7

Anti-symmetric calculus

In almost every respect there exists a strong analogy between symmetric and anti-symmetric tensors, between bosons and fermions. It is often convenient to stress this analogy in terminology and notation.

Symmetric tensors over a vector space can be treated as polynomial functions on its dual. Such functions can be multiplied, differentiated and integrated, and we can change their variables.

There exists a similar language in the case of anti-symmetric tensors. It has been developed mostly by Berezin, hence it is sometimes called the *Berezin calculus*. It is often used by physicists, because it allows them to treat bosons and fermions within the same formalism.

Anti-symmetric calculus has a great appeal – it often allows us to express the analogy between the bosonic and fermionic cases in an elegant way. On the other hand, readers who see it for the first time can find it quite confusing and strange. Therefore, we devote this chapter to a presentation of elements of anti-symmetric calculus.

Note that the main goal of this chapter is to present a certain intriguing notation. Essentially no new concepts of independent importance are introduced here. Therefore, a reader in a hurry can probably skip this chapter on the first reading.

This chapter can be viewed as a continuation of Chap. 3, and especially of Sect. 3.6. In particular, we will use the anti-symmetric multiplication, differentiation and the Hodge star introduced already in Chap. 3.

7.1 Basic anti-symmetric calculus

Let $\mathcal{Y}$ be a vector space over $\mathbb{K}$ of dimension m. Let v denote the generic variable in $\mathcal{Y}^{\#}$ and y the generic variable in $\mathcal{Y}$. We remind the reader that $\Gamma_{\mathrm{a}}^{n}(\mathcal{Y})$ denotes the n-th anti-symmetric tensor power of $\mathcal{Y}$.

7.1.1 Functional notation

Recall from Subsect. 3.5.1 that $\Psi \in \Gamma_{\mathrm{a}}^{n}(\mathcal{Y})$ can be considered as a multi-linear anti-symmetric form

$$\mathcal{Y}^{\#} \times \cdots \times \mathcal{Y}^{\#} \ni (v_1, \dots, v_n) \mapsto \Psi(v_1, \dots, v_n) = \langle \Psi | v_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} v_n \rangle. \quad (7.1)$$

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When we want to stress the meaning of an anti-symmetric tensor as a multi-linear form, we often write $\mathrm{Pol}_{\mathrm{a}}^n(\mathcal{Y}^\#)$ instead of $\Gamma_{\mathrm{a}}^n(\mathcal{Y})$.

Definition 7.1 *It is convenient to write* $\Psi(v)$ *for (7.1), where* v *stands for the* generic name of the variable in $\mathcal{Y}^\#$ *and not for an individual element of* $\mathcal{Y}^\#$. *We will call it the* functional notation.

(We mentioned this notation already in Subsect. 3.5.1).

Sometimes we will consider a vector space with a different name, and then we will change the generic name of its dual variable used in the functional notation. For instance, $\Phi \in \mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}_i)$, resp. $\Psi \in \mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}_1 \oplus \mathcal{Y}_2)$, in the functional notation will be written as $\Phi(v_i)$, resp. $\Psi(v_1, v_2)$.

Remark 7.2 *Note that the same symbols have a different meaning in (7.1) and in the functional notation. In (7.1),* v_i *stands for an "individual element of* $\mathcal{Y}^\#$*". In the functional notation,* v_i *is the "name of the generic variable".*

7.1.2 Change of variables in anti-symmetric polynomials

Let $\mathcal{Y}_1, \mathcal{Y}_2$ be two finite-dimensional vector spaces. As mentioned above, v_1, v_2 will denote the generic variables in $\mathcal{Y}_1^\#$ and $\mathcal{Y}_2^\#$.

Consider $r \in L(\mathcal{Y}_1, \mathcal{Y}_2)$ and $\Psi \in \mathrm{Pol}_{\mathrm{a}}^n(\mathcal{Y}_1^\#)$. Then $\Gamma(r)\Psi$, understood as a multi-linear functional, acts as

$$\mathcal{Y}^\# \times \cdots \times \mathcal{Y}^\# \ni (v_1, \dots, v_n) \mapsto \Gamma(r)\Psi(v_1, \dots, v_n) = \Psi(r^\# v_1, \dots, r^\# v_n). \quad (7.2)$$

Definition 7.3 *The functional notation for* $\Gamma(r)\Psi$ *is* $(\Gamma(r)\Psi)(v_2)$ *or, as suggested by (7.2),* $\Psi(r^\# v_2)$.

For example, let

$$\begin{aligned} j : \mathcal{Y} &\to \mathcal{Y} \oplus \mathcal{Y} \\ y &\mapsto y \oplus y, \end{aligned} \quad (7.3)$$

so that $j^\#(v_1, v_2) = v_1 + v_2$. Then the two possible functional notations for $\Gamma(j)\Psi$ are $(\Gamma(j)\Psi)(v_1, v_2)$ or $\Psi(v_1 + v_2)$.

7.1.3 Multiplication and differentiation operators

Definition 7.4 *If* $\Psi_1, \Psi_2 \in \Gamma_{\mathrm{a}}(\mathcal{Y})$, *then* $\Psi_1 \otimes_{\mathrm{a}} \Psi_2$ *will be denoted simply by* $\Psi_1 \cdot \Psi_2$, *if we consider* Ψ_1, Ψ_2 *as elements of* $\mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}^\#)$. *The functional notation will be either* $\Psi_1 \cdot \Psi_2(v)$ *or* $\Psi_1(v)\Psi_2(v)$.

Recall that in Subsect. 3.5.2 we defined *multiplication* and *differentiation operators*. For $\Psi \in \mathrm{Pol}_{\mathrm{a}}^n(\mathcal{Y}^\#)$ they are given by

$$\begin{aligned} y(v)\Psi &:= y \otimes_{\mathrm{a}} \Psi, && y \in \mathcal{Y}, \\ w(\nabla_v)\Psi &:= n\langle w| \otimes 1\!\!1_{\mathcal{Y}}^{\otimes(n-1)}\Psi, && w \in \mathcal{Y}^\#. \end{aligned}$$

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Therefore, v can be given the meaning of a $\mathcal{Y}^{\#}$-vector of anti-commuting operators on $\mathrm{Pol_a}(\mathcal{Y}^{\#})$. Similarly, ∇_v is a $\mathcal{Y}$-vector of anti-commuting operators on $\mathrm{Pol_a}(\mathcal{Y}^{\#})$.

Let $(e_1, \dots, e_m)$ be a basis in $\mathcal{Y}$ and $(e^1, \dots, e^m)$ be the corresponding dual basis in $\mathcal{Y}^{\#}$. The following operator on $\mathrm{Pol_a}(\mathcal{Y}^{\#} \oplus \mathcal{Y}^{\#})$ is clearly independent of the choice of the basis:

$$v_1 \cdot \nabla_{v_2} := \sum_{i=1}^{m} e_i(v_1) e^i(\nabla_{v_2}).$$

As an exercise in anti-symmetric calculus, it is instructive to check the following analog of *Taylor's formula*:

Proposition 7.5 *Let* $\Psi \in \mathrm{Pol_a}(\mathcal{Y}^{\#})$. *Then*

$$\Psi(v_1 + v_2) = \mathrm{e}^{v_1 \cdot \nabla_{v_2}} \Psi(v_2).$$

Note that $(v_1 \cdot \nabla_{v_2})^p = 0$ for $p > \dim \mathcal{Y}$, so the exponential is well defined.

Proof of Prop. 7.5. Let

$$d = \begin{bmatrix} 0 & \mathbb{1} \\ 0 & 0 \end{bmatrix} \in L(\mathcal{Y} \oplus \mathcal{Y}),$$

$$j_2 = \begin{bmatrix} 0 \\ \mathbb{1} \end{bmatrix}, \quad j = \begin{bmatrix} \mathbb{1} \\ \mathbb{1} \end{bmatrix} \in L(\mathcal{Y}, \mathcal{Y} \oplus \mathcal{Y}).$$

Then $\mathrm{e}^d j_2 = j$. This implies that $\Gamma(j) = \mathrm{e}^{\mathrm{d}\Gamma(d)} \Gamma(j_2)$. If we fix a basis $(e_1, \dots, e_m)$ of $\mathcal{Y}$, then $d = \sum\limits_{i=1}^{m} |e_i \oplus 0\rangle\langle 0 \oplus e^i|$. Hence,

$$\begin{aligned} \mathrm{d}\Gamma(d) &= \sum_{i=1}^{m} a^*(e_i \oplus 0) a(0 \oplus e^i) \\ &= \sum_{i=1}^{m} e_i(v_1) e^i(\nabla_{v_2}) \ = \ v_1 \cdot \nabla_{v_2}, \end{aligned}$$

where we have used the functional notation for creation and annihilation operators:

$$a^*(e_i \oplus 0) = e_i(v_1), \quad a(0 \oplus e^i) = e^i(\nabla_{v_2}).$$

But

$$\Gamma(j)\Psi(v_1, v_2) = \Psi(v_1 + v_2), \quad \Gamma(j_2)\Psi(v_1, v_2) = \Psi(v_2). \qquad \square$$

7.1.4 Berezin integrals

Recall that in Subsects. 3.5.2 and 3.5.3 we defined the left and right differentiation. Even though it sounds a little strange, the right differentiation will be renamed as integration.

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Let us be more precise. Let $\mathcal{Y}_1$ be a subspace of $\mathcal{Y}$ of dimension m_1. Its generic variable will be denoted v_1. Fix a volume form on $\mathcal{Y}_1$, that is, let $\Xi_1 \in \mathrm{Pol}_{\mathrm{a}}^{m_1}(\mathcal{Y}_1)$ be a non-zero form.

Definition 7.6 *The* partial right Berezin integral over $\mathcal{Y}_1$ of $\Psi \in \mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}^{\#})$ *is defined as*

$$\int \Psi(v)\mathrm{d}v_1 := \Xi_1(\overleftarrow{\nabla}_v)\Psi(v). \tag{7.4}$$

Note that (7.4) depends only on $(\mathcal{Y}/\mathcal{Y}_1)^{\#} \simeq \mathcal{Y}_1^{\mathrm{an}}$, where the superscript an stands for the annihilator (see Def. 1.11). Thus the Berezin integral produces an element of $\mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}_1^{\mathrm{an}})$.

In particular, if we take a volume form Ξ on $\mathcal{Y}$, i.e. a non-zero element of $\mathrm{Pol}_{\mathrm{a}}^{m}(\mathcal{Y})$, then the *right Berezin integral over* $\mathcal{Y}$

$$\int \Psi(v)\mathrm{d}v = \langle \Xi | \Psi \rangle \tag{7.5}$$

yields a number.

Let $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$. The generic variable on $\mathcal{Y}^{\#} = \mathcal{Y}_1^{\#} \oplus \mathcal{Y}_2^{\#}$ is denoted $v = (v_1, v_2)$. Fix volume forms $\Xi_i \in \mathrm{Pol}_{\mathrm{a}}^{m_i}(\mathcal{Y}_i)$. Equip $\mathcal{Y}^{\#}$ with the volume form $\Xi = \Xi_2 \wedge \Xi_1$. The corresponding Berezin integrals are denoted $\int \cdot\, \mathrm{d}v_i$, and $\int \cdot\, \mathrm{d}v$. Then we have the following version of the Fubini theorem:

$$\int \Psi(v)\mathrm{d}v = \int \left(\int \Psi(v_1, v_2)\mathrm{d}v_1 \right) \mathrm{d}v_2. \tag{7.6}$$

Thus, we can omit the parentheses and denote (7.6) by $\int\int \Psi(v_1, v_2)\mathrm{d}v_1\mathrm{d}v_2$.

Definition 7.7 *Apart from the right Berezin integral one considers the* partial left Berezin integral over $\mathcal{Y}_1$. *For $\Psi \in \mathrm{Pol}_{\mathrm{a}}^{n}(\mathcal{Y}^{\#})$, the left and right integrals are related to one another by*

$$\int \mathrm{d}v_1 \Psi(v) = (-1)^{m_1 n} \int \Psi(v)\mathrm{d}v_1.$$

In particular, we have the *left Berezin integral over* $\mathcal{Y}$:

$$\int \mathrm{d}v \Psi(v) = (-1)^{m} \int \Psi(v)\mathrm{d}v.$$

The following identities are easy to check for $\Psi \in \mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}^{\#})$:

$$\begin{aligned} \int \Phi(\nabla_v)\Psi(v)\mathrm{d}v &= 0, \qquad \Phi \in \mathrm{Pol}_{\mathrm{a}}^{\geq 1}(\mathcal{Y}); \\ \int \Psi(v+w)\mathrm{d}v &= \int \Psi(v)\mathrm{d}v, \qquad w \in \mathcal{Y}^{\#}; \\ \int \Psi(mv)\mathrm{d}v &= (\det m) \int \Psi(v)\mathrm{d}v, \quad m \in L(\mathcal{Y}^{\#}). \end{aligned} \tag{7.7}$$

Remark 7.8 *The identities of (7.7) are essentially the same as their analogs in the case of the usual integral described in (3.50) except for one important*

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difference: the determinant in the formula for the change of variables has the opposite power.

This is related to another difference between the Berezin and the usual integral. In the Berezin integral, such as (7.5), the natural meaning of the symbol dv *is a fixed volume form on* $\mathcal{Y}$. *In the usual integral, in the analogous situation, its meaning would be a volume form (or actually the corresponding density) on* $\mathcal{Y}^{\#}$.

Remark 7.9 *In the definition of the Berezin integral it does not matter whether the space* $\mathcal{Y}$ *is real or complex. However, if we want to have a closer analogy with the usual integral, we should assume that it is real. In this case, we can allow* $\Psi \in \mathbb{C}\mathrm{Pol_a}(\mathcal{Y}^{\#})$ *in (7.4), so that we can integrate complex polynomials.*

7.1.5 Berezin calculus in coordinates

So far, our presentation of anti-symmetric calculus has been coordinate-free. In most of the literature, it is introduced in a different way. One assumes from the very beginning that coordinates have been chosen and all definitions are coordinate-dependent. This approach has its advantages; in particular, it is a convenient way to check various identities. In this subsection we describe the anti-symmetric calculus in coordinates.

Definition 7.10 $v_1, \ldots, v_m$ *denote symbols satisfying the relations*

$$v_i v_j = -v_j v_i. \tag{7.8}$$

They are called Grassmann *or* anti-commuting variables. *If* $I = \{i_1, \ldots, i_p\}$ *with* $1 \le i_1 < \cdots < i_p \le m$, *we set* $\prod\limits_{i \in I} v_i := v_{i_1} \cdots v_{i_p}$.

The space of expressions

$$\sum_{I \subset \{1,\ldots,m\}} \alpha_I \prod_{i \in I} v_i, \quad \alpha_I \in \mathbb{K}$$

is an algebra naturally isomorphic to $\mathrm{Pol_a}(\mathbb{K}^m)$.

Remark 7.11 *Recall that in Remark 7.2 we distinguished two meanings of symbols* $v_1, v_2 \ldots$. *The same symbols are used in Def. 7.10 with a third meaning. They stand for* anti-commuting variables in $\mathbb{K}^m$ *(generators of the algebra* $\mathrm{Pol_a}(\mathbb{K}^m)$*). The first meaning was as* individual vectors in $\mathcal{Y}^{\#}$; *see e.g. (7.1) and (7.12). The second was as the* generic variables in $\mathcal{Y}_i^{\#}$; *see (7.2).*

Definition 7.12 *Let* $I \subset \{1, \ldots, m\}$. *We denote by* $\mathrm{sgn}(I)$ *the signature of* $(i_1, \ldots, i_p, i_{p+1}, \ldots, i_m)$, *where*

$$I = \{i_1, \ldots, i_p\}, \quad I^c := \{1, \ldots, m\} \backslash \{i_1, \ldots, i_p\} = \{i_{p+1}, \ldots, i_m\},$$

with $i_1 < \cdots < i_p$ *and* $i_{p+1} < \cdots < i_m$.

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Definition 7.13 *The* Hodge star operator *is defined as*

$$\theta v_{i_1} \cdots v_{i_p} := \operatorname{sgn}(I)\, v_{i_m} \cdots v_{i_{m-p+1}}, \tag{7.9}$$

where $\{i_1, \ldots, i_m\}$ *and* $\operatorname{sgn}(I)$ *are as in Def. 7.12.*

Definition 7.14 *For* $i = 1, \ldots, m$, v_i *will denote not only an element of* $\mathrm{Pol_a}(\mathbb{K}^m)$, *but also the* operator of left multiplication by v_i *acting on* $\mathrm{Pol_a}(\mathbb{K}^m)$. *These operators clearly satisfy the relations (7.8). We consider also the* partial derivatives ∇_{v_i} *satisfying the relations*

$$[\nabla_{v_i}, \nabla_{v_i}]_+ = 0, \quad [\nabla_{v_i}, v_j]_+ = \delta_{ij}.$$

The action of the partial derivatives on the variables is given by

$$\nabla_{v_i} 1 = 0, \qquad \nabla_{v_i} v_j = \delta_{ij}.$$

We introduce also the Berezin integral w.r.t. the variable v_i. *Its notation consists of two symbols:* $\int$ *and* $\mathrm{d}v_i$. *The rules of manipulating with* $\mathrm{d}v_i$ *are*

$$\mathrm{d}v_i \mathrm{d}v_j = -\mathrm{d}v_j \mathrm{d}v_i, \quad \mathrm{d}v_i v_j = -v_j \mathrm{d}v_i.$$

The rules of evaluating the integrals are

$$\int \mathrm{d}v_i = 0, \quad \int v_i \mathrm{d}v_j = \delta_{ij}.$$

For example, if $\sigma \in S_m$, then

$$\int v_{\sigma(1)} \cdots v_{\sigma(p)} \mathrm{d}v_m \cdots \mathrm{d}v_1 = \begin{cases} 0, & \text{if } p < m, \\ \operatorname{sgn}(\sigma), & \text{if } p = m. \end{cases}$$

Now let $\mathcal{Y}$ be a vector space of dimension m. If we fix a basis $(e_1, ..., e_m)$ of $\mathcal{Y}$, we can identify $\mathcal{Y}$ and $\mathcal{Y}^\#$ with $\mathbb{K}^m$, and hence $\mathrm{Pol_a}(\mathcal{Y}^\#)$ and $\mathrm{Pol_a}(\mathcal{Y})$ with $\mathrm{Pol_a}(\mathbb{K}^m)$. We see that v_i coincide with $e_i(v)$, ∇_{v_i} with $e^i(\nabla_v)$, and the Hodge star defined in (7.9) coincides with the Hodge star defined in Subsect. 3.6.2. If we use the volume form $e^m \wedge \cdots \wedge e^1$ on $\mathcal{Y}$, then

$$\int \Psi(v)\mathrm{d}v = \int \Psi(v_1, \ldots, v_m)\mathrm{d}v_m \cdots \mathrm{d}v_1,$$
$$\int \mathrm{d}v \Psi(v) = \int \mathrm{d}v_m \cdots \mathrm{d}v_1 \Psi(v_1, \ldots, v_m).$$

7.1.6 Differential operators and convolutions

The Hodge star operator transforms differentiation into convolution:

Theorem 7.15 *Let* $\Psi, \Phi \in \mathrm{Pol_a}(\mathcal{Y}^\#)$. *Then*

$$(\theta\Psi)(\nabla_v)\Phi(v) = (-1)^m \int \mathrm{d}w \Psi(w)\Phi(v + w).$$

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If $\dim \mathcal{Y}$ *is even, then the formula simplifies to*

$$(\theta\Psi)(\nabla_v)\Phi(v) = \int \Psi(w)\Phi(v+w)\mathrm{d}w.$$

Proof We fix a basis $(e_1, \dots, e_m)$ of $\mathcal{Y}$ and use the anti-symmetric calculus in coordinates. Without loss of generality we can assume that $\Phi(v) = v_1 \cdots v_n$ and $\Psi(v) = \prod_{j\in J} v_j$. Then, using the notation of Def. 7.12,

$$\Psi(w)\Phi(v+w) = \sum_{I\subset\{1,\dots,n\}} \mathrm{sgn}(I) \prod_{j\in J} w_j \prod_{i\in I} w_i \prod_{k\in\{1,\dots,n\}\backslash I} v_k. \tag{7.10}$$

The Berezin integral

$$\int \mathrm{d}w\Psi(w)\Phi(v+w) \tag{7.11}$$

is non-zero only if $J = \{j_1, \dots, j_p, n+1, \dots, m\}$. The only term on the r.h.s. of (7.10) giving a non-zero contribution corresponds to $I = \{j_{p+1}, \dots, j_n\}$. We have

$$\begin{aligned}\int \mathrm{d}w_m \cdots \mathrm{d}w_1 \ & \mathrm{sgn}(j_{p+1}, \dots, j_n, j_1, \dots, j_p, n+1, \dots, m) \\ & \times w_{j_1} \cdots w_{j_p} \cdot w_{n+1} \cdots w_m \cdot w_{j_{p+1}} \cdots w_{j_n} v_{j_1} \cdots v_{j_p} \\ &= (-1)^m \, \mathrm{sgn}(j_{p+1}, \dots, j_n, j_1, \dots, j_p, n+1, \dots, m) \\ & \times \mathrm{sgn}(j_1, \dots, j_p, n+1, \dots, m, j_{p+1}, \dots, j_n) v_{j_1} \cdots v_{j_p}.\end{aligned}$$

On the other hand, using that

$$\theta\Psi(y) = \mathrm{sgn}(j_1, \dots, j_p, n+1, \dots, m, j_{p+1}, \dots, j_n) y_{j_n} \cdots y_{j_{p+1}}$$

and

$$\Phi(v) = \mathrm{sgn}(j_{p+1}, \dots, j_n, j_1, \dots, j_p, n+1, \dots, m)\, v_{j_{p+1}} \cdots v_{j_n} \cdot v_{j_1} \cdots v_{j_p},$$

we get

$$\begin{aligned}(\theta\Psi)(\nabla_v)\Psi(v) &= \mathrm{sgn}(j_{p+1}, \dots, j_n, j_1, \dots, j_p, n+1, \dots, m) \\ & \times \mathrm{sgn}(j_1, \dots j_p, n+1, \dots, m, j_{p+1}, \dots, j_n)\, v_{j_1} \cdots v_{j_p}.\end{aligned}$$

This proves the first statement of the theorem. If m is even, then the left and right Berezin integrals coincide, which proves the second statement. □

7.1.7 Anti-symmetric exponential

Definition 7.16 *The* anti-symmetric exponential of $\Phi \in \mathrm{Pol}_\mathrm{a}(\mathcal{Y}^\#)$ *is defined as*

$$\mathrm{e}^\Phi(v) := \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^n(v).$$

(Note that the series terminates after a finite number of terms.)

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If at least one of the terms Φ_1, Φ_2 is even, then

$$\mathrm{e}^{\Phi_1+\Phi_2}(v) = \mathrm{e}^{\Phi_1}\mathrm{e}^{\Phi_2}(v).$$

The following propositions justify the analogy between the Hodge star operator and the Fourier transform.

Let $\mathcal{Y}$ be a vector space equipped with the volume form Ξ. Let us equip $\mathcal{Y}^{\#}$ with the volume form Ξ^{dual}.

Proposition 7.17 *Let* $\Psi \in \mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}^{\#})$. *Then*

$$\theta\Psi(y) = (-1)^m \int \mathrm{d}v \Psi(v) \cdot \mathrm{e}^{v\cdot y},$$
$$\Psi(v) = (-1)^m \int \mathrm{d}y \theta\Psi(y) \cdot \mathrm{e}^{y\cdot v}.$$

In particular, if m *is even, then*

$$\theta\Psi(y) = \int \Psi(v) \cdot \mathrm{e}^{v\cdot y}\mathrm{d}v,$$
$$\Psi(v) = \int \theta\Psi(y) \cdot \mathrm{e}^{y\cdot v}\mathrm{d}y.$$

Proof We use the anti-symmetric calculus in coordinates and assume that $\Psi(v) = v_1 \cdots v_p$. We have

$$\mathrm{e}^{v\cdot y} = \mathrm{e}^{\sum_{i=1}^m v_i \cdot y_i} = \sum_{K\subset\{1,\dots,m\}} \prod_{i\in K} v_i \cdot y_i.$$

This yields

$$\int \mathrm{d}v\Psi(v)\mathrm{e}^{v\cdot y} = \int \mathrm{d}v_m \cdots \mathrm{d}v_1 v_1 \cdots v_p \cdot v_{p+1} \cdot y_{p+1} \cdots v_m \cdot y_m$$
$$= \int \mathrm{d}v_m \cdots \mathrm{d}v_1 v_1 \cdots v_m y_m \cdots y_{p+1} = (-1)^m y_m \cdots y_{p+1} = \theta\Psi(y).$$

The second identity can be proved similarly, using that $\mathrm{d}y = \mathrm{d}y_1 \cdots \mathrm{d}y_m$. □

7.1.8 Anti-symmetric Gaussians

Let $\zeta \in \mathrm{Pol}_{\mathrm{a}}^2(\mathcal{Y}^{\#}) \simeq L_{\mathrm{a}}(\mathcal{Y}^{\#}, \mathcal{Y})$.

Definition 7.18 *The functional notation for*

$$\mathcal{Y}^{\#} \times \mathcal{Y}^{\#} \ni (v_1, v_2) \mapsto v_1 \cdot \zeta v_2 \tag{7.12}$$

will be either $\zeta(v)$ *or, more often,* $v \cdot \zeta v$. *The functional notation for* e^{ζ} *will be either* $\mathrm{e}^{\zeta}(v)$ *or* $\mathrm{e}^{v\cdot\zeta v}$.

The following proposition should be compared to (4.11) and (4.14), the corresponding identities for the usual Gaussians.

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Proposition 7.19 *Let $\mathcal{Y}$ be a vector space of even dimension equipped with a volume form. Then*

(1) $\left(\theta \mathrm{e}^{\frac{1}{2}\zeta}\right)(y) = \int \mathrm{e}^{y\cdot v}\mathrm{e}^{\frac{1}{2}v\cdot\zeta v}\mathrm{d}v = \mathrm{Pf}(\zeta)\mathrm{e}^{\frac{1}{2}y\cdot\zeta^{-1}y}$.
(2) $\mathrm{e}^{\frac{1}{2}\nabla_v\cdot\zeta^{-1}\nabla_v}\Phi(v) = \mathrm{Pf}(\zeta)^{-1}\int \mathrm{e}^{\frac{1}{2}w\cdot\zeta w}\Phi(v+w)\mathrm{d}w, \quad \Phi \in \mathrm{Pol_a}(\mathcal{Y}^{\#})$.
(3) $\int \mathrm{e}^{\frac{1}{2}v\cdot\zeta v}\mathrm{d}v = \mathrm{Pf}(\zeta)$.

Proof Let us consider ζ as an element of $L_{\mathrm{a}}(\mathcal{Y}^{\#},\mathcal{Y})$. Let us equip $\mathcal{Y}$ with a Euclidean structure ν compatible with the volume form Ξ and note that $\zeta\nu$ is an anti-self-adjoint operator on $\mathcal{Y}$. Applying Corollary 2.85, we can find a basis $(e_1,\dots,e_{2m})$ of $\mathcal{Y}$ such that

$$\zeta = \sum_{i=1}^{m}\mu_i\left(|e_{2i-1}\rangle\langle e_{2i}| - |e_{2i}\rangle\langle e_{2i-1}|\right). \tag{7.13}$$

Note that

$$\mathrm{Pf}(\zeta) = \prod_{i=1}^{m}\mu_i.$$

We can rewrite (7.13) as

$$\frac{1}{2}\zeta = \sum_{i=1}^{m}\zeta_i,$$

where $\zeta_i = \mu_i e_{2i-1}\cdot e_{2i}$. Since $\zeta_i^2 = 0$ and $\zeta_i\zeta_j = \zeta_j\zeta_i$, we have

$$\mathrm{e}^{\frac{1}{2}\zeta} = \sum_{I\subset\{1,\dots,m\}}\prod_{i\in I}\zeta_i.$$

Now

$$\theta\prod_{i\in I}\zeta_i = \left(\prod_{i=1}^{m}\mu_i\right)\prod_{i\in I^{\mathrm{c}}}\mu_i^{-1}e^{2i}\cdot e^{2i-1}.$$

This yields $\theta\mathrm{e}^{\frac{1}{2}\zeta} = \mathrm{Pf}(\zeta)\mathrm{e}^{\frac{1}{2}\zeta^{-1}}$. By Prop. 7.17, we know that

$$\theta\mathrm{e}^{\frac{1}{2}\zeta}(y) = \int \mathrm{e}^{\frac{1}{2}v\cdot\zeta v}\mathrm{e}^{v\cdot y}\mathrm{d}v. \tag{7.14}$$

The two exponentials in the integral commute since they are both of even degree, and the function on the l.h.s. is an even function of y, which proves that (7.14) equals $\int \mathrm{e}^{y\cdot v}\mathrm{e}^{\frac{1}{2}v\cdot\zeta v}\mathrm{d}v$.

(2) follows from (1) and statement (1) of Thm. 7.15 for $\Psi(v) = \mathrm{e}^{\frac{1}{2}v\cdot\zeta v}$.

(3) follows from (2) for $\Phi = 1$. □

7.2 Operators and anti-symmetric calculus

Throughout the section $\mathcal{X}$ is a vector space with $\dim\mathcal{X} = d$. Anti-symmetric calculus is especially useful in the context of the space $\mathcal{X}\oplus\mathcal{X}^{\#}$. This space has an even dimension and a natural volume form, which is helpful in the context of anti-symmetric calculus. We will see that the space $\mathrm{Pol}_{\mathrm{a}}(\mathcal{X} \oplus \mathcal{X}^{\#})$ is well suited to describe linear operators on $\Gamma_{\mathrm{a}}(\mathcal{X}^{\#}) = \mathrm{Pol}_{\mathrm{a}}(\mathcal{X})$.

7.2.1 Berezin integral on $\mathcal{X} \oplus \mathcal{X}^{\#}$

In Subsect. 1.1.16, and then in Subsect. 3.6.4, we considered symplectic spaces of the form $\mathcal{X}^{\#} \oplus \mathcal{X}$ and $\mathcal{X} \oplus \mathcal{X}^{\#}$. They can be viewed as dual to one another. The canonical symplectic form on $\mathcal{X}^{\#} \oplus \mathcal{X}$ is denoted by ω. Consequently, the canonical symplectic form on $\mathcal{X} \oplus \mathcal{X}^{\#}$ is denoted by ω^{-1}. The corresponding Liouville forms are defined as $\frac{1}{d!} \wedge^d \omega$, resp. $\frac{1}{d!} \wedge^d \omega^{-1}$. If we choose a volume form Ξ on $\mathcal{X}$ and the volume form Ξ^{dual} on $\mathcal{X}^{\#}$, then the Liouville volume forms on both $\mathcal{X}^{\#} \oplus \mathcal{X}$ and $\mathcal{X} \oplus \mathcal{X}^{\#}$ are $\Xi^{\mathrm{dual}} \wedge \Xi$.

The generic variable of $\mathcal{X}$ will be denoted by x and of $\mathcal{X}^{\#}$ by ξ. The corresponding Berezin integrals will be denoted by $\int \cdot \mathrm{d}x$, resp. $\int \cdot \mathrm{d}\xi$. Hence the Berezin integral of $\Phi \in \mathrm{Pol}_{\mathrm{a}}(\mathcal{X} \oplus \mathcal{X}^{\#})$ w.r.t. the Liouville volume form will be denoted by

$$\int \Phi(x, \xi) \mathrm{d}\xi \mathrm{d}x.$$

If we fix a basis $(e_1, \dots, e_d)$ of $\mathcal{X}$ and if $(e^1, \dots, e^d)$ is the dual basis of $\mathcal{X}^{\#}$, then the symplectic form ω on $\mathcal{X}^{\#} \oplus \mathcal{X}$ and ω^{-1} on $\mathcal{X} \oplus \mathcal{X}^{\#}$ is

$$\sum_{i=1}^{d} e_i \wedge e^i. \tag{7.15}$$

The volume forms on $\mathcal{X}$, resp. $\mathcal{X}^{\#}$ are $e^d \wedge \cdots \wedge e^1$, resp. $e_1 \wedge \cdots \wedge e_d$, which, inside Berezin, integrals, is written as $\mathrm{d}x^d \cdots \mathrm{d}x^1$, resp. $\mathrm{d}\xi_1 \cdots \mathrm{d}\xi_d$.

Definition 7.20 *We will use the following shorthand functional notation:*

$$\begin{aligned} x \cdot \xi &:= \sum x^i \xi_i = \sum_{i=1}^{d} e^i(x) \cdot e_i(\xi) \\ &= -\frac{1}{2}(x, \xi) \cdot \omega^{-1}(x, \xi), \\ \nabla_x \cdot \nabla_\xi &:= \nabla_{x^i} \cdot \nabla_{\xi_i} = \sum_{i=1}^{d} e_i(\nabla_x) \cdot e^i(\nabla_\xi) \\ &= \frac{1}{2}(\nabla_x, \nabla_\xi) \cdot \omega(\nabla_x, \nabla_\xi), \end{aligned}$$

where we have used various notational conventions to express the same object.

As an application we have the following proposition:

Proposition 7.21

$$\mathrm{e}^{t \nabla_x \cdot \nabla_\xi} \Phi(x, \xi) = t^d \int \mathrm{e}^{t^{-1}(\xi - \xi') \cdot (x - x')} \Phi(x', \xi') \mathrm{d}x' \mathrm{d}\xi'.$$

Proof By (7.15), $\mathrm{Pf}(\omega^{-1}) = 1$. Hence the proposition follows from Prop. 7.19 applied to $\zeta = t\omega^{-1}$. $\square$

7.2.2 Operators on the space of anti-symmetric polynomials

Let $B \in L\big(\mathrm{Pol_a}(\mathcal{X})\big)$.

Definition 7.22 *The* Bargmann kernel of B *is an element of* $\mathrm{Pol_a}(\mathcal{X} \oplus \mathcal{X}^{\#})$, *denoted* B^{Bar}, *obtained from* $\frac{1}{\sqrt{N!}} B \frac{1}{\sqrt{N!}}$ *by the following identification:*

$$\begin{aligned} L\big(\mathrm{Pol_a}(\mathcal{X})\big) &\simeq \mathrm{Pol_a}(\mathcal{X}) \otimes \mathrm{Pol_a}(\mathcal{X})^{\#} \\ \simeq\ \mathrm{Pol_a}(\mathcal{X}) \otimes \mathrm{Pol_a}(\mathcal{X}^{\#}) &\simeq \mathrm{Pol_a}(\mathcal{X} \oplus \mathcal{X}^{\#}). \end{aligned} \tag{7.16}$$

In the first identification we use the identification of $L(\mathcal{V})$ with $\mathcal{V} \otimes \mathcal{V}^{\#}$ described in Subsect. 3.1.8. The second involves the identification of $\mathrm{Pol_a}(\mathcal{X})^{\#}$ *with* $\mathrm{Pol_a}(\mathcal{X}^{\#})$*; see (3.4). The third is the exponential law for anti-symmetric tensor algebras; see Subsect. 3.5.4.*

Note that B^{Bar} is the fermionic analog of the Bargmann kernel of an operator introduced in Def. 9.51.

Let us compute the Bargmann kernel in a basis. Recall that we have the following notation: for $I = \{i_1, \dots, i_n\} \subset \{1, \dots, d\}$ with $i_1 < \cdots < i_n$,

$$e_I := e_{i_1} \cdots e_{i_n}, \qquad e^I := e^{i_n} \cdots e^{i_1}.$$

In the functional notation these are written as

$$e_I(\xi) := e_{i_1}(\xi) \cdots e_{i_n}(\xi), \quad e^I(x) := e^{i_n}(x) \cdots e^{i_1}(x).$$

We saw in Subsect. 3.3.6 that $\{e_I \ : \ I \subset \{1, \dots, d\}\}$ is a basis of $\mathrm{Pol_a}(\mathcal{X})^{\#}$, and $\{\#I! e^I \ : \ I \subset \{1, \dots, d\}\}$ is the dual basis of $\mathrm{Pol_a}(\mathcal{X})$. Clearly, $B \in L\big(\mathrm{Pol_a}(\mathcal{X})\big)$ can be written in terms of its matrix elements as

$$B = \sum_{I,J \subset \{1,\dots,d\}} B_{I,J} \#I! |e^I\rangle\langle e_J|,$$

for

$$B_{I,J} = \#J! \langle e_I | B e^J \rangle.$$

Thus

$$\frac{1}{\sqrt{N!}} B \frac{1}{\sqrt{N!}} = \sum_{I,J \subset \{1,\dots,d\}} B_{I,J} \sqrt{\#I!} |e^I\rangle\langle e_J| \frac{1}{\sqrt{\#J!}}.$$

Therefore, the identification (7.16) leads to the formula

$$B^{\mathrm{Bar}}(x, \xi) = \sum_{I,J \subset \{1,\dots,d\}} B_{I,J} \sqrt{\#I!} e^I(x) \cdot e_J(\xi) \frac{1}{\sqrt{\#J!}}. \tag{7.17}$$

Recall that Θ_{a}^k denotes the projection onto $\mathrm{Pol}_{\mathrm{a}}^k(\mathcal{X}\oplus\mathcal{X}^{\#})$ (see Def. 3.24). Recall also that in Subsect. 3.5.7 we introduced the following notation: if $\Phi\in\mathrm{Pol}_{\mathrm{a}}(\mathcal{X}^{\#})$, $\Psi\in\mathrm{Pol}_{\mathrm{a}}(\mathcal{X})$, then we write

$$\Psi^{\mathrm{mod}} := \frac{1}{\sqrt{N!}}\Psi, \qquad \Phi^{\mathrm{mod}} := \frac{1}{\sqrt{N!}}\Phi.$$

Theorem 7.23 (1) *Let $B\in L\big(\mathrm{Pol}_{\mathrm{a}}(\mathcal{X})\big)$, $0\le k\le d$. Then*

$$\mathrm{Tr}\,B\Theta_{\mathrm{a}}^k = \frac{1}{(d-k)!}\int (x\cdot\xi)^{d-k}B^{\mathrm{Bar}}(x,\xi)\mathrm{d}x\mathrm{d}\xi,$$
$$\mathrm{Tr}\,B = \int \mathrm{e}^{x\cdot\xi}B^{\mathrm{Bar}}(x,\xi)\mathrm{d}x\mathrm{d}\xi.$$

(2) *Let $\Phi\in\mathrm{Pol}_{\mathrm{a}}(\mathcal{X}^{\#})$, $\Psi\in\mathrm{Pol}_{\mathrm{a}}(\mathcal{X})$. Then*

$$\langle\Phi|\Theta_{\mathrm{a}}^k\Psi\rangle = \frac{1}{(d-k)!}\int (x\cdot\xi)^{d-k}\Psi^{\mathrm{mod}}(x)\Phi^{\mathrm{mod}}(\xi)\mathrm{d}x\mathrm{d}\xi,$$
$$\langle\Phi|\Psi\rangle = \int \mathrm{e}^{x\cdot\xi}\Psi^{\mathrm{mod}}(x)\Phi^{\mathrm{mod}}(\xi)\mathrm{d}x\mathrm{d}\xi.$$

Proof Using the basis of $\mathcal{X}$ and $\mathcal{X}^{\#}$, we can write

$$\frac{1}{(d-k)!}(x\cdot\xi)^{d-k} = \sum_{\#K=d-k}\prod_{i\in K}e^i(x)\cdot e_i(\xi).$$

By (7.17),

$$\begin{aligned}&\frac{1}{(d-k)!}(x\cdot\xi)^{d-k}B^{\mathrm{Bar}}(x,\xi)\\ &= \sum_{\#K=d-k}\prod_{i\in K}e^i(x)\cdot e_i(\xi)\sum_{I,J}B_{I,J}\sqrt{\#I!}e^I(x)\cdot e_J(\xi)\frac{1}{\sqrt{\#J!}}.\end{aligned}\tag{7.18}$$

In the integral of (7.18), only the terms of degree (d,d) contribute. Therefore, we can replace (7.18) by

$$\sum_{\#I=k}\prod_{i\in I^{\mathrm{c}}}e^i(x)\cdot e_i(\xi)B_{I,I}e^I(x)\cdot e_I(\xi).$$

Since $e^I\cdot e_I = \prod\limits_{i\in I}e^i(x)\cdot e_i(\xi)$ and $\prod\limits_{i=1}^{d}e^i\cdot e_i = e^d\cdots e^1\cdot e_1\cdots e_d$, we get

$$\begin{aligned}\frac{1}{(d-k)!}\int (x\cdot\xi)^{d-k}B^{\mathrm{Bar}}(x,\xi)\mathrm{d}x\mathrm{d}\xi &= \sum_{\#I=k}B_{I,I}\\ &= \mathrm{Tr}(B\Theta_{\mathrm{a}}^k).\end{aligned}$$

This proves the first statement of (1). The second follows by taking the sum over $1\le k\le d$.

(2) follows from (1) by noting that if $B=|\Psi\rangle\langle\Phi|$, then $B^{\mathrm{Bar}}(x,\xi)=\Psi^{\mathrm{mod}}(x)\cdot\Phi^{\mathrm{mod}}(\xi)$ and $\mathrm{Tr}|\Psi\rangle\langle\Phi| = \langle\Phi|\Psi\rangle$. □

7.2.3 Integral kernel of an operator

Let $B \in L\big(\mathrm{Pol_a}(\mathcal{X})\big)$. It is easy to see that there exists a unique $B(\cdot,\cdot) \in \mathrm{Pol_a}(\mathcal{X} \oplus \mathcal{X})$ such that for $\Psi \in \mathrm{Pol_a}(\mathcal{X})$

$$B\Psi(x) = \int B(x,y)\Psi(y)\mathrm{d}y,$$

where we use y as the generic variable in the second copy of $\mathcal{X}$.

Definition 7.24 *We will call $B(x,y)$ the* integral kernel of B *(w.r.t. the volume form Ξ).*

Clearly, if $\mathcal{X}$ is real, the integral kernel introduced in the above definition is the fermionic analog of the usual integral kernel, such as in Thm. 4.24.

7.2.4 x, ∇_x-quantization

Definition 7.25 *We define the x, ∇_x-*quantization*, resp. the ∇_x, x-*quantization *as the maps*

$$\mathrm{Pol_a}(\mathcal{X} \oplus \mathcal{X}^{\#}) \ni b \mapsto \mathrm{Op}^{x,\nabla_x}(b) \in L\big(\mathrm{Pol_a}(\mathcal{X})\big),$$
$$\mathrm{Pol_a}(\mathcal{X} \oplus \mathcal{X}^{\#}) \ni b \mapsto \mathrm{Op}^{\nabla_x,x}(b) \in L\big(\mathrm{Pol_a}(\mathcal{X})\big),$$

defined as follows: Let $b_1 \in \mathrm{Pol_a}(\mathcal{X})$, $b_2 \in \mathrm{Pol_a}(\mathcal{X}^{\#})$. Then for $b(x,\xi) = b_1(x)b_2(\xi)$ we set

$$\mathrm{Op}^{x,\nabla_x}(b) := b_1(x)b_2(\nabla_x),$$

and for $b(x,\xi) = b_2(\xi)b_1(x)$ we set

$$\mathrm{Op}^{x,\nabla_x}(b) := b_2(\nabla_x)b_1(x).$$

We extend the definition to $\mathrm{Pol_a}(\mathcal{X} \oplus \mathcal{X}^{\#})$ by linearity.

If $\mathcal{X}$ is real, the (fermionic) ∇_x, x- and x, ∇_x-quantizations introduced above are parallel to the (bosonic) D, x- and x, D-quantizations discussed in Subsect. 4.3.1. If $\mathcal{X}$ is complex, they essentially coincide with the fermionic Wick and anti-Wick quantizations, which will be discussed in Subsect. 13.3.1.

Theorem 7.26 *Assume that d is even.*

(1) *Let $b \in \mathrm{Pol_a}(\mathcal{X} \oplus \mathcal{X}^{\#})$. Then the integral kernels of the quantizations of b are*

$$\mathrm{Op}^{x,\nabla_x}(b)(x,y) = \int b(x,\xi)\mathrm{e}^{(x-y)\cdot\xi}\mathrm{d}\xi,$$
$$\mathrm{Op}^{\nabla_x,x}(b)(x,y) = \int b(y,\xi)\mathrm{e}^{(x-y)\cdot\xi}\mathrm{d}\xi.$$

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(2) *If* $b_+, b_- \in \mathrm{Pol_a}(\mathcal{X} \oplus \mathcal{X}^{\#})$ *and* $\mathrm{Op}^{x,\nabla_x}(b_+) = \mathrm{Op}^{\nabla_x,x}(b_-)$, *then*

$$\begin{aligned} b_+(x,\xi) &= \mathrm{e}^{\nabla_x \cdot \nabla_\xi} b_-(x,\xi) \\ &= \int b_-(x_1,\xi_1)\mathrm{e}^{(\xi-\xi_1)\cdot(x-x_1)}\mathrm{d}x_1\mathrm{d}\xi_1. \end{aligned}$$

(3) *If* $b_1, b_2 \in \mathrm{Pol_a}(\mathcal{X} \oplus \mathcal{X}^{\#})$ *and* $\mathrm{Op}^{x,\nabla_x}(b_1)\mathrm{Op}^{x,\nabla_x}(b_2) = \mathrm{Op}^{x,\nabla_x}(b)$, *then*

$$\begin{aligned} b(x,\xi) &= \mathrm{e}^{\nabla_{x_2} \cdot \nabla_{\xi_1}} b_1(x_1,\xi_1) b_2(x_2,\xi_2)\Big|_{\substack{x_1 = x_2 = x, \\ \xi_1 = \xi_2 = \xi}} \\ &= \int \mathrm{e}^{(\xi-\xi_1)\cdot(x-x_1)} b_1(x,\xi_1) b_2(x_1,\xi)\mathrm{d}x_1\mathrm{d}\xi_1. \end{aligned}$$

Proof We will give a proof of (1) for the x, ∇_x-quantization. We can assume that $b(x,\xi) = b_1(x)b_2(\xi)$. Then using Thm. 7.15 and Prop. 7.17, we obtain

$$\begin{aligned} b_1(x)b_2(\nabla_x)\Psi(x) &= \int b_1(x)\theta^{-1}b_2(y)\Psi(x+y)\mathrm{d}y \\ &= \int b_1(x)b_2(\xi)\mathrm{e}^{\xi\cdot y}\Psi(x+y)\mathrm{d}\xi\mathrm{d}y \\ &= \int b_1(x)b_2(\xi)\mathrm{e}^{(x-y)\cdot\xi}\Psi(y)\mathrm{d}\xi\mathrm{d}y. \end{aligned}$$

□

7.3 Notes

The material of this chapter is based on the work of Berezin (1966, 1983).

https://doi.org/10.1017/9781009290876.008 Published online by Cambridge University Press