Proceedings of the Royal Society of Edinburgh, page 1 of 7
DOI:10.1017/prm.2023.96

Fixed point conditions for non-coprime actions

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(Received 13 March 2023; accepted 20 August 2023)

In the setting of finite groups, suppose J acts on N via automorphisms so that the induced semidirect product $N \rtimes J$ acts on some non-empty set Ω, with N acting transitively. Glauberman proved that if the orders of J and N are coprime, then J fixes a point in Ω. We consider the non-coprime case and show that if N is abelian and a Sylow p-subgroup of J fixes a point in Ω for each prime p, then J fixes a point in Ω. We also show that if N is nilpotent, $N \rtimes J$ is supersoluble, and a Sylow p-subgroup of J fixes a point in Ω for each prime p, then J fixes a point in Ω.

Keywords: Non-coprime actions; conjugacy of complements; supersoluble groups

2020 *Mathematics Subject Classification* 05E18; 20E22; 20E45; 20F16; 20J06; 55N45

1. Introduction

Suppose a finite group J acts via automorphisms on a finite group N and the induced semi-direct product $G = N \rtimes J$ acts on some non-empty set Ω where the action of N is transitive. Glauberman showed that if each supplement H of N in G splits over $N \cap H$ and each complement of N in G is conjugate to J, then there exists a J-invariant element $\omega \in \Omega$. Consequently, if the orders of J and N are coprime so that the Schur–Zassenhaus theorem applies, a fixed point always exists [**4**, Thm. 4]. In this note, we consider the non-coprime case and establish some conditions for the existence of a fixed point.

Given an action as described above, consider the stabiliser $G_\alpha \leqslant G$ fixing an arbitrary point $\alpha \in \Omega$. As N is transitive, G_α supplements N in G. In this context, J fixes an element of Ω if and only if the following two conditions are met. Firstly, we must ensure G_α splits over $N \cap G_\alpha$ so that there exists some complement J'. As $G/N \cong G_\alpha/(N \cap G_\alpha)$, it will follow that J' also complements N in G. Secondly, we require that $J' = g^{-1}Jg$ for some $g \in G$ so that J fixes $g \cdot \alpha$. For the latter requirement, we concern ourselves with conditions for two specific complements in a semidirect product to be conjugate.

To this end, we say two subgroups H and H' are *locally conjugate* in a group G if for each prime p, a Sylow p-subgroup of H is conjugate to a Sylow p-subgroup of H'. Losey and Stonehewer showed that if H and H' are locally conjugate supplements of some normal nilpotent subgroup N in a soluble group G, then H and H' are conjugate if either G/N is nilpotent or N is abelian [**7**]. Evans and Shin further showed that if N is abelian, then G need not be soluble [**3**].

We first restrict N to be abelian and use a decomposition result from group cohomology to provide an alternate proof of:

LEMMA 1.1 (Evans and Shin). *In a finite group, two complements of a normal abelian subgroup are conjugate if and only if they are locally conjugate.*

We use this, along with Gaschütz's result that a finite group G splits over an abelian subgroup N if and only if for each prime p, a Sylow p-subgroup S of G splits over $N \cap S$, to show:

THEOREM 1.2. *Given a finite group J acting via automorphisms on a finite abelian group N, suppose the induced semidirect product $N \rtimes J$ acts on some non-empty set Ω where the action of N is transitive. If for each prime p, a Sylow p-subgroup of J fixes an element of Ω, then there exists some J-invariant element $\omega \in \Omega$.*

This had previously been shown using elementary arguments for the special case that J is supersoluble [**2**, Cor. 2]. The theorem implies:

COROLLARY 1.3. *Let G be a finite split extension over an abelian subgroup N. If for each prime p there is a Sylow p-subgroup S of G such that any two complements of $N \cap S$ in S are conjugate, then any two complements of N in G are G-conjugate.*

This extends a result of D. G. Higman [**5**, Cor. 2] that requires the complements of $N \cap S$ in S to be conjugate *within* S.

We then consider nilpotent N and supersoluble $N \rtimes J$. We adapt our approach for lemma 1.1 to nonabelian cohomology and demonstrate:

LEMMA 1.4. *In a finite supersoluble group, two complements of a normal nilpotent subgroup are conjugate if and only if they are locally conjugate.*

With this, we then show:

THEOREM 1.5. *Given a finite group J acting via automorphisms on a finite nilpotent group N, suppose the induced semidirect product $N \rtimes J$ is supersoluble and acts on some non-empty set Ω where the action of N is transitive. If for each prime p, a Sylow p-subgroup of J fixes an element of Ω, then there exists some J-invariant element $\omega \in \Omega$.*

The theorem also implies an analogue of corollary 1.3 that we state and prove in § 3.

1.1. Outline

We proceed as follows. In the remainder of this section, we introduce notation and some conventions from group cohomology. In the next section, we restrict N to be abelian and prove theorem 1.2. We then restrict N to be nilpotent and $N \rtimes J$ to be supersoluble in § 3 and prove theorem 1.5, before concluding in § 4.

1.2. Notation and conventions

All groups in this note are assumed finite. A subgroup $K \leqslant G$ supplements $N \lhd G$ if $G = NK$ and complements N if it both supplements N and the intersection $N \cap K$ is trivial. We denote conjugation by $g^\gamma = \gamma^{-1} g \gamma$ for $g, \gamma \in G$ and otherwise let groups act from the left. For a prime p, we let $\mathrm{Syl}_p(G)$ denote the set of Sylow p-subgroups of a group G.

We rely on rudimentary notions from group cohomology that can be found in the texts of Brown [**1**] and Serre [**8**]. Given a group J acting on a group N via automorphisms, crossed homomorphisms or 1-cocycles are maps $\varphi : J \to N$ satisfying $\varphi(jj') = \varphi(j)\varphi(j')^{j^{-1}}$ for all $j, j' \in J$. Two such maps φ and φ' are cohomologous if there exists $n \in N$ such that $\varphi'(j) = n^{-1}\varphi(j)n^{j^{-1}}$ for all $j \in J$; in this case, we write $\varphi \sim \varphi'$. We take the first cohomology $H^1(J, N)$ to be the pointed set $Z^1(J, N)$ of crossed homomorphisms modulo this equivalence. The distinguished point corresponds to the equivalence class containing the map taking each element of J to the identity of N. Our interest in this set stems primarily from the well-known bijective correspondence [**8**, Exer. 1 in §I.5.1] between it and the N-conjugacy classes of complements to N in $N \rtimes J$. Specifically, for each $\varphi \in Z^1(J, N)$, the subgroup $F(\varphi) = \{\varphi(j)j\}_{j \in J}$ complements N in NJ and all such complements may be written in this way. Two crossed homomorphisms yield conjugate complements under F if and only if they are cohomologous, so F induces the desired correspondence.

For a subgroup $K \leqslant J$, we let $\varphi|_K$ denote the restriction of $\varphi \in Z^1(J, N)$ to K and $\mathrm{res}^J_K : H^1(J, N) \to H^1(K, N)$ be the map induced in cohomology. For $\varphi \in Z^1(K, N)$ and $j \in J$, define $\varphi^j(x) = \varphi(x^{j^{-1}})^j$. We call φ J-invariant if $\mathrm{res}^K_{K \cap K^j} \varphi \sim \mathrm{res}^{K^j}_{K \cap K^j} \varphi^j$ for all $j \in J$ and let $\mathrm{inv}_J\, H^1(K, N)$ denote the set of J-invariant elements in $H^1(K, N)$. For any $\varphi \in Z^1(J, N)$, we have $\varphi^j(x) = n^{-1}\varphi(x)n^{x^{-1}}$ where $n = \varphi(j^{-1})$ so that $\varphi^j \sim \varphi$. In particular, $\mathrm{res}^J_K\, H^1(J, N) \subseteq \mathrm{inv}_J\, H^1(K, N)$.

2. N is abelian

In this section, we restrict N to be abelian so that $H^1(J, N)$ takes the form of an abelian group. We first prove lemma 1.1 as stated in § 1.

Proof of lemma 1.1. Suppose we are given locally conjugate complements J and J' of a normal abelian subgroup N in some group G. As any element $g \in G$ may be uniquely written $g = jn$ for $j \in J$ and $n \in N$, for each prime p we have $J'_p = (J_p)^n$ for some $J_p \in \mathrm{Syl}_p(J)$, $J'_p \in \mathrm{Syl}_p(J')$, and $n \in N$. Let $\varphi' \in Z^1(J, N)$ denote the crossed homomorphism corresponding to J'. It suffices to show that $\varphi' \sim 1$, where $1 \in Z^1(J, N)$ denotes the map taking each element of J to the identity of N. Through the p-primary decomposition of $H^1(J, N)$, we have the isomorphism [**1**, §III.10]:

$$H^1(J, N) \cong \oplus_{p \in \mathcal{D}} \mathrm{inv}_J\, H^1(J_p, N) \tag{2.1}$$

where $\mathcal{D}$ is the set of prime divisors of $|J|$ and the J_p are those given above. For every $p \in \mathcal{D}$, we see that $\varphi'|_{J_p} \sim 1|_{J_p}$ as J_p and J'_p are N-conjugate complements of N in NJ_p. Thus, φ' maps to the identity in each direct summand on the right-hand side of (2.1) and we may conclude $\varphi' \sim 1$ so that J and J' are conjugate. □

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We can now use the lemma and Gaschütz's theorem to prove theorem 1.2.

Proof of theorem 1.2. Given J, N, and Ω as described in the hypotheses of the theorem, let $G = N \rtimes J$ denote the induced semidirect product and consider the stabiliser subgroup G_α for some fixed $\alpha \in \Omega$. As N acts transitively, any $g \in G$ may be written $g \cdot \alpha = n \cdot \alpha$ for some $n \in N$, so that $n^{-1}g \in G_\alpha$. Thus, $G = NG_\alpha$.

We claim G_α splits over $N \cap G_\alpha$. For any prime p, there exists by hypothesis some $n \in N$ and $P \in \mathrm{Syl}_p(J)$ such that $P^n \leqslant G_\alpha$. Let $L \in \mathrm{Syl}_p(N \cap G_\alpha)$. As $|G_\alpha| = |N \cap G_\alpha|\,[G : N]$, it follows that $S = LP^n \in \mathrm{Syl}_p(G_\alpha)$ so P^n complements $S \cap N = L$ in S. As the choice of prime p was arbitrary, we may apply Gaschütz's theorem to conclude that G_α splits over $N \cap G_\alpha$.

Let J' complement $N \cap G_\alpha$ in G_α. As $G/N \cong G_\alpha/(N \cap G_\alpha)$, it follows that J' also complements N in G. Lemma 1.1 then implies that $J' = J^g$ for some $g \in G$ so that J fixes $\omega = g \cdot \alpha$. □

Finally, we outline how corollary 1.3 follows from theorem 1.2.

Proof of corollary 1.2. Given a group G satisfying the hypotheses of the corollary, suppose J and J' each complement N in G. Then G acts on the cosets $\Omega = G/J'$ in such a way that we may apply theorem 1.2 to infer that J fixes gJ' for some $g \in G$. Therefore, J and J' are conjugate. As the choice of complements was arbitrary, we may conclude. □

3. N is nilpotent and $N \rtimes J$ is supersoluble

In this section, we suppose that N is nilpotent and $N \rtimes J$ is supersoluble. Consequently, N decomposes as the direct sum $N \cong \oplus_{p\in\mathcal{D}} N_p$ over its characteristic Sylow p-subgroups N_p where $\mathcal{D}$ denotes the set of prime divisors of $|N|$. Direct calculations show that the natural projections $N \to N_p$ induce an isomorphism of pointed sets

$$H^1(J, N) \cong \oplus_{p\in D} H^1(J, N_p). \tag{3.1}$$

To parse the components on the right-hand side of (3.1), we introduce the following:

PROPOSITION 3.1. *Suppose a group J acts on a p-group N via automorphisms, so that the induced semidirect product $N \rtimes J$ is supersoluble. Then $\mathrm{res}^J_{J_p} : H^1(J,\, N) \to \mathrm{inv}_J\, H^1(J_p,\, N)$ is an isomorphism for $J_p \in \mathrm{Syl}_p(J)$.*

Proof. We induct on the order of J. If J itself is a p-group, the conclusion is immediate. If p is not a divisor of $|J|$, the lemma follows from the Schur–Zassenhaus theorem. Otherwise, let $Q \lhd J$ be a Sylow q-subgroup where q is the largest prime divisor of $|J|$ [**6**, Exer. 3B.10] so that $J \cong Q \rtimes M$ for some Hall q'-subgroup $M \leqslant J$. Consider the inflation–restriction exact sequence [**8**, §I.5.8],

$$1 \to H^1(J/Q, N^Q) \to H^1(J, N) \xrightarrow{\mathrm{res}^J_Q} H^1(Q, N)^{J/Q} \tag{3.2}$$

where N^Q denotes the elements of N fixed by Q.

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If $q \neq p$, then $H^1(Q, N)$ is trivial so that $H^1(J, N) \cong H^1(M, N^Q)$. In the supersoluble group NQ, Q is a Sylow q-subgroup for the largest prime divisor of $|NQ|$, so that $Q \lhd NQ$ and $N^Q = N$. Consequently, $H^1(J, N) \cong H^1(M, N)$. We claim that res^J_M affords this isomorphism. It suffices to show that res^J_M is surjective. For any $\varphi \in Z^1(M, N)$, we may define $\tilde{\varphi} : J \to N$ by $\tilde{\varphi}(qm) = \varphi(m)$ for $q \in Q$ and $m \in M$. This map is well-defined as $J \cong Q \rtimes M$. For $q, q' \in Q$ and $m, m' \in M$, we have $\tilde{\varphi}(qmq'm') = \varphi(mm') = \varphi(m)\varphi(m')^{m^{-1}} = \tilde{\varphi}(qm)\tilde{\varphi}(q'm')^{(qm)^{-1}}$, where the last equality follows from the fact that elements of N commute with elements of Q. Thus, $\tilde{\varphi} \in Z^1(J, N)$. As $\tilde{\varphi}|_M = \varphi$, we conclude res^J_M is surjective.

Exchanging M for a conjugate if necessary, we may assume that $J_p \leqslant M$. As $\operatorname{res}^M_{J_p}$ is injective by induction, it follows that the composition $\operatorname{res}^J_{J_p} = \operatorname{res}^M_{J_p} \circ \operatorname{res}^J_M$ is also injective. On the other hand,

$$\operatorname{inv}_J H^1(J_p, N) \subseteq \operatorname{inv}_M H^1(J_p, N) = \operatorname{res}^M_{J_p} H^1(M, N) \subseteq \operatorname{res}^J_{J_p} H^1(J, N)$$

where the equality above follows from the inductive hypothesis, so that $\operatorname{res}^J_{J_p}$ is surjective.

Otherwise, $q = p$, so that $J_p = Q$ is a Sylow p-subgroup of J. In this case, $H^1(J/Q, N^Q)$ is trivial in (3.2) and so $\operatorname{res}^J_{J_p}$ is injective. As $H^1(Q, N)^{J/Q} = \operatorname{inv}_J H^1(Q, N)$, it remains to show that this map is surjective. For M-invariant $\varphi \in Z^1(J_p, N)$, define $\tilde{\varphi} : J \to N$ by $\tilde{\varphi}(hm) = \varphi(h)$ for $h \in J_p$ and $m \in M$. Then for any $h, h' \in J_p$ and $m, m' \in M$, we have $\tilde{\varphi}(hmh'm') = \varphi(h(h')^{m^{-1}}) = \varphi(h)\varphi((h')^{m^{-1}})^{h^{-1}} = \varphi(h)\varphi(h')^{m^{-1}h^{-1}} = \tilde{\varphi}(hm)\tilde{\varphi}(h'm')^{(hm)^{-1}}$ where the third equality follows from φ being M-invariant. As $J \cong J_p \rtimes M$, we conclude that $\tilde{\varphi} \in Z^1(J, N)$. Clearly, $\operatorname{res}^J_{J_p} \tilde{\varphi} \sim \varphi$ so that $\operatorname{res}^J_{J_p}$ is surjective. □

For each prime p, we may apply proposition 3.1 to the component for p in (3.1) and find that $H^1(J, N_p) \cong \operatorname{inv}_J H^1(J_p, N_p) \cong \operatorname{inv}_J H^1(J_p, N)$ for some $J_p \in \operatorname{Syl}_p(J)$. In particular, it follows that:

Proposition 3.2. *Given a group J acting on a nilpotent group N via automorphisms so that $N \rtimes J$ is supersoluble, the restriction maps $\operatorname{res}^J_{J_p}$ induce an isomorphism of pointed sets $H^1(J, N) \cong \oplus_{p \in \mathcal{D}} \operatorname{inv}_J H^1(J_p, N)$ where $\mathcal{D}$ denotes the set of prime divisors of $|J|$ and $J_p \in \operatorname{Syl}_p(J)$ for each $p \in \mathcal{D}$.*

We are now prepared to provide a proof of lemma 1.4.

Proof of lemma 1.4. In a supersoluble group G, suppose J and J' are locally conjugate complements of a normal nilpotent subgroup N. As in lemma 1.1, we have for each prime p that some $J_p \in \operatorname{Syl}_p(J)$ and $J'_p \in \operatorname{Syl}_p(J')$ are conjugate by an element of N. Let $\varphi' \in Z^1(J, N)$ denote the map corresponding to J'. As the isomorphism in proposition 3.2 is induced by restriction maps, it takes the identity $1 \in H^1(J, N)$ to $\oplus_{p \in \mathcal{D}} 1|_{J_p}$. Thus, as $\varphi'|_{J_p} \sim 1|_{J_p}$ for each $p \in \mathcal{D}$, we may apply proposition 3.2 to conclude $\varphi' \sim 1$ so that J and J' are conjugate. □

We now use lemma 1.4 to show:

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Proposition 3.3. *Let H be a subgroup of some supersoluble $G \cong N \rtimes J$ where N is nilpotent. If for each prime p, H contains a conjugate of some $S \in \mathrm{Syl}_p(J)$, then H contains a conjugate of J and so splits over $N \cap H$.*

Proof. The hypotheses imply that H supplements N in G. We induct on the order of G. If N is trivial or if H is a p-group, the conclusion follows immediately. If multiple primes divide $|N|$, then for some prime p, HN_p must be a strict subgroup of G for $N_p \in \mathrm{Syl}_p(N)$; otherwise H would contain a Sylow subgroup of G for each prime and we would have $H = G$. Let p be such a prime. Induction in G/N_p implies $J^g \leqslant HN_p$ for some $g \in G$. Switching to a conjugate of H if necessary, we may assume that g is trivial and apply the inductive hypothesis in HN_p to conclude $J^{g'} \leqslant H$ for some $g' \in G$. We now proceed under the assumption that N is a q-subgroup for some prime q.

Let $A \leqslant N$ be a minimal normal subgroup of G; as G is supersoluble, it will have prime order q. If $A \leqslant H$, then in G/A, induction implies that $J^g A \leqslant HA = H$ for some $g \in G$ so that $J^g \leqslant H$.

Otherwise, $A \cap H$ is trivial. Without loss, $J_q \leqslant H$ for some $J_q \in \mathrm{Syl}_q(J)$. In G/A, induction implies that a conjugate of JA/A is contained in HA/A. Let $\overline{K}$ denote this conjugate. Switching to a different conjugate if necessary, we may assume that $J_qA/A \leqslant \overline{K}$. Let $\varphi : h \mapsto hA/A$ denote the isomorphism from H to HA/A and consider $K = \varphi^{-1}(\overline{K})$. It follows that $J_q \leqslant K$ and $|K| = |J|$ so that $K \leqslant H$ complements N in G. As N is a q-group, a Sylow p-subgroup of J will be conjugate to a Sylow p-subgroup of K for primes $p \neq q$. Lemma 1.4 then implies that J and $K \leqslant H$ are conjugate in G. □

We now prove theorem 1.5.

Proof of theorem 1.5. Given J, N, and Ω as described in the hypotheses of the theorem, let $G = N \rtimes J$ denote the induced semidirect product and consider G_α for some $\alpha \in \Omega$. As N acts transitively, $G = NG_\alpha$. For each prime p, the hypotheses of the theorem imply $(J_p)^{n_p} \leqslant G_\alpha$ for some $J_p \in \mathrm{Syl}_p(J)$ and $n_p \in N$, so that proposition 3.3 implies G_α contains a conjugate of J, say J^g for $g \in G$. It follows that J fixes $\omega = g \cdot a$. □

This in turn implies:

Corollary 3.4. *Let G be a supersoluble split extension over a nilpotent subgroup N. If for each prime p there is a Sylow p-subgroup S of G such that any two complements of $S \cap N$ in S are conjugate, then any two complements of N in G are conjugate.*

Proof. Suppose arbitrary J and J' complement N in G. Then G acts on the cosets $\Omega = G/J'$ in such a way that we may apply theorem 1.5 to infer that J fixes gJ' for some $g \in G$. Consequently, J and J' are conjugate, and we may conclude. □

4. Concluding remarks

In their paper, Losey and Stonehewer exhibited a soluble group $G \cong N \rtimes J$ with N nilpotent and J supersoluble and a second complement J' to N in G such that

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J and J' are locally conjugate but not conjugate [7]. Thus, lemma 1.4 cannot be extended to supersoluble complements of a normal nilpotent subgroup in a soluble group.

Acknowledgements

The author thanks Elizabeth Crites, the editor Alex Bartel, and an anonymous reviewer for thoughtful and detailed feedback on the manuscript.

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