Canad. Math. Bull. Vol. **23** (3) 1980

ON THE BOUNDEDNESS AND RANGE OF THE EXTENDED HANKEL TRANSFORMATION

BY
P. G. ROONEY

1. **Introduction.** For $1 \le p < \infty$, $\mu \in \mathbb{R}$, let $\mathcal{L}_{\mu,p}$ denote the collection of functions f, measurable on $(0, \infty)$ and such that

$$\|f\|_{\mu,p} = \left\{ \int_0^\infty |x^\mu f(x)|^p \, dx/x \right\}^{1/p} < \infty.$$

Let C_0 be the collection of functions continuous and compactly supported on $(0, \infty)$; it is known that C_0 is dense in $\mathcal{L}_{\mu,p}$—see [2; Lemma 2.2]. If X and Y are Banach spaces, denote by $[X, Y]$ the collection of bounded linear operators from X into Y, abbreviating $[X, X]$ to $[X]$.

In [2] and [3] we studied the Hankel transformation on $\mathcal{L}_{\mu,p}$. Here if $\nu > -1$, $f \in C_0$, the Hankel transformation of order ν, H_ν is defined by

$$(H_\nu f)(x) = \int_0^\infty (xt)^{1/2} J_\nu(xt) f(t) \, dt,$$

and by continuous extension on $\mathcal{L}_{\mu,p}$ when justified. In [2], as an application of a Mellin multiplier technique, we showed that if $1 < p < \infty$, $\gamma(p) \le \mu < \nu + \frac{3}{2}$, where

$$\gamma(p) = \max(p^{-1}, p'^{-1}),$$

then for all $q \ge p$ such that $q'^{-1} \le \mu$, $H_\nu \in [\mathcal{L}_{\mu,p}, \mathcal{L}_{1-\mu,q}]$, while in [3] we gave a complete description of $H_\nu(\mathcal{L}_{\mu,p})$.

The Hankel transformation H_ν has been extended to $\nu \in \mathbb{R}$, $\nu \ne -1, -3, \ldots$ follows. For $m \ge 0$, let

$$J_{\nu,m}(x) = \sum_{k=m}^{\infty} \frac{(-1)^k (\frac{1}{2}x)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)} = J_\nu(x) - \sum_{k=1}^{m-1} \frac{(-1)^k (\frac{1}{2}x)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)};$$

$J_{\nu,m}$ is sometimes called a "cut" Bessel function. If $\nu \in \mathbb{R}$, $\nu \ne -1, -3, \ldots$, there is a least integer $m \ge 0$ such that $\nu + 2m > -1$, and then for $f \in C_0$, we define

$$(H_\nu f)(x) = \int_0^\infty (xt)^{1/2} J_{\nu,m}(xt) f(t) \, dt.$$

This extended Hankel transformation has been considerably studied; see [1], for example.

Received by the editors August 14, 1978.

https://doi.org/10.4153/CMB-1980-045-8 Published online by Cambridge University Press

Our object in this paper is to obtain the boundedness properties of the extended Hankel transformation on the $\mathscr{L}_{\mu,p}$ spaces, and to characterize its range on these spaces. Our technique will be that of [2], as used in [2; §7] and in [3]. The boundedness is shown in section 2 below, while the range is characterized in section 3; section 4 contains some concluding remarks.

The reader should note that $\mathscr{L}_{\mu,p}$ is slightly different from the space $L_{\mu,p}$ defined in [2], and make the necessary adjustments in the statements of the theorems of [2].

2. **Boundedness.** The following theorem gives the boundedness properties of the extended Hankel transformation on the $\mathscr{L}_{\mu,p}$ spaces, $p>1$.

THEOREM 1. *Suppose* $1<p<\infty$, $\gamma(p)\le\mu<\nu+2m+\frac{3}{2}$. *Then for all* $q\ge p$ *so that* $q'^{-1}\le\mu$, $H_\nu\in[\mathscr{L}_{\mu,p},\mathscr{L}_{1-\mu,q}]$.

Proof. We may suppose $\nu<-1$; for if $\nu>-1$, $m=0$ and the result is known—see [2; § 7]. Now if $\nu<-1$, then $-1<\nu+2m<1$; for, as m is the least non-negative integer such that $\nu+2m>-1$, and if $\nu+2m>1$, then $\nu+2(m-1)>-1$, a contradiction, while if $\nu+2m=1$, then the condition $\nu\ne-1,-3,\ldots,$ is violated.

We use [2; Theorem 3(a)] with $S_1=H_\nu$, $S_2=H_\eta$ where $\eta=|\nu+2m|$. Clearly $\eta>-1$. From [1; §§ 2 and 3], S_1 and $S_2\in[\mathscr{L}_{1/2},2]$ and

$$\omega_1(t)=2^{it}\frac{\Gamma(\frac{1}{2}(\nu+1+it))}{\Gamma(\frac{1}{2}(\nu+1-it))},\qquad \omega_2(t)=2^{it}\frac{\Gamma(\frac{1}{2}(\eta+1+it))}{\Gamma(\frac{1}{2}(\eta+1-it))},$$

and thus

$$\frac{\omega_1(t)}{\omega_2(t)}=\frac{\Gamma(\frac{1}{2}(\nu+1+it))\Gamma(\frac{1}{2}(\eta+1-it))}{\Gamma(\frac{1}{2}(\eta+1+it))\Gamma(\frac{1}{2}(\nu+1-it))}.$$

Let

$$m(s)=\frac{\Gamma(\frac{1}{2}(\nu+\frac{1}{2}+s))\Gamma(\frac{1}{2}(\eta+\frac{3}{2}-s))}{\Gamma(\frac{1}{2}(\eta+\frac{1}{2}+s))\Gamma(\frac{1}{2}(\nu+\frac{3}{2}-s))}.$$

Then m is holomorphic in the strip $S=\{s\mid\alpha(m)<\operatorname{Re}s<\beta(m)\}$ where $\alpha(m)=-(2m+\nu)-\frac{1}{2}$ and $\beta(m)=-(2m+\nu)+\frac{3}{2}$, since $\eta+\frac{3}{2}\ge-(2m+\nu)+\frac{3}{2}$. Also since $|\Gamma(x+iy)|\sim\sqrt{2\pi}\,|y|^{x-1/2}e^{-\pi|y|/2}$ as $|y|\to\infty$, uniformly in x for x in any bounded interval, then $|m(\sigma+it)|\sim1$ as $|t|\to\infty$, uniformly in σ for $\sigma_1\le\sigma\le\sigma_2$, where $\alpha(m)<\sigma_1\le\sigma_2<\beta(m)$, and hence on the closed strip $\sigma_1\le\operatorname{Re}s\le\sigma_2$, $m(s)$ is bounded. Further since from [2; p. 1100],

$$\Gamma'(z)=\Gamma(z)(\log z-(2z)^{-1}+O(|z|^{-2}))$$

as $z\to\infty$ in $|\arg z|\le\pi-\delta$, and m is bounded

$$|m'(\sigma+it)|=O(|t|^{-2})\quad\text{as}\quad|t|\infty.$$

https://doi.org/10.4153/CMB-1980-045-8 Published online by Cambridge University Press

Thus $m\in\mathscr{A}$—see [2; Definition 3.1]. Also since $-1<\nu+2m<1$, $\alpha(m)<\frac{1}{2}<\beta(m)$.

Now by [2; § 7], if $1<p<\infty$, $\gamma(p)\le\mu<\eta+\frac{3}{2}$, then for all $q\ge p$ with $q'^{-1}\le\mu$, $H_\eta\in[\mathscr{L}_{\mu,p},\mathscr{L}_{1-\mu,q}]$. Hence, by [2; Theorem 3(a)], if the above conditions on p, q and μ are satisfied, and in addition $\nu+2m-\frac{1}{2}<\mu<+2m+\frac{3}{2}$, $H_\nu\in[\mathscr{L}_{\mu,p},\mathscr{L}_{1-\mu,q}]$. But $\nu+2m-\frac{1}{2}<\frac{1}{2}\le\gamma(p)$, and since $\eta\ge\nu+2m$, $\nu+2m+\frac{3}{2}\le\eta+\frac{3}{2}$. Thus if $1<p<\infty$, $\gamma(p)\le\mu<\nu+2m+\frac{3}{2}$, then for all $q\ge p$ such that $q'^{-1}\le\mu$, $H_\nu\in[\mathscr{L}_{\mu,p},\mathscr{L}_{1-\mu,q}]$.

3. **The range of** H_ν. We could have said something about the range of H_ν already, for [2; Theorem 3(a)] also says that under the conditions of Theorem 1, $H_\nu(\mathscr{L}_{\mu,p})\subseteq H_\eta(\mathscr{L}_{\mu,p})$, and the range of H_η on $\mathscr{L}_{\mu,p}$ was characterized by us recently—see [3]. However, except in one isolated case, we can be much more precise, as the following theorem shows.

THEOREM 2. *Suppose* $1<p<\infty$, $\gamma(p)\le\mu<\nu+2m+\frac{3}{2}$, $\eta=|\nu+2m|$. *Then except when* $\mu=-(\nu+2m)+\frac{3}{2}$, $\nu<-1$,

$$H_\nu(\mathscr{L}_{\mu,p})=H_\eta(\mathscr{L}_{\mu,p}).$$

Proof. For $\nu>-1$, the result is either obvious ($\nu\ge0$) or contained in [3, Theorem 1]. Hence we may assume $\nu<-1$. The proof for $\nu<-1$ is a continuation of that of Theorem 1, using [2; Theorem 3(c)]. For this we need to study

$$1/m(s)=\frac{\Gamma(\frac{1}{2}(\eta+\frac{1}{2}+s))\Gamma(\frac{1}{2}(\nu+\frac{3}{2}-s))}{\Gamma(\frac{1}{2}(\nu+\frac{1}{2}+s))\Gamma(\frac{1}{2}(\eta+\frac{3}{2}-s))}.$$

Now $\Gamma(\frac{1}{2}(\nu+\frac{3}{2}-s))$ is holomorphic in each of the strips $S_r=\{\nu+2r-\frac{1}{2}<\text{Re}\, s<\nu+2r+\frac{3}{2}\}$, $r=1,2,\ldots,$ and in the half-plane $S_0=\{\text{Re}\, s<\nu+\frac{3}{2}\}$. The intersection of these strips with the strip S depends on whether $\nu+2m=0$, $\nu+2m>0$, or $\nu+2m<0$, and thus we must divide our proof into three cases.

Case (i). $\nu+2m=0$. In this case, $\eta=0$, and $\Gamma(\frac{1}{2}(\eta+\frac{1}{2}+s))$ is holomorphic in $\text{Re}\, s>-\frac{1}{2}$. Also $S_m=S$, $S_r\cap S=\emptyset$, $r\ne m$. Hence we may take $\alpha(m^{-1})=\alpha(m)=-\frac{1}{2}$, $\beta(m^{-1})=\beta(m)=\frac{3}{2}$, and by the same argument as given for m in the proof of Theorem 1, or since m^{-1} is the same function as m with ν and η interchanged, $m^{-1}\in\mathscr{A}$. Thus by [2; Theorem 3(c)], if $1<p<\infty$, $\gamma(p)\le\mu<\frac{3}{2}$, $-\frac{1}{2}<\mu<\frac{3}{2}$, $H_\nu(\mathscr{L}_{\mu,p})=H_0(\mathscr{L}_{\mu,p})=H_\eta(\mathscr{L}_{\mu,p})$. The condition $-\frac{1}{2}<\mu<\frac{3}{2}$ is clearly superfluous since $\gamma(p)\ge\frac{1}{2}$, and thus the result of our Theorem is true in this case.

Case (ii). $\nu+2m>0$. In this case $\eta=\nu+2m$, and $\Gamma(\frac{1}{2}(\eta+\frac{1}{2}+s))$ is holomorphic in $\text{Re}\, s>-(\nu+2m)-\frac{1}{2}=\alpha(m)$. Also, since $\alpha(m)=-(\nu+2m)-\frac{1}{2}<\nu+2m-\frac{1}{2}<-(\nu+2m)+\frac{3}{2}=\beta(m)$, and since the right hand boundary of S_{m-1}

https://doi.org/10.4153/CMB-1980-045-8 Published online by Cambridge University Press

and the left hand boundary of S_m are the lines Re $s=\nu+2m-\frac{1}{2}$, it follows that $S_r\cap S=\emptyset$ unless $r=m-1$ or $r=m$. Thus there are two possible choices for $\alpha(m^{-1})$ and $\beta(m^{-1})$ namely $\alpha_1(m^{-1})=-(\nu+2m)-\frac{1}{2}$, $\beta_1(m^{-1})=\nu+2m-\frac{1}{2}$, and $\alpha_2(m^{-1})=\nu+2m-\frac{1}{2}$, $\beta_2(m^{-1})=\nu+2m+\frac{3}{2}$. Relative to each of the intervals $\alpha_j(m^{-1})<\text{Re } s<\beta_j(m^{-1})$, $j=1,2$, $1/m\in\mathcal{A}$ by the same argument as in Case (i). Hence by [2; Theorem 3(c)], if $1<p<\infty$, $\gamma(p)\le\mu<\nu+2m+\frac{3}{2}$, and either $\max(\nu+2m-\frac{1}{2},\ -(\nu+2m)-\frac{1}{2})<\mu<\min(\nu+2m+\frac{3}{2},\ -(\nu+2m)+\frac{3}{2})$ or $\max(\nu+2m-\frac{1}{2},\ -(\nu+2m+\frac{3}{2})<\mu<\nu+2m+\frac{3}{2}$, $H_\nu(\mathcal{L}_{\mu,p})=H_\eta(\mathcal{L}_{\mu,p})$. But since $\nu+2m>0$, these last two conditions on μ come down to $\nu+2m-\frac{1}{2}<\mu<-(\nu+2m)+\frac{3}{2}$ and $-(\nu+2m)+\frac{3}{2}<\mu<\nu+2m+\frac{3}{2}$, and thus since $\nu+2m-\frac{1}{2}<\frac{1}{2}\le\gamma(p)$, if $1<p<\infty$, $\gamma(p)\le\mu<\nu+2m+\frac{3}{2}$, then except when $\mu=-(\nu+2m)+\frac{3}{2}$, $H_\nu(\mathcal{L}_{\mu,p})=H_\eta(\mathcal{L}_{\mu,p})$, proving the theorem in this case.

Case (iii). $\nu+2m<0$. In this case $\eta=-(\nu+2m)$, and $\Gamma(\frac{1}{2}(\eta+\frac{1}{2}+s))$ is holomorphic in Re $s>\nu+2m-\frac{1}{2}$. Also since $\alpha(m)=-(\nu+2m)-\frac{1}{2}<\nu+2m+\frac{3}{2}<-(\nu+2m)+\frac{3}{2}=\beta(m)$, and since the right hand boundary of S_m and the left hand boundary of S_{m+1} is the line Re $s=\nu+2m+\frac{3}{2}$, it follows that $S_r\cap S=\emptyset$ unless $r=m$ or $r=m+1$. Thus again there are two possible values of $\alpha(m^{-1})$ and $\beta(m^{-1})$ namely $\alpha_1(m^{-1})=\nu+2m-\frac{1}{2}$, $\beta_1(m^{-1})=\nu+2m+\frac{3}{2}$, and $\alpha_2(m^{-1})=\nu+2m+\frac{3}{2}$, $\beta_2(m^{-1})=\nu+2m+\frac{7}{2}$. Relative to each of the intervals $\alpha_j<\text{Re } s<\beta_j$, $j=1,2$, $1/m\in\mathcal{A}$ by the same argument as in Case (i). Hence by [2; Theorem 3(c)], if $1<p<\infty$, $\gamma(p)\le\mu<\nu+2m+\frac{3}{2}$, and either

$$\max(\nu+2m-\tfrac{1}{2},-(\nu+2m)-\tfrac{5}{2})<\mu<\min(\nu+2m+\tfrac{3}{2},-(\nu+2m)-\tfrac{1}{2})$$

or

$$\max(\nu+2m-\tfrac{1}{2},-(\nu+2m)-\tfrac{1}{2})<\mu<\min(\nu+2m+\tfrac{3}{2},-(\nu+2m)+\tfrac{3}{2}),$$

$H_\nu(\mathcal{L}_{\mu,p})=H_\eta(\mathcal{L}_{\mu,p})$. But $\min(\nu+2m+\frac{3}{2},-(\nu+2m)-\frac{1}{2})=-(\nu+2m)-\frac{1}{2}<\frac{1}{2}\le\gamma(p)$, $\max(\nu+2m-\frac{1}{2},\ -(\nu+2m)-\frac{1}{2})=-(\nu+2m)-\frac{1}{2}\le\gamma(p)$, and $\min(\nu+2m+\frac{3}{2},-(\nu+2m)+\frac{3}{2})=\nu+2m+\frac{3}{2}$, so that if $1<p<\infty$, $\gamma(p)\le\mu<\nu+2m+\frac{3}{2}$, $H_\nu(\mathcal{L}_{\mu,p})=H_\eta(\mathcal{L}_{\mu,p})$, and Case (iii) is proved.

COROLLARY. $1<p<\infty$, $\gamma(p)\le\mu<\nu+2m+\frac{3}{2}$, *then except in the case* $\nu<-1$, $\mu=-(\nu+2m)+\frac{3}{2}$

$$H_\nu(\mathcal{L}_{\mu,p})=(I_{\mu-\gamma}F_c)(\mathcal{L}_{\gamma,p}),$$

where for $f\in\mathcal{L}_{\mu,p}$ *with* $\mu<1$, *and* $\alpha\ge0$

$$(I_\alpha f)(x)=\frac{2x^{-\alpha+1}}{\Gamma(\alpha)}\int_0^x(x^2-t^2)^{\alpha-1}f(t)\,dt,\qquad \alpha>0.$$
$$=f(x),\qquad \alpha=0,$$

F_c *is the Fourier cosine transformation, that is,* $F_c=H_{-1/2}$, *and* $\gamma=\gamma(p)$.

Proof. This follows from Theorem 2, and [3; Theorem 2].

4. **Conclusion.** The reader should note that the condition in both theorems that $\gamma(p) < \nu + 2m + \frac{3}{2}$ imposes limitations on the values of p allowed if $\nu + 2m < -\frac{1}{2}$. For example, if $\nu + 2m = -\frac{3}{4}$ the condition becomes $\frac{4}{3} < p < 4$.

The exceptional case, $\nu < -1$, $\mu = -(\nu + 2m) + \frac{3}{2}$, which necessarily implies $\nu + 2m > 0$, does not seem amenable to our techniques here, though certainly in this case $H_\nu(\mathscr{L}_{\mu,p}) \subseteq H_\eta(\mathscr{L}_{\mu,p})$, as mentioned earlier. Since this case corresponds to a pole of $1/m$, it seems most likely that in this case $H_\nu(\mathscr{L}_{\mu,p})$ is some proper subset of $H_\eta(\mathscr{L}_{\mu,p})$.

References

1. H. Kober, *Hankelsche Transformationen*, Quart. J. Math. **8** (Ser. 2, 1937), 186–199.

2. P. G. Rooney, *A technique for studying the boundedness and extendability of certain types of operators*, Can. J. Math. **25** (1973), 1090–1102.

3. ——, On the range of the *Hankel transformation*, Bull. Lond. Math. Soc. 11 (1979), 45–48.

University of Toronto.