Siegel families with application to class fields

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We investigate certain families of meromorphic Siegel modular functions on which Galois groups act in a natural way. By using Shimura’s reciprocity law we construct some algebraic numbers in the ray class fields of CM-fields in terms of special values of functions in these Siegel families.

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1. Introduction

For a positive integer $N$ let $\mathcal{H}_N$ be the field of meromorphic modular functions of level $N$ (defined on $\mathbb{H} = \{ \tau \in \mathbb{C} | \text{Im}(\tau) > 0 \}$) whose Fourier coefficients belong to the $N$th cyclotomic field. As is well known, $\mathcal{H}_N$ is a Galois extension of $\mathcal{H}_1$ whose Galois group is isomorphic to $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ (see [11, §6.1–6.2]). Now, let $N \geq 2$ and consider a set

$$V_N = \{ \mathbf{v} \in \mathbb{Q}^2 | \text{N} \mathbf{v} \in \mathbb{Z}^2 \}$$

as the index set. We call a family $\{f_\mathbf{v}(\tau)\}_{\mathbf{v} \in V_N}$ of functions in $\mathcal{H}_N$ a Fricke family of level $N$ if each $f_\mathbf{v}(\tau)$ depends only on $\pm \mathbf{v}$ (mod $\mathbb{Z}^2$) and satisfies

$$f_\mathbf{v}(\tau)^{\alpha} = f_{\alpha \mathbf{v}}(\tau) \quad (\alpha \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}).$$

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where $\alpha^T$ means the transpose of $\alpha$. For example, Siegel functions of one variable form such a Fricke family of level $N$ [8, ch. 2, proposition 1.3] (see also [4] or [6]).

Let $K$ be an imaginary quadratic field with the ring of integers $\mathcal{O}_K$, and let $f$ be a proper non-trivial ideal of $\mathcal{O}_K$. We denote by $\text{Cl}(f)$ and $K_f$ the ray class group modulo $f$ and its corresponding ray class field modulo $f$, respectively. If $\{f_\nu(\tau)\}_\nu$ is a Fricke family of level $N$ in which every $f_\nu(\tau)$ is holomorphic on $\mathbb{H}$, then we can assign to each ray class $C \in \text{Cl}(f)$ an algebraic number $f_C(f)$ as a special value of a function in $\{f_\nu(\tau)\}_\nu$. Furthermore, we attain by Shimura’s reciprocity law that $f_C(f)$ belongs to $K_f$ and satisfies

$$f_C(f)^{\sigma(D)} = f_C(fD)$$

where $\sigma$ is the Artin reciprocity map for $f$ (see [8, ch. 11, theorem 1.1]).

In this paper we shall define a Siegel family $\{h_M(Z)\}_M$ of level $N$ consisting of meromorphic Siegel modular functions of (higher) genus $g$ and level $N$, which is a generalization of a Fricke family of level $N$ in the case when $g = 1$ (definition 3.1). It turns out that every Siegel family of level $N$ is induced from a meromorphic Siegel modular function for the congruence subgroup $\Gamma_1(N)$ with rational Fourier coefficients (theorem 3.5).

Let $K$ be a CM-field and let $f = NO_K$. Given a Siegel family $\{h_M(Z)\}_M$ of level $N$, we shall introduce a number $h_C(f)$ as a special value of a function in $\{h_M(Z)\}_M$ for each ray class $C \in \text{Cl}(f)$ (definition 5.4). Under certain assumptions on $K$ (assumption 5.1) we shall prove that if $h_C(f)$ is finite, then it lies in the ray class field $K_f$ whose Galois conjugates are of the same form (theorem 7.2 and corollary 7.3). To this end, we assign a principally polarized abelian variety to each non-trivial ideal of $\mathcal{O}_K$, and apply Shimura’s reciprocity law to $h_C(f)$.

On the other hand, we note that there is a remarkable paper by Grant [2] in which he generalized a classical formula of Eisenstein and obtained classes of S-units by evaluating abelian functions at the intersections of divisors on the Jacobian of the curve $y^2 = x^5 + 1$. We hope that our invariant $h_C(f)$ obtained from a Siegel family in theorem 4.3 will contribute further towards finding a higher-dimensional analogue of an elliptic unit.

2. Actions on Siegel modular functions

First, we shall describe the Galois group between fields of meromorphic Siegel modular functions in a concrete way.

Let $g$ be a positive integer and let

$$\eta_g = \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix}.$$ 

For every commutative ring $R$ with unity we define

$$\text{GSp}_{2g}(R) = \{ \alpha \in \text{GL}_{2g}(R) \mid \alpha^T \eta_g \alpha = \nu(\alpha) \eta_g \text{ with } \nu(\alpha) \in R^\times \},$$

$$\text{Sp}_{2g}(R) = \{ \alpha \in \text{GSp}_{2g}(R) \mid \nu(\alpha) = 1 \}.$$ 

Let

$$G = \text{GSp}_{2g}(\mathbb{Q})$$
and let $G_h$ be the adelization of $G$, let $G_0$ be its non-Archimedean part and let $G_\infty$ be its Archimedean part. One can extend the multiplier map $\nu: G \to \mathbb{Q}_\kappa^\times$ continuously to the map $\nu: G_h \to \mathbb{Q}_\kappa^\times$ and set
\[
G_\infty^+ = \{ \alpha \in G_\infty \mid \nu(\alpha) > 0 \}, \quad G_h^+ = G_0 G_\infty^+, \quad G_+ = G \cap G_h^+.
\]
Furthermore, let
\[
\Delta = \left\{ \begin{bmatrix} I_g & O_g \\ O_g & S I_g \end{bmatrix} \mid s \in \prod_p \mathbb{Z}_p^\times \right\},
\]
\[
U_1 = \prod_p \text{GSp}_{2g}(\mathbb{Z}_p) \times G_\infty^+,
\]
\[
U_N = \{ x \in U_1 \mid x_p \equiv I_{2g} \pmod{N \cdot M_{2g}(\mathbb{Z}_p)} \text{ for all rational primes } p \}
\]
for every positive integer $N$. Then we have
\[
U_N \subseteq U_1 \subseteq G_h^+ \quad \text{and} \quad G_h^+ = U_N \Delta G_+
\]
(see [13, lemma 8.3(1)]).

Note that the group $G_\infty^+$ acts on the Siegel upper half-space
\[
\mathbb{H}_g = \{ Z \in M_g(\mathbb{C}) \mid Z^T = Z, \text{Im}(Z) \text{ is positive definite} \}
\]
by
\[
\alpha(Z) = (AZ + B)(CZ + D)^{-1} \quad (\alpha \in G_\infty^+, \ Z \in \mathbb{H}_g),
\]
where $A, B, C, D$ are $g \times g$ block matrices of $\alpha$. Let $\mathcal{F}_N$ be the field of meromorphic Siegel modular functions of genus $g$ for the congruence subgroup
\[
\Gamma(N) = \{ \gamma \in \text{Sp}_{2g}(\mathbb{Z}) \mid \gamma \equiv I_{2g} \pmod{N \cdot M_{2g}(\mathbb{Z})} \}
\]
of the symplectic group $\text{Sp}_{2g}(\mathbb{Z})$ whose Fourier coefficients belong to the $N$th cyclotomic field $\mathbb{Q}(\zeta_N) = e^{2\pi i/N}$. That is, if $f \in \mathcal{F}_N$, then
\[
f(Z) = \frac{\sum_h e(h)c(h e(\text{tr}(hZ))/N)}{\sum_h d(h)c(h e(\text{tr}(hZ))/N)} \text{ for some } c(h), d(h) \in \mathbb{Q}(\zeta_N),
\]
where the denominator and numerator of $f$ are Siegel modular forms of the same weight, $h$ runs over all $g \times g$ positive semi-definite symmetric matrices over half integers with integral diagonal entries, and $e(w) = e^{2\pi i w}$ for $w \in \mathbb{C}$ [5, § 4, theorem 1]. Let
\[
\mathcal{F} = \bigcup_{N=1}^{\infty} \mathcal{F}_N.
\]

**Proposition 2.1.** There exists a homomorphism $\tau: G_h^+ \to \text{Aut}(\mathcal{F})$ satisfying the following properties. Let
\[
f(Z) = \frac{\sum_h e(h)c(h e(\text{tr}(hZ))/N)}{\sum_h d(h)c(h e(\text{tr}(hZ))/N)} \in \mathcal{F}_N.
\]

(i) If $\alpha \in G_+ = \{ \alpha \in G \mid \nu(\alpha) > 0 \}$, then
\[
f^\tau(\alpha) = f \circ \alpha.
\]
Proof. See [13, theorem 8.10].

Since

\[ U_N(Q^\times G_{\infty +})/Q^\times G_{\infty +} \simeq U_N/(U_N \cap Q^\times G_{\infty +}) \simeq \begin{cases} U_1/ \pm G_{\infty +} & \text{if } N = 1, \\ U_N/G_{\infty +} & \text{if } N > 1, \end{cases} \]

we see by proposition 2.1(iii) and (iv) that \( \mathcal{F}_N \) is a Galois extension of \( \mathcal{F}_1 \) with

\[ \text{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq U_1/ \pm U_N. \]  

(2.1)

**Proposition 2.2.** We have

\[ \text{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/(\pm I_{2g}). \]

**Proof.** Let \( \alpha \in U_1 \). Take a matrix \( A \) in \( M_{2g}(\mathbb{Z}) \) for which \( A \equiv \alpha_p \pmod{N \cdot M_{2g}(\mathbb{Z}_p)} \) for all rational primes \( p \). Define a matrix \( \psi(\alpha) \in M_{2g}(\mathbb{Z}/N\mathbb{Z}) \) by the image of \( A \) under the natural reduction \( M_{2g}(\mathbb{Z}) \to M_{2g}(\mathbb{Z}/N\mathbb{Z}) \). Then, by the Chinese remainder theorem, \( \psi(\alpha) \) is well defined and independent of the choice of \( A \). Furthermore, let \( t \) be an integer relatively prime to \( N \) such that \( t \equiv \nu(\alpha_p) \pmod{N \cdot M_{2g}(\mathbb{Z}_p)} \) for all rational primes \( p \). We then derive that

\[ t_\eta \equiv \nu(\alpha_p) \eta_p \equiv \alpha_p^T \eta_p \alpha_p \equiv A^T \eta_p A \equiv \psi(\alpha)^T \eta_p \psi(\alpha) \pmod{N \cdot M_{2g}(\mathbb{Z}_p)} \]

for all rational primes \( p \), and hence \( \psi(\alpha) \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \). Thus, we obtain a group homomorphism

\[ \psi: U_1 \to \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}). \]

Let \( \beta \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \) and take a preimage \( B \) of \( \beta \) under the natural reduction \( M_{2g}(\mathbb{Z}) \to M_{2g}(\mathbb{Z}/N\mathbb{Z}) \). Since \( \nu(\beta) \in (\mathbb{Z}/N\mathbb{Z})^\times \) and

\[ B^T \eta \beta \equiv \beta^T \eta \beta \equiv \nu(\beta) \eta \pmod{N \cdot M_{2g}(\mathbb{Z})}, \]

\( B \) belongs to \( \text{GSp}_{2g}(\mathbb{Z}_p) \) for every rational prime \( p \) dividing \( N \). Let \( \alpha = (\alpha_p)_p \) be the element of \( \prod_p \text{GSp}_{2g}(\mathbb{Z}_p) \) given by

\[ \alpha_p = \begin{cases} B & \text{if } p | N, \\ I_{2g} & \text{otherwise}. \end{cases} \]

We then see that \( \alpha \in U_1 \) and \( \psi(\alpha) = \beta \). Thus, \( \psi \) is surjective.
Clearly, $U_N$ is contained in $\ker(\psi)$. Let $\gamma \in \ker(\psi)$. Since $\gamma_p \equiv I_{2g} \pmod{N \cdot M_{2g}(\mathbb{Z}_p)}$ for all rational primes $p$, we get $\gamma \in U_N$, and hence $\ker(\psi) = U_N$. Therefore, $\psi$ induces an isomorphism $U_1/U_N \simeq \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$, from which we achieve, by (2.1),

$$\text{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq U_1/\pm U_N \simeq \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}.$$ 

\[ \square \]

**Remark 2.3.** We have the decomposition

$$\text{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \simeq G_N \cdot \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\},$$

where

$$G_N = \left\{ \begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix} \bigg| \nu \in (\mathbb{Z}/N\mathbb{Z})^\times \right\}.$$ 

By proposition 2.1 one can describe the action of $\text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ on $\mathcal{F}_N$ as follows.

Let $f(Z) = \sum_h c(h)e(\text{tr}(hZ)/N) / \sum_h d(h)e(\text{tr}(hZ)/N) \in \mathcal{F}_N$.

(i) An element

$$\beta = \begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix}$$

of $G_N$ acts on $f$ by

$$f^\beta = \frac{\sum_h c(h)^{\sigma}e(\text{tr}(hZ)/N)}{\sum_h d(h)^{\sigma}e(\text{tr}(hZ)/N)},$$

where $\sigma$ is the automorphism of $\mathbb{Q}(\zeta_N)$ satisfying $\zeta_{N,\sigma} = \zeta_N^\nu$.

(ii) An element $\gamma$ of $\text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ acts on $f$ by

$$f^\gamma = f \circ \gamma',$$

where $\gamma'$ is any preimage of $\gamma$ under the natural reduction

$$\text{Sp}_{2g}(\mathbb{Z}) \to \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}.$$ 

### 3. Siegel families of level $N$

By making use of the description of $\text{Gal}(\mathcal{F}_N/\mathcal{F}_1)$ in § 2 we shall introduce a generalization of a Fricke family in higher dimensional cases.

Let $N \geq 2$. For $\alpha \in M_{2g}(\mathbb{Z})$ we denote by $\tilde{\alpha}$ its reduction modulo $N$. Define a set

$$\mathcal{V}_N = \left\{ \frac{1}{N} \begin{bmatrix} A^T \\ B^T \end{bmatrix} \bigg| \alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z}) \text{ such that } \tilde{\alpha} \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \right\}.$$ 

Let $\alpha, \beta$ be elements of $M_{2g}(\mathbb{Z})$ satisfying $\alpha, \beta \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. If $M$ is an element of $\mathcal{V}_N$ induced from $\alpha$, then it is straightforward that $\beta^TM$ is also an element of $\mathcal{V}_N$ given by the product $\alpha \beta$. 

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DEFINITION 3.1. We call a family \( \{ h_M(Z) \}_{M \in \mathcal{V}} \) a Siegel family of level \( N \) if it satisfies the following:

(S1) each \( h_M(Z) \) belongs to \( \mathcal{F}_N \);

(S2) \( h_M(Z) \) depends only on \( \pm M \pmod{M_{2g \times g}(Z)} \);

(S3) \( h_M(Z)^\sigma = h_{\sigma T} M(Z) \) for all \( \sigma \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \simeq \text{Gal}(\mathcal{F}_N/\mathcal{F}_1) \).

By \( S_N \) we mean the set of such Siegel families of level \( N \).

REMARK 3.2. Let \( \{ h_M(Z) \}_{M \in S_N} \).

(i) The property (S3) yields a right action of the group \( \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \) on \( \{ h_M(Z) \}_{M} \).

(ii) We let \( M = (1/N) \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{V} \),

and so there is a matrix

\[
\alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z})
\]

such that \( \tilde{\alpha} \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \). Considering \( \tilde{\alpha} \) as an element of \( \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \) we obtain

\[
h_{(1/N) \begin{bmatrix} I_g & O_g \\ * & I_g \end{bmatrix}}(Z) = h_{\tilde{\alpha} \begin{bmatrix} I_g & O_g \\ * & I_g \end{bmatrix}}(Z) = h_M(Z).
\]

Thus, the action of \( \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \) on \( \{ h_M(Z) \}_{M} \) is transitive.

Let

\[
\Gamma^1(N) = \left\{ \gamma \in \text{Sp}_{2g}(\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} I_g & O_g \\ * & I_g \end{bmatrix} \pmod{N \cdot M_{2g}(\mathbb{Z})} \right\},
\]

and let \( \mathcal{F}^1_N(\mathbb{Q}) \) be the field of meromorphic Siegel modular functions for \( \Gamma^1(N) \) with rational Fourier coefficients.

LEMMA 3.3. If \( \{ h_M(Z) \}_{M \in S_N} \), then

\[
h_{\begin{bmatrix} (1/N) I_g \\ O_g \end{bmatrix}}(Z) \in \mathcal{F}^1_N(\mathbb{Q}).
\]

Proof. For any

\[
\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma^1(N)
\]

we deduce by (S2) and (S3) that

\[
h_{\begin{bmatrix} (1/N) I_g \\ O_g \end{bmatrix}}(\gamma(Z)) = h_{\begin{bmatrix} (1/N) I_g \\ O_g \end{bmatrix}}(Z)^\gamma = h_{\gamma T} \begin{bmatrix} (1/N) I_g \\ O_g \end{bmatrix}(Z)
\]

\[
= h_{\begin{bmatrix} (1/N) A^T \\ B^T \end{bmatrix}}(Z) = h_{\begin{bmatrix} (1/N) I_g \\ O_g \end{bmatrix}}(Z)
\]
because

\[ A \equiv I_g, \quad B \equiv O_g \pmod{N \cdot M_g(Z)}. \]

Thus, \( h \left[ \frac{1}{(1/N)I_g} \right] (Z) \) is modular for \( \Gamma^1(N) \).

For every \( \nu \in (\mathbb{Z}/N\mathbb{Z})^\times \) we see by \((S2)\) and \((S3)\) that

\[
h \left[ \frac{1}{(1/N)I_g} \right] (Z) = h \left[ \frac{1}{(1/N)I_g} \right] (Z),
\]

which implies that \( h \left[ \frac{1}{(1/N)I_g} \right] (Z) \) has rational Fourier coefficients. This proves the lemma. \( \square \)

One can consider \( S_N \) as a field under the binary operations

\[
\{ h_M(Z) \} \cdot \{ k_M(Z) \} = \{ (h_M + k_M)(Z) \},
\]

\[
\{ h_M(Z) \} \cdot \{ k_M(Z) \} = \{ (h_Mk_M)(Z) \}.
\]

By lemma 3.3 we get the ring homomorphism

\[
\phi_N : S_N \rightarrow \mathcal{F}_N^1(Q)
\]

\[
\{ h_M(Z) \} \mapsto h \left[ \frac{1}{(1/N)I_g} \right] (Z).
\]

**Lemma 3.4.** If \( M \in \mathcal{V}_N \), then there exists

\[
\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(Z)
\]

such that \( \tilde{\gamma} \in \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \) and

\[
M = \left( \frac{1}{N} \right) \begin{bmatrix} A^T \\ B^T \end{bmatrix}.
\]

**Proof.** Let

\[
\alpha = \begin{bmatrix} A & B \\ U & V \end{bmatrix} \in M_{2g}(Z)
\]

such that \( \tilde{\alpha} \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \) and

\[
M = \left( \frac{1}{N} \right) \begin{bmatrix} A^T \\ B^T \end{bmatrix}.
\]

In \( M_{2g}(\mathbb{Z}/N\mathbb{Z}) \), decompose \( \tilde{\alpha} \) as

\[
\tilde{\alpha} = \begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix} \begin{bmatrix} A & B \\ \nu^{-1}U & \nu^{-1}V \end{bmatrix} \quad \text{with} \quad \nu = \nu(\tilde{\alpha}) \in (\mathbb{Z}/N\mathbb{Z})^\times
\]

so that

\[
\begin{bmatrix} A & B \\ \nu^{-1}U & \nu^{-1}V \end{bmatrix}
\]
belongs to $\text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Since the reduction $\text{Sp}_{2g}(\mathbb{Z}) \to \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ is surjective (see [10]), we can take $\gamma \in M_{2g}(\mathbb{Z})$ satisfying
\[
\tilde{\gamma} = \begin{bmatrix} A & B \\ \nu^{-1}U & \nu^{-1}V \end{bmatrix}.
\]

**Theorem 3.5.** $\mathcal{S}_N$ and $\mathcal{F}_N^1(\mathbb{Q})$ are isomorphic via $\phi_N$.

**Proof.** Since $\mathcal{S}_N$ and $\mathcal{F}_N^1(\mathbb{Q})$ are fields, it suffices to show that $\phi_N$ is surjective.

Let $h(Z) \in \mathcal{F}_N^1(\mathbb{Q})$. For each $M \in \mathcal{V}_N$, take any $\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z})$ such that $\tilde{\gamma} \in \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ and
\[
M = (1/N) \begin{bmatrix} A^T \\ B^T \end{bmatrix}
\]
by using lemma 3.4. We set
\[
h_M(Z) = h(Z)^{\tilde{\gamma}}.
\]
We claim that $h_M(Z)$ is independent of the choice of $\gamma$. Indeed, if $\gamma' = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in M_{2g}(\mathbb{Z})$ such that $\tilde{\gamma}' \in \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$, then we attain in $M_{2g}(\mathbb{Z}/N\mathbb{Z})$ that
\[
\tilde{\gamma}' \tilde{\gamma}^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} D^T & -B^T \\ -C^T & A^T \end{bmatrix} = \begin{bmatrix} I_g & O_g \\ * & I_g \end{bmatrix}
\]
by the fact $\tilde{\gamma}, \tilde{\gamma}' \in \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Let $\delta$ be an element of $\text{Sp}_{2g}(\mathbb{Z})$ such that $\tilde{\delta} = \tilde{\gamma}' \tilde{\gamma}^{-1}$. We then achieve
\[
h(Z)^{\tilde{\gamma}} = (h(Z)^{\tilde{\gamma}' \tilde{\gamma}^{-1}})^{\tilde{\gamma}} = h(\delta(Z))^{\tilde{\gamma}} = h(Z)^{\tilde{\gamma}}
\]
because $h(Z)$ is modular for $\Gamma^1(N)$ and $\delta \in \Gamma^1(N)$.

Now, for any
\[
\sigma = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}
\]
with $\nu = \nu(\sigma)$ we derive that
\[
h_M(Z)^{\sigma} = h(Z)^{\tilde{\gamma} \sigma}
\]
\[
= h(Z)^{\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix}}
\]
\[
= h(Z)^{\begin{bmatrix} I_g & O_g \\ * & I_g \end{bmatrix} \begin{bmatrix} A^P & B^R \\ C^Q & D^S \end{bmatrix}}
\]
\[
= h(Z)^{\begin{bmatrix} A^P & B^R \\ C^Q & D^S \end{bmatrix}}
\]
\[
= h(Z)^{\begin{bmatrix} A^P + BR & A^Q + BS \\ \nu^{-1}(CP + DR) & \nu^{-1}(CQ + DS) \end{bmatrix}}.
\]
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\[ h(Z) = h(\nu^{-1}(CP+DB) \cdot \nu^{-1}(CQ+DS)) \]

since \( h(Z) \) has rational Fourier coefficients

\[ h \left( (AP+BR)^T (AQ+BS)^T \right) \]

\[ h \left( \rho^T R^T \left[ A^T \right] (Z) \right) \]

\[ h_{\sigma M}(Z). \]

This shows that the family \( \{h_M(Z)\}_M \) belongs to \( S_N \). Furthermore, since

\[ \phi_N(\{h_M(Z)\}_M) = h(1/N) \left[ \begin{array}{cc} I_g & O_g \\ O_g & I_g \end{array} \right] (Z) = h(Z) \left[ \begin{array}{cc} I_g & O_g \\ O_g & I_g \end{array} \right] = h(Z), \]

\( \phi \) is surjective as desired.

**Remark 3.6.**

(i) By proposition 2.2 and remark 2.3 we obtain

\[ \text{Gal}(\mathcal{F}_N/\mathcal{F}_N^1(\mathbb{Q})) \cong G_N \cdot \left\{ \gamma \in \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \mid \gamma = \pm \left[ \begin{array}{cc} I_g & O_g \\ O_g & I_g \end{array} \right] \right\}. \]

(ii) Let \( \mathcal{F}_{1,N}(\mathbb{Q}) \) be the field of meromorphic Siegel modular functions for

\[ \Gamma_1(N) = \left\{ \gamma \in \text{Sp}_{2g}(\mathbb{Z}) \mid \gamma \equiv \left[ \begin{array}{cc} I_g & O_g \\ O_g & I_g \end{array} \right] \pmod{N \cdot M_{2g}(\mathbb{Z})} \right\} \]

with rational Fourier coefficients. If we set

\[ \omega = \left[ \begin{array}{cc} (1/\sqrt{N})I_g & O_g \\ O_g & \sqrt{N}I_g \end{array} \right], \]

then we know that \( \omega \in \text{Sp}_{2g}(\mathbb{R}) \) and

\[ \omega \left[ \begin{array}{cc} A & B \\ C & D \end{array} \right] \omega^{-1} = \left[ \begin{array}{cc} A & (1/N)B \\ NC & D \end{array} \right] \text{ for all } \left[ \begin{array}{cc} A & B \\ C & D \end{array} \right] \in \text{Sp}_{2g}(\mathbb{R}). \]

This implies

\[ \omega \Gamma_1(N) \omega^{-1} = \Gamma_1(N), \]

and so \( \mathcal{F}_{1,N}(\mathbb{Q}) \) and \( \mathcal{F}_{1,N}^1(\mathbb{Q}) \) are isomorphic via

\[ \mathcal{F}_{1,N}(\mathbb{Q}) \to \mathcal{F}_{1,N}^1(\mathbb{Q}) \]

\[ h(Z) \mapsto (h \circ \omega)(Z) = h((1/N)Z). \]

**4. An example of a Siegel family**

In this section, we shall give a concrete example of a Siegel family by means of theta constants.

Let \( g \) be a positive integer. For

\[ \mathbf{v} = \left[ \begin{array}{c} \mathbf{v}_u \\ \mathbf{v}_f \end{array} \right] \in \mathbb{Q}^{2g} \]
Remark 7.1. It was shown by Igusa (see [3, theorem 2]) that $\theta_v(Z)$ is identically zero if and only if every entry of the vector $v$ is in $(1/2)\mathbb{Z}$ and $e(2v_a^T a) = -1$. Let

$$S_\pm = \left\{ a = \begin{bmatrix} a_a \\ a_f \end{bmatrix} \in \{0, 1/2\}^{2g} \mid e(2a_a^T a_f) = -1 \right\} \text{ and } S_+ = \{0, 1/2\}^{2g} \setminus S_-.$$ 

Now, let $v \in \mathbb{Q}^{2g}$ with exact denominator $N \geq 2$. We define

$$\Theta_v(Z) = 2^{4N} e(-2^g N (2^g - 1)(2^g + 1)v_a^T v_f) \frac{\prod_{\lambda \in S_-} \theta_{a-\lambda v}(Z)^{4N(2^g+1)}}{\prod_{b \in S_+} \theta_b(Z)^{4N(2^g-1)}} \quad (Z \in \mathbb{H}_g)$$

(see [7, definition 4.2]).

**Proposition 4.1.** The function $\Theta_v(Z)$ depends only on $\pm v$ (mod $\mathbb{Z}^{2g}$). Moreover, it belongs to $\mathcal{F}_N$ and satisfies that

$$\Theta_v(Z)^\sigma = \Theta_{\sigma^* v}(Z)$$

for every $\sigma \in \text{GSp}_{2g}(\mathbb{Z}/NZ) / \{\pm I_{2g}\} \simeq \text{Gal}(\mathcal{F}_N / \mathcal{F}_1)$.

**Proof.** See [7, lemma 4.4].

**Remark 4.2.** One can readily verify that if $g \geq 2$, then $\Theta_v(Z)$ is identically zero if and only if $N = 2$.

**Theorem 4.3.** If $r \in \mathbb{Q}^g$ with exact denominator $N \geq 3$, then $\{\Theta_{M(Nr)}\}_{M \in \mathcal{V}_N}$ is a Siegel family of level $N$.

**Proof.** For any $\gamma \in G^1(N)$ we derive by proposition 4.1 that

$$\Theta_{[\mathfrak{r}]^\gamma}(Z) = \Theta_{[\mathfrak{r}]^\gamma}(Z)^\gamma = \Theta_{[\mathfrak{r}]^\gamma}(Z) = \Theta_{[\mathfrak{r}^{\mathfrak{I}}]}(Z) = \Theta_{[\mathfrak{r}]}(Z).$$

This shows that $\Theta_{[\mathfrak{r}]}(Z)$ is modular for $G^1(N)$. Furthermore, for any $v \in (\mathbb{Z}/NZ)^\times$, by proposition 4.1 we see that

$$\Theta_{[\mathfrak{r}]}(Z)_{\left[ \begin{array}{cc} I_v & O_s \\ O_v & I_s \end{array} \right]} = \Theta_{[\mathfrak{r}]}(Z)_{\left[ \begin{array}{cc} I_v & O_s \\ O_v & I_s \end{array} \right]} = \Theta_{[\mathfrak{r}]}(Z).$$

Thus, $\Theta_{[\mathfrak{r}]}(Z)$ has rational Fourier coefficients, and hence $\Theta_{[\mathfrak{r}]}(Z)$ belongs to $\mathcal{F}_N^1(\mathbb{Q})$.

For each $M \in \mathcal{V}_N$, we can take an element

$$\gamma_M = \left[ \begin{array}{cc} A & B \\ C & D \end{array} \right]$$

of $M_{2g}(\mathbb{Z})$ such that $\tilde{\gamma}_M \in \text{Sp}_{2g}(\mathbb{Z}/NZ)$ and

$$M = (1/N) \left[ \begin{array}{c} A^T \\ B^T \end{array} \right].$$
by lemma 3.4. Then, by the proof of theorem 3.5, the family \( \{ \Theta_{[r]}(Z) \}_{M \in V_N} \) turns out to be a Siegel family of level \( N \). Lastly, we obtain by proposition 4.1 that

\[
\Theta_{[r]}(Z) = \Theta_{\gamma r}[\tilde{\gamma}]_{M}(Z) = \Theta_{\gamma T}[A^T B^T D^T r](Z) = \Theta_{M(Nr)}(Z).
\]

This completes the proof.

5. Special values associated with a Siegel family

As an application of a Siegel family of level \( N \) we shall construct a number associated with each ray class modulo \( N \) of a CM-field. Let \( n \) be a positive integer, \( K \) be a CM-field with \( [K: \mathbb{Q}] = 2^n \) and \( \{ \varphi_1, \ldots, \varphi_n \} \) be a set of embeddings of \( K \) into \( \mathbb{C} \) such that \( (K, \{ \varphi_i \}_{i=1}^n) \) is a CM-type. We fix a finite Galois extension \( L \) of \( \mathbb{Q} \) containing \( K \), and set

\[
S = \{ \sigma \in \text{Gal}(L/\mathbb{Q}) \mid \sigma|_K = \varphi_i \text{ for some } i \in \{1, 2, \ldots, n\} \},
\]

\[
S^* = \{ \sigma^{-1} \mid \sigma \in S \},
\]

\[
H^* = \{ \gamma \in \text{Gal}(L/\mathbb{Q}) \mid \gamma S^* = S^* \}.
\]

Let \( K^* \) be the subfield of \( L \) corresponding to the subgroup \( H^* \) of \( \text{Gal}(L/\mathbb{Q}) \), and let \( \{ \psi_1, \ldots, \psi_g \} \) be the set of all embeddings of \( K^* \) into \( \mathbb{C} \) arising from the elements of \( S^* \). Then we know that \( (K^*, \{ \psi_j \}_{j=1}^g) \) is a primitive CM-type and

\[
K^* = \mathbb{Q}\left( \sum_{i=1}^n a^{\varphi_i} \mid a \in K \right)
\]

(see [12, §8.3, proposition 28]). We call this CM-type \( (K^*, \{ \psi_j \}_{j=1}^g) \) the reflex of \( (K, \{ \varphi_i \}_{i=1}^n) \). Using this CM-type we define an embedding

\[
\Psi: K^* \to \mathbb{C}^g
\]

\[
a \mapsto \begin{bmatrix} a^{\psi_1} \\ \vdots \\ a^{\psi_g} \end{bmatrix}.
\]

For each purely imaginary element \( c \) of \( K^* \) we associate an \( \mathbb{R} \)-bilinear form

\[
E_c: \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{R}
\]

\[
(u, v) \mapsto \sum_{j=1}^g c^{\psi_i} (u_j \bar{v}_j - \bar{u}_j v_j)
\]

\[
\left( \begin{array}{c} u \\ v \end{array} \right) = \left( \begin{array}{c} u_1 \\ \vdots \\ u_g \\ v_1 \\ \vdots \\ v_g \end{array} \right).
\]

Then, one can readily check that

\[
E_c(\Psi(a), \Psi(b)) = \text{Tr}_{K^*/\mathbb{Q}}(cab) \quad \text{for all } a, b \in K^*
\]

(5.1)

by using the fact \( a^{\psi_j} = \bar{a}^{\psi_j} \) for all \( a \in K^* \) (1 ≤ \( j \) ≤ \( g \)).
ASSUMPTION 5.1. In what follows we assume the following conditions.

(i) \((K^*)^\times = K\).

(ii) There is a purely imaginary element \(\xi\) of \(K^*\) and a \(\mathbb{Z}\)-basis \(\{a_1, \ldots, a_{2g}\}\) of the lattice \(\Psi(O_K^*)\) in \(\mathbb{C}^g\) for which

\[
[E_\xi(a_i, a_j)]_{1 \leq i, j \leq 2g} = \begin{bmatrix}
O_g & -I_g \\
I_g & O_g
\end{bmatrix}.
\]

In this case, we say that the complex torus \((\mathbb{C}^g/\Psi(O_K^*), E_\xi)\) is a principally polarized abelian variety with a symplectic basis \(\{a_1, \ldots, a_{2g}\}\). See [12, § 6.2].

(iii) \(f = NO_K\) for an integer \(N \geq 2\).

REMARK 5.2. The assumption 5.1(i) is equivalent to saying that \((K, \{\varphi_i\}_{i=1}^n)\) is a primitive CM-type, namely, the abelian varieties of this CM-type are simple [12, § 8.2, proposition 26].

By assumption 5.1(i) one can define a group homomorphism

\[
g: K^\times \to (K^*)^\times
\]

\[
d \mapsto \prod_{i=1}^n d_i \varphi_i,
\]

and extend it continuously to the homomorphism \(g: K^\times_a \to (K^*)^\times\) of idele groups. It is also known that for each fractional ideal \(a\) of \(K\) there is a fractional ideal \(\mathcal{G}(a)\) of \(K^*\) such that [12, § 8.3]

\[
\mathcal{G}(a) \mathcal{O}_L = \prod_{i=1}^n (a \mathcal{O}_L)^{d_i}.
\]

Let \(\mathcal{C}\) be a given ray class in \(\text{Cl}(f)\). Take any integral ideal \(c\) in \(\mathcal{C}\), and let

\[
\mathcal{N}(c) = \mathcal{N}_K/\mathcal{O}(c) = |\mathcal{O}_K/c|.
\]

LEMMA 5.3. \((\mathbb{C}^g/\Psi(\mathcal{G}(c)^{-1}), E_{\xi\mathcal{N}(c)})\) is also a principally polarized abelian variety.

Proof. It follows from (5.1) that

\[
E_{\xi\mathcal{N}(c)}(\Psi(\mathcal{G}(c)^{-1}), \Psi(\mathcal{G}(c)^{-1})) = \text{Tr}_{K^*/\mathcal{O}}(\xi \mathcal{N}(c)^{-1} \mathcal{G}(c)^{-1} \mathcal{G}(c)^{-1})
\]

\[
= \text{Tr}_{K^*/\mathcal{O}}(\xi \mathcal{G}(c)^{-1})
\]

\[
= E_\xi(\Psi(\mathcal{G}(c)^{-1}), \Psi(\mathcal{G}(c)^{-1}))
\]

\[
\subseteq \mathbb{Z}
\]

because \(E_\xi\) is a Riemann form on \(\mathbb{C}^g/\Psi(O_K^*)\). Thus, \(E_{\xi\mathcal{N}(c)}\) defines a Riemann form on \(\mathbb{C}^g/\Psi(\mathcal{G}(c)^{-1})\).

Now, let \(\{b_1, \ldots, b_{2g}\}\) be a symplectic basis of the abelian variety \((\mathbb{C}^g/\Psi(\mathcal{G}(c)^{-1}), E_{\xi\mathcal{N}(c)})\) so that

\[
\Psi(\mathcal{G}(c)^{-1}) = \sum_{j=1}^{2g} \mathbb{Z}b_j \quad \text{and} \quad [E_{\xi\mathcal{N}(c)}(b_i, b_j)]_{1 \leq i, j \leq 2g} = \begin{bmatrix}
O_g & -\mathcal{E} \\
\mathcal{E} & O_g
\end{bmatrix},
\]
where

\[
E = \begin{bmatrix}
\varepsilon_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \varepsilon_g
\end{bmatrix}
\]

is a \(g \times g\) diagonal matrix for some positive integers \(\varepsilon_1, \ldots, \varepsilon_g\) satisfying \(\varepsilon_1 | \cdots | \varepsilon_g\). Furthermore, let \(b_1, \ldots, b_{2g}\) be elements of \(G(c)^{-1}\) such that \(b_j = \Psi(b_j)\) (\(1 \leq j \leq 2g\)). Since \(O_K^\ast \subseteq G(c)^{-1}\), we have

\[
[a_1 \cdots a_{2g}] = [b_1 \cdots b_{2g}] \alpha \quad \text{for some } \alpha \in M_{2g}(\mathbb{Z}) \cap GL_{2g}(\mathbb{Q}),
\]

and hence

\[
\begin{bmatrix}
a_1^\psi_1 & \cdots & a_{2g}^\psi_1 \\
\vdots & \ddots & \vdots \\
a_1^\psi_g & \cdots & a_{2g}^\psi_g
\end{bmatrix}
= \begin{bmatrix}
b_1^\psi_1 & \cdots & b_{2g}^\psi_1 \\
\vdots & \ddots & \vdots \\
b_1^\psi_g & \cdots & b_{2g}^\psi_g
\end{bmatrix} \alpha.
\]

Taking determinant and squaring gives rise to the identity

\[
\Delta_{K^\ast/\mathbb{Q}}(a_1, \ldots, a_{2g}) = \Delta_{K^\ast/\mathbb{Q}}(b_1, \ldots, b_{2g}) \det(\alpha)^2.
\]

It then follows that

\[
det(\alpha)^2 = \frac{\left| \Delta_{K^\ast/\mathbb{Q}}(a_1, \ldots, a_{2g}) \right|}{\left| \Delta_{K^\ast/\mathbb{Q}}(b_1, \ldots, b_{2g}) \right|} = \frac{d_{K^\ast/\mathbb{Q}}(O_K^\ast)}{d_{K^\ast/\mathbb{Q}}(\mathcal{G}(c))^{-1}}
= \mathcal{N}_{K^\ast/\mathbb{Q}}(\mathcal{G}(c))
= \mathcal{N}_{K^\ast/\mathbb{Q}}(\mathcal{G}(c)\mathcal{G}(c)^{-1})
= \mathcal{N}(c)^{2g}, \tag{5.3}
\]

where \(d_{K^\ast/\mathbb{Q}}\) stands for the discriminant of a fractional ideal of \(K^\ast\) [9, ch. III, proposition 13]. Furthermore, we deduce by (5.2) that

\[
\mathcal{N}(c) \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix} = \left[ \mathcal{N}(c)E_{\xi}(a_i, a_j) \right]_{1 \leq i, j \leq 2g}
= \left[ E_{\xi\mathcal{N}(c)}(a_i, a_j) \right]_{1 \leq i, j \leq 2g}
= \alpha^T \left[ E_{\xi\mathcal{N}(c)}(b_i, b_j) \right]_{1 \leq i, j \leq 2g} \alpha
= \alpha^T \begin{bmatrix} O_g & -\mathcal{E} \\ \mathcal{E} & O_g \end{bmatrix} \alpha.
\]

By taking the determinant we get \(\mathcal{N}(c)^{2g} = det(\alpha)^2(\varepsilon_1 \cdots \varepsilon_g)^2\), which, by (5.3), yields that \(\varepsilon_1 = \cdots = \varepsilon_g = 1\), and so \(\mathcal{E} = I_g\). Therefore, \((\mathbb{C}^g/\mathcal{G}(c)^{-1}), E_{\xi\mathcal{N}(c)}\) becomes a principally polarized abelian variety. \(\square\)
In this section we shall show that the value \( h \) belonging to principally polarized abelian variety \((\mathbb{C}^g / \Psi(G(\mathfrak{c})^{-1}), E_{\xi N(\mathfrak{c})})\), and let \( b_1, \ldots, b_{2g} \) be elements of \( G(\mathfrak{c})^{-1} \) such that \( b_j = \Psi(b_j) \) (\( 1 \leq j \leq 2g \)). We then have

\[
[a_1 \cdots a_{2g}] = [b_1 \cdots b_{2g}] \alpha \quad \text{for some} \quad \alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z}) \cap \text{GSp}_{2g}(\mathbb{Q}).
\]

(5.4)

Since \( \nu(\alpha) = N(\mathfrak{c}) \) is relatively prime to \( N \), the reduction \( \tilde{\alpha} \) of \( \alpha \) modulo \( N \) belongs to \( \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \). Let \( Z_\mathfrak{c}^* \) be the CM-point associated with the symplectic basis \( \{ b_1, \ldots, b_{2g} \} \), namely

\[
Z_\mathfrak{c}^* = [b_{g+1} \cdots b_{2g}]^{-1} [b_1 \cdots b_g],
\]

which belongs to \( \mathbb{H}_g \) [1, proposition 8.1.1].

**Definition 5.4.** Let \( \{ h_M(Z) \}_{M} \in S_N \). For a given ray class \( \mathcal{C} \in \text{Cl}(f) \) we define

\[
h_{\mathcal{C}}(\mathcal{C}) = h_{(1/N)[B]}(Z_\mathfrak{c}^*).
\]

**Remark 5.5.** Here, the index matrix

\[
(1/N) \begin{bmatrix} B \\ D \end{bmatrix}
\]

is obtained using the fact that

\[
\left( \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix} \right)^T = \begin{bmatrix} B^T & D^T \\ -A^T & -C^T \end{bmatrix}.
\]

**6. Well-definedness of \( h_{\mathcal{C}}(\mathcal{C}) \)**

In this section we shall show that the value \( h_{\mathcal{C}}(\mathcal{C}) \) given in definition 5.4 depends only on the ray class \( \mathcal{C} \), and hence it is independent of the choice of a symplectic basis and an integral ideal in \( \mathcal{C} \).

**Proposition 6.1.** The value \( h_{\mathcal{C}}(\mathcal{C}) \) does not depend on the choice of a symplectic basis \( \{ b_1, \ldots, b_{2g} \} \) of \((\mathbb{C}^g / \Psi(G(\mathfrak{c})^{-1}), E_{\xi N(\mathfrak{c})})\).

**Proof.** Let \( \{ \tilde{b}_1, \ldots, \tilde{b}_{2g} \} \) be another symplectic basis of \((\mathbb{C}^g / \Psi(G(\mathfrak{c})^{-1}), E_{\xi N(\mathfrak{c})})\).

Thus,

\[
[\tilde{b}_1 \cdots \tilde{b}_{2g}] = [b_1 \cdots b_{2g}] \beta \quad \text{for some} \quad \beta = \begin{bmatrix} P \\ Q \\ R \\ S \end{bmatrix} \in \text{GL}_{2g}(\mathbb{Z}).
\]

(6.1)

We then derive

\[
\begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix} = E_{\xi N(\mathfrak{c})}(\tilde{b}_1, \tilde{b}_j)_{1 \leq i, j \leq 2g}
\]

\[
= \beta^T E_{\xi N(\mathfrak{c})}(b_1, b_j)_{1 \leq i, j \leq 2g} \beta
\]

\[
= \beta^T \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix} \beta,
\]

(5.4)
which shows that $\beta \in \text{Sp}_{2g}(\mathbb{Z})$. Since

$$[a_1 \cdots a_{2g}] = [b_1 \cdots b_{2g}] \alpha = [\hat{b}_1 \cdots \hat{b}_{2g}] \beta^{-1} \alpha$$

by (5.4) and (6.1), the special value obtained by $\{\hat{b}_1, \ldots, \hat{b}_{2g}\}$ is

$$h_{(1/N)\beta^{-1}[B_D]}(\hat{Z}_\xi^*),$$

where $\hat{Z}_\xi^*$ is the CM-point corresponding to $\{\hat{b}_1, \ldots, \hat{b}_{2g}\}$.

On the other hand, we attain that

$$\hat{Z}_\xi^* = \left[\hat{b}_{g+1} \cdots \hat{b}_{2g}\right]^{-1} \left[\hat{b}_1 \cdots \hat{b}_g\right]$$

$$= \left(\left[\hat{b}_1 \cdots \hat{b}_g\right] Q + \left[\hat{b}_{g+1} \cdots \hat{b}_{2g}\right] S\right)^{-1}$$

$$\times \left(\left[\hat{b}_1 \cdots \hat{b}_g\right] P + \left[\hat{b}_{g+1} \cdots \hat{b}_{2g}\right] R\right) \text{ by (6.1)}$$

$$= (P^T \left[\hat{b}_1 \cdots \hat{b}_g\right]^T + R^T \left[\hat{b}_{g+1} \cdots \hat{b}_{2g}\right]^T)$$

$$\times (Q^T \left[\hat{b}_1 \cdots \hat{b}_g\right]^T + S^T \left[\hat{b}_{g+1} \cdots \hat{b}_{2g}\right]^T)^{-1}, \text{ since } (\hat{Z}_\xi^*)^T = \hat{Z}_\xi^*$$

$$= (P^T \left[\hat{b}_{g+1} \cdots \hat{b}_{2g}\right]^{-1} \left[\hat{b}_1 \cdots \hat{b}_g\right]^T + R^T)$$

$$\times (Q^T \left[\hat{b}_{g+1} \cdots \hat{b}_{2g}\right]^{-1} \left[\hat{b}_1 \cdots \hat{b}_g\right]^T + S^T)^{-1}$$

$$= (P^T Z_\xi^* + R^T)(Q^T Z_\xi^* + S^T)^{-1} \text{ because } (Z_\xi^*)^T = Z_\xi^*$$

$$= \beta^T(Z_\xi^*). \quad (6.2)$$

Thus, we deduce that

$$h_{(1/N)\beta^{-1}[B_D]}(\hat{Z}_\xi^*) = h_{(1/N)\beta^{-1}[B_D]}(\beta^T(Z_\xi^*)) \text{ by (6.2)}$$

$$= (h_{(1/N)\beta^{-1}[B_D]}(Z))^{\beta^T}|_{Z=Z_\xi^*}$$

$$= h_{(1/N)(\beta^T)^{-1}[B_D]}(Z_\xi^*) \text{ by the property (S3) of } h_M(Z)_{\mathcal{M}}$$

$$= h_{(1/N)[B_D]}(Z_\xi^*).$$

This proves that the value $h_{\xi}(C)$ is independent of the choice of a symplectic basis of $(\mathbb{C}^g/\Psi(\mathcal{G}(\xi)^{-1}), E_{\xi N(\xi)})$.

**Remark 6.2.** One can analogously readily show that $h_{\xi}(C)$ does not depend on the choice of a symplectic basis $\{a_1, \ldots, a_{2g}\}$ of $(\mathbb{C}^g/\Psi(\mathcal{O}_K), E_{\xi})$.

**Proposition 6.3.** $h_{\xi}(C)$ does not depend on the choice of an integral ideal $\mathfrak{c}$ in $C$.

**Proof.** Let $\mathfrak{c}'$ be another integral ideal in the class $C$, and hence

$$\mathfrak{c}'^{-1} = (1 + a)\mathcal{O}_K \text{ for some } a \in \mathfrak{a}^{-1}, \quad (6.3)$$

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where \( a \) is an integral ideal of \( K \) relatively prime to \( f \). Since \( 1 \in c^{-1} \) and \( (1 + a) \in c'c^{-1} \subseteq c^{-1} \), we get \( a \in c^{-1} \). Thus, we derive that

\[
aac \subseteq fc \cap a \quad \text{by the facts that } a \in fa^{-1} \text{ and } a \in c^{-1} \\
\subseteq f \cap a \\
= fa \quad \text{because } f \text{ and } a \text{ are relatively prime},
\]

from which it follows that \( a \in fc^{-1} \). Using the fact that \( f = N\mathcal{O}_K \) yields

\[
g(1 + a) = \prod_{i=1}^{n} (1 + (a + c^{-1}O_L)) \subseteq K^* \cap (1 + N\mathcal{O}(c)}^{-1}O_L)
= 1 + N\mathcal{G}(c)^{-1}. \tag{6.4}
\]

Let

\[
b'_j = g(1 + a)^{-1}b_j \quad \text{and} \quad b'_j = \Psi(b'_j) \quad (1 \leq j \leq 2g).
\]

We know that \( \{b'_1, \ldots, b'_{2g}\} \) is a \( \mathbb{Z} \)-basis of the lattice \( \Psi(\mathcal{G}(c')^{-1}) \) in \( \mathbb{C}^g \) and

\[
b'_j = Tb_j \quad \text{with } T = \begin{bmatrix}
(1 + a)^{-1}\psi_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & (1 + a)^{-1}\psi_g
\end{bmatrix}. \tag{6.6}
\]

Furthermore, we get that

\[
\left[ E_{\xi N(c')} (b'_i, b'_j) \right]_{1 \leq i, j \leq 2g}
= \left[ Tr_{K'/\mathbb{Q}(\xi N(c')) b'_i b'_j} \right]_{1 \leq i, j \leq 2g} \quad \text{by (5.1)}
= \left[ Tr_{K'/\mathbb{Q}(\xi N(c')) g(1 + a)^{-1}b'_i g(1 + a)^{-1}b'_j} \right]_{1 \leq i, j \leq 2g} \quad \text{by (6.5)}
= \left[ Tr_{K'/\mathbb{Q}(\xi N(c')) N_{K'/\mathbb{Q}}(1 + a)^{-1}b'_i b'_j} \right]_{1 \leq i, j \leq 2g}
= \left[ Tr_{K/Q} (\xi N(c)) b'_i b'_j \right]_{1 \leq i, j \leq 2g} \quad \text{by (6.3) and the fact that } N_{K/Q}(1 + a) > 0
= \left[ E_{\xi N(c)} (b_i, b_j) \right]_{1 \leq i, j \leq 2g} \quad \text{by (5.1)}
= \begin{bmatrix}
O_g & -I_g \\
I_g & O_g
\end{bmatrix}.
\]

Thus, \( \{b'_1, \ldots, b'_{2g}\} \) is a symplectic basis of \( (\mathbb{C}^g/\Psi(\mathcal{G}(c')^{-1}), E_{\xi N(c)}) \), and its associated CM-point \( Z_c \) is given by

\[
Z_c = \left[ b'_{g+1} \cdots b'_{2g} \right]^{-1} \left[ b'_1 \cdots b'_g \right]
= \left[ Tb_{g+1} \cdots Tb_{2g} \right]^{-1} \left[ Tb_1 \cdots Tb_g \right] \quad \text{by (6.6)}
= Z_c^*. \tag{6.7}
\]

Let \( \alpha = [a_{ij}] \), \( \alpha' = [a'_{ij}] \in M_{2g}(\mathbb{Z}) \) such that

\[
[a_1 \cdots a_{2g}] = \left[ b_1 \cdots b_{2g} \right] \alpha \quad \text{and} \quad \left[ b'_1 \cdots b'_{2g} \right] \alpha' = \left[ b'_1 \cdots b'_{2g} \right]. \tag{6.8}
\]
For each $1 \leq i \leq 2g$ we obtain that
\[
\sum_{j=1}^{2g} a'_{ji} b_j = g(1 + a) \sum_{j=1}^{2g} a'_{ji} b_j \quad \text{by (6.5)}
\]
\[
= a_i g(1 + a) \quad \text{by (6.8)}
\]
\[
\in a_i (1 + NG(c)^{-1}) \quad \text{by (6.4)}
\]
\[
\subseteq a_i + NG(c)^{-1} \quad \text{because } a_i \in \mathcal{O}_K
\]
\[
= \sum_{j=1}^{2g} a_{ji} b_j + N \sum_{j=1}^{2g} \mathbb{Z} b_j \quad \text{by (6.8)}.
\]

This yields $\alpha \equiv \alpha' \pmod{N \cdot M_{2g}(\mathbb{Z})}$, and hence
\[
(1/N) \alpha \equiv (1/N) \alpha' \pmod{M_{2g}(\mathbb{Z})}.
\]

Now, the result follows from (6.7), (6.9) and the property (S2) of \{h_M(Z)\}_M. \qed

7. Galois actions on $h_f(C)$

Finally, we shall show that if $h_f(C)$ is finite, then it lies in the ray class field $K_1$ and satisfies the natural transformation formula under the Artin reciprocity map for $f$.

Let $r: K^* \rightarrow M_{2g}(\mathbb{Q})$ be the regular representation with respect to the ordered basis $\{a_1, \ldots, a_{2g}\}$ of $K^*$ over $\mathbb{Q}$ given by
\[
\begin{bmatrix}
a_1 \\
\vdots \\
a_{2g}
\end{bmatrix}
= r(a)
\begin{bmatrix}
a_1 \\
\vdots \\
a_{2g}
\end{bmatrix}
(a \in K^*).
\]

Then it can be extended to the map $r: (K^*)_\mathbb{A} \rightarrow M_{2g}(\mathbb{Q}_\mathbb{A})$ of adele rings.

**Lemma 7.1** (Shimura’s reciprocity law). Let $f$ be an element of $\mathcal{F}$ that is finite at $Z^*_\mathbb{A}$.

(i) The special value $f(Z^*_\mathbb{A})$ lies in $K_{ab}$.

(ii) For every $s \in K^*_\mathbb{A}$ we have $r(g(s)) \in G_{\mathbb{A}+}$ and
\[
f(Z^*_\mathbb{A})^{[s,K]} = f^r(r(g(s)^{-1}))(Z^*_\mathbb{A}).
\]

**Proof.** See [13, lemma 9.5 and theorem 9.6]. \qed

**Theorem 7.2.** If $h_f(C)$ is finite, then it belongs to $K_1$. Furthermore, it satisfies
\[
h_f(C)^{\sigma_1(D)} = h_f(CD) \quad \text{for every } D \in \text{Cl}(f),
\]
where $\sigma_1$ is the Artin reciprocity map for $f$.

**Proof.** Since $h_f(C)$ belongs to $K_{ab}$ by lemma 7.1(i), there is a sufficiently large positive integer $M$ so that $N|M$ and $h_f(C) \in K_m$ with $m = M\mathcal{O}_K$. Take an integral
ideal \( \mathfrak{d} \) in \( \mathcal{D} \) relatively prime to \( \mathfrak{m} \) by using the surjectivity of the natural map \( \text{Cl}(\mathfrak{m}) \to \text{Cl}(\mathfrak{f}) \). Let \( \{d_1, \ldots, d_{2g}\} \) be a symplectic basis of the principally polarized abelian variety \( (\mathbb{C}^g/\Psi(G(\mathfrak{d})^{-1}), E_{\xi(N(\mathfrak{d}))}) \), and let \( d_1, \ldots, d_{2g} \) be elements of \( G(\mathfrak{d})^{-1} \) such that \( d_j = \Psi(d_j) \ (1 \leq j \leq 2g) \). Since \( G(\mathfrak{c})^{-1} \subseteq G(\mathfrak{d})^{-1} \), we get

\[
[b_1 \cdots b_{2g}] = [d_1 \cdots d_{2g}] \delta \quad \text{for some } \delta \in M_{2g}(\mathbb{Z}) \cap GL_{2g}(\mathbb{Q}). \quad (7.2)
\]

We then have that

\[
\begin{bmatrix}
O_g & -I_g \\
I_g & O_g
\end{bmatrix} = [E_{\xi(N(\mathfrak{c}))}(b_i, b_j)]_{1 \leq i, j \leq 2g} = \delta^T [E_{\xi(N(\mathfrak{c}))}(d_i, d_j)]_{1 \leq i, j \leq 2g} \delta \quad \text{by (7.2)}
\]

\[
= \delta^T [N(\mathfrak{c})N(\mathfrak{d})^{-1}E_{\xi(N(\mathfrak{d}))}(d_i, d_j)]_{1 \leq i, j \leq 2g} \delta
\]

\[
= N(\mathfrak{d})^{-1} \delta^T \begin{bmatrix}
O_g & -I_g \\
I_g & O_g
\end{bmatrix} \delta.
\]

This claims that

\[
\delta \in M_{2g}(\mathbb{Z}) \cap G_+ \quad \text{with } \nu(\delta) = N(\mathfrak{d}). \quad (7.3)
\]

Furthermore, if we let \( Z^*_\mathfrak{d} \) be the CM-point associated with \( \{d_1, \ldots, d_{2g}\} \), then we obtain

\[
Z^*_\mathfrak{d} = (\delta^{-1})^T (Z^*_\mathfrak{e}) \quad (7.4)
\]

in a similar way to the argument in the proof of proposition 6.1.

Let \( s = (s_p)_p \) be an idele of \( K \) such that

\[
s_p = \begin{cases} 
1 & \text{if } p|\mathfrak{m}, \\
\mathfrak{d}_p & \text{if } p|\mathfrak{M}.
\end{cases} \quad (5.5)
\]

If we set \( \mathcal{D} \) to be the ray class in \( \text{Cl}(\mathfrak{m}) \) containing \( \mathfrak{d} \), then by (5.5) we attain

\[
[s, K]_{|\mathfrak{m}} = \sigma_{\mathfrak{m}}(\mathcal{D}), \quad (6.6)
\]

\[
g(s)^{-1}_p(\mathfrak{O}_K) = G(\mathfrak{d})^{-1} \quad \text{for all rational primes } p. \quad (7.7)
\]

It then follows from (7.1)–(7.7) that for every rational prime \( p \), the entries of each of the vectors

\[
r(g(s)^{-1})_p \begin{bmatrix}
b_1 \\
\vdots \\
b_{2g}
\end{bmatrix} \quad \text{and} \quad (\delta^{-1})^T \begin{bmatrix}
b_1 \\
\vdots \\
b_{2g}
\end{bmatrix}
\]

form a basis of \( G(\mathfrak{d})^{-1} = G(\mathfrak{c})^{-1} G(\mathfrak{d})^{-1} \). So, there exists a matrix \( u = (u_p)_p \in \prod_p GL_{2g}(\mathbb{Z}_p) \) satisfying

\[
r(g(s)^{-1}) = u(\delta^{-1})^T. \quad (7.8)
\]

Since \( \delta^T \) and

\[
\begin{bmatrix}
I_g & O_g \\
O_g & N(\delta)I_g
\end{bmatrix}
\]
can be viewed as elements of $\text{GSp}_{2g}(\mathbb{Z}/\mathbb{M})$ by (7.3), there exists a matrix $\gamma \in \text{Sp}_{2g}(\mathbb{Z})$ such that

$$\delta^T \equiv \begin{bmatrix} I_g & O_g \\ O_g & N(\delta)I_g \end{bmatrix} \gamma \pmod{M \cdot M_{2g}(\mathbb{Z})} \quad (7.9)$$

owing to the surjectivity of the reduction $\text{Sp}_{2g}(\mathbb{Z}) \to \text{Sp}_{2g}(\mathbb{Z}/\mathbb{M})$. Since

$$r(g(s)^{-1})_p = I_{2g} \quad \text{for all } p \mid M$$

by (7.5), we get $u_p = \delta^T$ for all $p \mid M$ by (7.8). Hence, we deduce using (7.9) that

$$u_p \gamma^{-1} \equiv \begin{bmatrix} I_g & O_g \\ O_g & N(\delta)I_g \end{bmatrix} \quad (\text{mod } M \cdot M_{2g}(\mathbb{Z}_p)) \quad \text{for all rational primes } p. \quad (7.10)$$

On the other hand, we have by (5.4) and (7.2) that

$$\begin{bmatrix} a_1 & \cdots & a_{2g} \\ b_1 & \cdots & b_{2g} \end{bmatrix} = \begin{bmatrix} b_1 & \cdots & b_{2g} \end{bmatrix} \delta^{-1}(\delta \alpha) = \begin{bmatrix} d_1 & \cdots & d_{2g} \end{bmatrix} (\delta \alpha). \quad (7.11)$$

Letting

$$\alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

we induce the following:

$$h_{1f}(\mathcal{C})^\sigma_m(\mathcal{D}) = h_{1f}(\mathcal{C})^{[s,K]} \quad \text{by (7.6)}$$

$$= h_{\{1/N\}}[B \mid D](Z_\gamma)^{[s,K]} \quad \text{by definition 5.4}$$

$$= h_{\{1/N\}}[B \mid D](Z)^{\tau(g(s)^{-1})}\big|_{Z=Z_\gamma} \quad \text{by lemma 7.1(ii)}$$

$$= h_{\{1/N\}}[B \mid D](Z)^{\tau(u(\delta^{-1})^T)}\big|_{Z=Z_\gamma} \quad \text{by (7.8)}$$

$$= h_{\{1/N\}}[B \mid D](Z)^{\tau(u^{-1})\tau(\gamma)\tau((\delta^{-1})^T)}\big|_{Z=Z_\gamma} \quad \text{by (7.9) and (S3)}$$

$$= h_{\{1/N\}}[B \mid D](Z)^{\tau((\delta^{-1})^T)}\big|_{Z=Z_\gamma} \quad \text{by (7.10) and (S3)}$$

$$= h_{\{1/N\}}[B \mid D](Z)^{\tau(\gamma)\tau((\delta^{-1})^T)}\big|_{Z=Z_\gamma} \quad \text{by (S3)}$$

$$= h_{\{1/N\}}[B \mid D](Z)^{\tau((\delta^{-1})^T)}\big|_{Z=Z_\gamma} \quad \text{by (7.9) and (S2)}$$

$$= h_{\{1/N\}}[B \mid D](\delta^{-1})^T(Z_\gamma^\ast) \quad \text{due to the fact that } \delta \in G_+ \text{ and by (A1)}$$

$$= h_{1f}(CD) \quad \text{by (7.4), (7.11) and definition 5.4.}$$

In particular, suppose that $\mathcal{D} = d0_K$ for some $d \in O_K$ such that $d \equiv 1 \pmod{f}$. Then $\mathcal{D}$ is the identity class of $\mathcal{C}(f)$, and so the above observation implies that $\sigma_m(\mathcal{D})$ leaves $h_{1f}(\mathcal{C})$ fixed. Therefore, we conclude that $h_{1f}(\mathcal{C})$ lies in $K_f$. \hfill $\Box$
Corollary 7.3. Let $H$ be a subgroup of $\text{Cl}(f)$ defined by

$$H = \langle D \in \text{Cl}(f) \mid D \text{ contains an integral ideal } \mathfrak{d} \text{ of } K \text{ for which } g(d) = g(d) \mathcal{O}_{K^*} \text{ for some } d \in \mathcal{O}_K \text{ such that } g(d) \equiv 1 \pmod{N\mathcal{O}_{K^*}} \rangle,$$

and let $K^H_f$ be the fixed field of $H$. If $h_f(C)$ is finite, then it belongs to $K^H_f$.

Proof. Let $C_0$ be the identity class of $\text{Cl}(f)$. Since $h_f(C_0) \in K_f$ by theorem 7.2, $K(h_f(C_0))$ is a Galois extension of $K$ as a subfield of $K_f$. Furthermore, since $h_f(C_0) = h_f(C)$ by theorem 7.2, $K(h_f(C_0))$ contains $h_f(C)$. Thus, it suffices to show that $h_f(C_0)$ belongs to $K^H_f$.

To this end, let $D$ be an element of $\text{Cl}(f)$ containing an integral ideal $\mathfrak{d}$ of $K$ for which

$$g(d) = g(d) \mathcal{O}_{K^*} \text{ for some } d \in \mathcal{O}_K \text{ such that } g(d) \equiv 1 \pmod{N\mathcal{O}_{K^*}}.$$ 

Now that

$$((\mathbb{C}^g/\Phi(g(d)^{-1}), E_{\mathcal{L}N(\mathfrak{d})}) = ((\mathbb{C}^g/\Phi(g(d)^{-1}\mathcal{O}_{K^*}), E_{\mathcal{L}N(d\mathcal{O}_K)}),$$

we obtain

$$h_f(C_0)^{\sigma_i(D)} = h_f(D) = h_f([d\mathcal{O}_K]),$$

where $[a]$ is the ray class containing $a$ for a fractional ideal $a$ of $K$. Moreover, since $g(d) \equiv 1 \pmod{N\mathcal{O}_{K^*}}$, we obtain

$$h_f([d\mathcal{O}_K]) = h_f([\mathcal{O}_K]) = h_f(C_0)$$

analogously to the proof of proposition 6.3. This proves that $h_f(C_0)$ belongs to $K^H_f$. 

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References


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