}$  corresponds via  $B$  to  $\tilde{\Upsilon} \in \mathfrak{g}_{x,-r}/\mathfrak{S}_+^\bullet$ .

**Theorem 7.1** *Assume that  $p \notin \text{bad}(\mathbf{G})$ , and that the characteristic of  $F$  does not divide  $|\pi_1(\mathbf{G}')|$ . Assume further that at least one of the following holds:*

1.  $\mathbf{G}$  is a form of  $\text{GL}_n$  or  $\text{SL}_n$ , or is almost simple and not of type  $A_n$ .
2.  $\mathbf{G}$  is a product of such groups and possibly a torus.
3. There is a group  $\mathbf{H}$  of the form above, and  $\mathbf{G}$  and  $\mathbf{H}$  are isogenous via central isogenies (possibly to or from a third group), both of which have kernels of order prime to  $p$ .
4.  $p$  does not divide  $k(\mathbf{G}) \cdot |\pi_1(\mathbf{G}')|$ .

Then

$$\mathcal{J}_G(\bar{\rho}) = \mathcal{J}_{C_G(a)}J.$$

**Remark 7.2** Given our assumption that  $p \notin \text{bad}(\mathbf{G})$ , the third hypothesis of the theorem is weaker than the fourth. The only reason for going to the trouble of stating it is to handle more groups with factors of type  $A_n$ .

**Corollary 7.3** *Suppose  $K$  is a compact subgroup of  $G$  containing  $J_+$ , and  $\sigma$  is a representation of  $K$  whose restriction to  $J_+$  is a multiple of  $\bar{\rho}$ . Then, assuming the hypotheses of the Theorem,*

$$\mathcal{J}_G(\sigma) \subset J\mathcal{J}_{G'}(\sigma|_{G' \cap K})J,$$

where  $G' = C_G(a)$ . ■

The theorem is immediate from the following:

**Proposition 7.4** *Assuming the hypotheses of the Theorem,  $\mathcal{J}_G(\tilde{\Upsilon}) = \mathcal{J}_{C_G(a)}J$ .*

If  $g \in \mathcal{J}_G(\tilde{\Upsilon})$ , then  $\text{Ad}(g)X_1 = X_2$  for some  $X_i \in \tilde{\Upsilon}$ . From Lemma 6.3, there exist elements  $k_i$  in  $J$  such that  $\text{Ad}(k_i)X_i \in C_{\mathfrak{g}}(a)$ . Thus  $\text{Ad}(g')X'_1 = X'_2$ , where  $X'_i = \text{Ad}(k_i)X_i$  and  $g' = k_2gk_1^{-1}$ .

The key step is therefore the following:

**Proposition 7.5** *Let  $a$  and  $\tilde{\Upsilon}$  be as above. Suppose  $X$  and  $\text{Ad}(g)X$  lie in  $\tilde{\Upsilon} \cap C_{\mathfrak{g}}(a)$ . Then, assuming the hypotheses of the theorem,  $g \in C_G(a)$ .*

This is proved in Section 8, and implies Proposition 7.4.

### 8 Proof of Proposition 7.5

Write  $Y = \text{Ad}(g)X$ .

From Remark 7.2, we may ignore the fourth hypothesis of Theorem 7.1.

Suppose that  $f: \mathbf{G} \rightarrow \mathbf{G}^{(1)}$  is a central isogeny whose kernel has order prime to  $p$ . From Lemma 2.4, the map  $df: \mathfrak{g} \rightarrow \mathfrak{g}^{(1)}$  is an isomorphism. Write the images of  $X, Y, a$ , and  $g$  under these maps as  $X^{(1)}$ , etc. Then  $\text{Ad}(g^{(1)})(X^{(1)}) = Y^{(1)}$ . Note that  $[X^{(1)}, a^{(1)}] = [Y^{(1)}, a^{(1)}] = 0$ , and  $X^{(1)}, Y^{(1)} \in \tilde{\mathbf{Y}}^{(1)}$  (in the obvious notation). Suppose that  $g^{(1)} \in C_{\mathbf{G}^{(1)}(\bar{F})}(a^{(1)})$ . Then  $g \in C_{\mathbf{G}(\bar{F})}(a)$  and hence  $g \in C_G(a)$ , as required.

Similar reasoning holds when we have a central isogeny  $f: \mathbf{G}^{(1)} \rightarrow \mathbf{G}$  whose kernel has order prime to  $p$  (and we let  $g^{(1)}$  be any preimage of  $g$ ). Thus, we may replace  $\mathbf{G}$  by a group satisfying the second hypothesis of Theorem 7.1, and it is a trivial matter to reduce to the case where  $\mathbf{G}$  is one of the groups listed in the first hypothesis of the theorem.

If  $\mathbf{G}$  is an almost simple group with root system  $\Phi$  not of type  $A_n$ , then, since our assumptions on  $p$  already imply that  $p$  does not divide  $z(\Phi)$ , we may use the reasoning above to replace  $\mathbf{G}$  with any group in its central isogeny class. Thus, we may restrict our attention to forms of the following groups:  $GL_n, SL_n, SO_n, Sp_{2n}$ , the simple exceptional groups.

For these groups, we will show that the proposition is a consequence of the existence of a rational representation  $(\rho, V)$  of  $\mathbf{G}(\bar{F})$  having certain good properties. We first describe these properties and then explain why this implies the proposition. We then produce such a representation for each of the groups that we need to consider.

**Required Properties of  $(\rho, V)$**  We assume that  $\rho: \mathbf{G}(\bar{F}) \rightarrow GL(V)$  is faithful. We also write  $\rho: \mathfrak{g} \otimes \bar{F} \rightarrow \mathfrak{gl}(V)$  for the associated representation of Lie algebras. Let  $(, )$  denote the trace form on  $\mathfrak{gl}(V)$ . We assume that  $(, )$  is nondegenerate when restricted to  $\mathfrak{g} \otimes \bar{F}$ . Thus,  $\mathfrak{gl}(V) = (\mathfrak{g} \otimes \bar{F}) \oplus (\mathfrak{g} \otimes \bar{F})^\perp$ , and each summand is  $\mathbf{G}(\bar{F})$ -stable.

Fix a maximal  $F$ -torus  $\mathbf{T} \subset \mathbf{G}$  such that  $a \in \mathfrak{t} = \text{Lie}(\mathbf{T}(F))$ . Then  $a \in \mathfrak{t}_{-r} \setminus \mathfrak{t}_{(-r)^+}$ . Having fixed a basis of  $V$ , we may assume that  $\rho$  maps  $\mathfrak{t} \otimes \bar{F}$  into the corresponding diagonal subalgebra  $\mathfrak{d}$  of  $\mathfrak{gl}(V)$ . The trace form  $(, )$  is nondegenerate on  $\mathfrak{d}$ . Since the form is nondegenerate on restriction to  $\mathfrak{g} \otimes \bar{F}$ , it remains nondegenerate on restriction to  $\mathfrak{t} \otimes \bar{F}$ . Therefore  $\mathfrak{d} = (\mathfrak{t} \otimes \bar{F}) \oplus \mathfrak{s}$  where  $\mathfrak{s} = (\mathfrak{t} \otimes \bar{F})^\perp \cap \mathfrak{d}$ .

We assume that  $\mathfrak{d}$  contains an  $\mathcal{O}$ -structure  $\mathfrak{d}_0$  which is compatible with the decomposition  $\mathfrak{d} = (\mathfrak{t} \otimes \bar{F}) \oplus \mathfrak{s}$ , i.e.,  $\mathfrak{d}_0 = (\mathfrak{t}_0 \otimes \mathcal{O}_{\bar{F}}) \oplus \mathfrak{s}_0$  where  $\mathfrak{s}_0 = (\mathfrak{t} \otimes \bar{F})^\perp \cap \mathfrak{d}_0$ . Fix a uniformizer  $\varpi$  for some splitting field of  $\mathbf{T}$  having minimal ramification degree  $e$ , and let  $\mathfrak{d}_i = \varpi^i \mathfrak{d}_0$ . It then follows that  $\mathfrak{d}_i = (\mathfrak{t}_{i/e} \otimes \mathcal{O}_{\bar{F}}) \oplus \mathfrak{s}_i$  for all  $i \in \mathbb{Z}$  (in the obvious notation). The decomposition  $\mathfrak{d}_0 = (\mathfrak{t}_0 \otimes \mathcal{O}_{\bar{F}}) \oplus \mathfrak{s}_0$  holds provided the discriminants of the  $\mathcal{O}$ -bilinear forms  $(, )|_{\mathfrak{d}_0 \times \mathfrak{d}_0}$  and  $(, )|_{\mathfrak{t}_0 \times \mathfrak{t}_0}$  belong to  $\mathcal{O}^\times$ . (This is the condition we verify when we specify  $(\rho, V)$  below.)

Finally, we need that the decompositions  $\mathfrak{gl}(V) = (\mathfrak{g} \otimes \bar{F}) \oplus (\mathfrak{g} \otimes \bar{F})^\perp$  and  $\mathfrak{d} = (\mathfrak{t} \otimes \bar{F}) \oplus \mathfrak{s}$  are compatible in the sense that  $\mathfrak{s} \subset (\mathfrak{g} \otimes \bar{F})^\perp$ .

**Proof of the Proposition Assuming the Existence of  $(\rho, V)$**  For any  $M \in \mathfrak{gl}(V)$  and any  $\lambda \in \bar{F}$ , let  $V_\lambda(M)$  denote its generalized  $\lambda$ -eigenspace. Note that  $\rho(a) \in \mathfrak{d}_{-er}$ . The eigenvalues of  $a$  (viewed as an element of  $\mathfrak{gl}(V)$  via  $\rho$ ) lie in  $\mathcal{P}_E^{-er}$ , but they need not be distinct mod  $\mathcal{P}_E^{1-er}$ . Let  $a'$  be any element of  $\mathfrak{d}_{-er}$  such that  $a \equiv a' \pmod{\mathfrak{d}_{1-er}}$  and such that the distinct eigenvalues of  $a'$  are distinct mod  $\mathcal{P}_E^{1-er}$ . Suppose  $\lambda$  is an eigenvalue of  $X$  (or  $Y$ ).



Then  $\lambda \equiv \mu \pmod{\mathcal{P}_E^{1-er}}$  for a unique eigenvalue  $\mu$  of  $a'$ .

It is not hard to check that  $V_\mu(a') = \bigoplus V_{\lambda_i}(X)$ , where the sum is taken over the set of all eigenvalues  $\lambda_i$  of  $X$  such that  $\lambda_i \equiv \mu \pmod{\mathcal{P}_E^{1-er}}$ . Replacing  $X$  by  $Y$ , a similar statement holds. The relation  $\text{Ad}(g)X = Y$  implies  $\text{Ad}(g)V_\lambda(X) = V_\lambda(Y)$ . Hence  $\text{Ad } g$  preserves the eigenspaces of  $a'$  and thus  $\text{Ad}(g)a' = a'$ . Now write  $a'$  as  $a_1 + a_2$ , where  $a_1 \in \mathfrak{t}_{-r} \otimes \bar{F}$  and  $a_2 \in \mathfrak{s}_{-er}$ . We have

$$\text{Ad}(g)a_1 + \text{Ad}(g)a_2 = a_1 + a_2.$$

Since  $a_2 \in \mathfrak{s} \subset (\mathfrak{g} \otimes \bar{F})^\perp$ , this implies  $\text{Ad}(g)a_1 = a_1$  (and  $\text{Ad}(g)a_2 = a_2$ ). In particular,  $g \in C_{G(\bar{F})}(a_1)$ .

Note

$$a' - a = (a_1 - a) + a_2 \in \mathfrak{d}_{1-er} = (\mathfrak{t}_{-r^+} \otimes \bar{F}) \oplus \mathfrak{s}_{1-er}.$$

Hence  $a_2 \in \mathfrak{s}_{1-er}$ , so  $a_1 \equiv a \pmod{\mathfrak{d}_{1-er}}$ .

Our hypotheses imply that the characteristic of  $F$  is not a torsion prime for  $G$ . Therefore, as observed in Section 1, the group  $C_{G(\bar{F})}(a_1)$  is connected, and is thus generated by  $T$  and the root groups  $U_\alpha$  for all  $\alpha$  such that  $d\alpha(a_1) = 0$ . The goodness of  $a$  implies that  $C_{G(\bar{F})}(a_1) \subset C_{G(\bar{F})}(a)$ . Thus,  $g \in C_{G(\bar{F})}(a)$ , and we are done.

**Description of  $(\rho, V)$**  If  $G(\bar{F})$  is isomorphic to  $GL_n(\bar{F})$ ,  $SL_n(\bar{F})$ ,  $SO_n(\bar{F})$ , or  $Sp_{2n}(\bar{F})$ , then we use the standard representations of these as classical groups. It is not hard to verify that these  $\rho$  have the required properties.

For the exceptional groups, let  $\rho$  be the adjoint representation on the Lie algebra  $\mathfrak{g} \otimes \bar{F}$ . We observe that the only primes dividing the discriminant of the Killing form with respect to a Chevalley basis are the bad primes for the corresponding root system (see [25, Corollary I.4.9]). This implies that  $\rho$  has all the required properties.

Proposition 7.5 and Theorem 7.1 now follow. ■

## 9 Intertwining Between Different $K$ -Types

Suppose that the hypotheses of Theorem 7.1 are satisfied.

For  $i = 1, 2$ , let  $\Upsilon_i \in \mathfrak{g}_{x_i, -r_i} / \mathfrak{g}_{x_i, (-r_i)^+}$  be a good coset, and let  $a_i \in \Upsilon_i$  be a good element. Let  $\rho_i$  be the corresponding character of  $G_{x_i, r_i}$ . Using the methods of Section 6, one can form subgroups  $J_i$  and  $J_{i+}$ , extend  $\rho_i$  to a character  $\bar{\rho}$  of  $J_{i+}$ , and show that a representation of  $G$  contains  $(G_{x_i, r_i}, \rho_i)$  if and only if it contains  $(J_{i+}, \bar{\rho}_i)$ . As before, let  $\tilde{\Upsilon}_i \subset \Upsilon_i$  be the coset corresponding to  $\bar{\rho}_i$ .

If  $g \in \mathcal{I}_G(\bar{\rho}_1, \bar{\rho}_2)$ , then  $\text{Ad}(g)X_1 = X_2$  for some  $X_i \in \tilde{\Upsilon}_i$ . From Lemma 6.3, there exist elements  $k_i \in J_i$  such that  $\text{Ad}(k_i)X_i \in C_{\mathfrak{g}}(a_i)$ . Thus,  $\text{Ad}(g')X'_1 = X'_2$ , where  $X'_i = \text{Ad}(k_i)X_i$  and  $g' = k_2 g k_1^{-1}$ . The following immediate consequence of Proposition 7.5 will be useful elsewhere.

**Proposition 9.1** *Suppose that  $a_1 = a_2 = a$ . Then  $\mathcal{I}_G(\bar{\rho}_1, \bar{\rho}_2) = J_2 C_G(a) J_1$ .*

More generally, suppose we do not assume any relationship between  $a_1$  and  $a_2$ . Let the  $X_i$  and  $X'_i$  be as above. Since we may take the  $X_i$  (and thus the  $X'_i$ ) to be regular and semisimple, we conclude that there exist Cartan subalgebras  $\mathfrak{t}_i$  such that  $a_i \in \mathfrak{t}_i$  and  $\text{Ad}(g')\mathfrak{t}_1 = \mathfrak{t}_2$ . Moreover, we must have  $r_1 = r_2$ .

We have proved the following:

**Proposition 9.2** *If there is no Cartan subalgebra containing both  $a_1$  and a conjugate of  $a_2$ , or if  $r_1 \neq r_2$ , then  $\bar{\rho}_1$  and  $\bar{\rho}_2$  have no intertwining.*

Here is a sharper result. When  $\mathbf{G}$  is a classical group, recall that in Definition 5.6 we constructed a set  $\mathcal{G} \subset \mathfrak{g}$  such that for any tamely ramified Cartan subalgebra  $\mathfrak{t}$ , each coset in  $\mathfrak{t}_{-r}/\mathfrak{t}_{(-r)^+}$  contains exactly one element of  $\mathcal{G}$ . We showed that elements of  $\mathcal{G}$  are good, though for this we required  $p$  to be odd if  $\mathbf{G}$  is not an inner form of  $\text{GL}_n$ .

**Proposition 9.3** *Suppose either that  $\mathbf{G}$  is an inner form of  $\text{GL}_n$ , or that  $p \neq 2$  and  $\mathbf{G}$  is a form of  $\text{GL}_n$ ,  $\text{Sp}_n$ , or  $\text{SO}_n$  (non-triviality). Suppose that the  $a_i$  are chosen to lie in the class  $\mathcal{G}$ . Then*

$$J_G(\bar{\rho}_1, \bar{\rho}_2) = J_2 C(a_1, a_2) J_1,$$

where  $C(a_1, a_2) = \{g \in G \mid \text{Ad}(g)a_1 = a_2\}$ . In particular, the intertwining set is empty if the  $a_i$  are not conjugate in  $G$ .

**Proof** It is clear that the intertwining set is at least as large as claimed. Suppose  $g \in J_G(\bar{\rho}_1, \bar{\rho}_2)$ . From reasoning at the beginning of the proof of Proposition 7.4, after multiplying on the left by an element of  $J_2$  and on the right by an element of  $J_1$ , we may assume that  $\text{Ad}(g)X_1 = X_2$ , for some  $X_i \in a_i + (\mathfrak{t}_i)_{(-r)^+}$ , where  $\mathfrak{t}_i$  is a Cartan subalgebra containing  $a_i$ . Since  $\text{Ad}(g)a_1$  and  $a_2$  both belong to  $\mathcal{G}$  and both belong to  $a_2 + (\mathfrak{t}_2)_{(-r)^+}$ , we must have  $\text{Ad}(g)a_1 = a_2$ . ■

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