ON A CENTRE-LIKE SUBSET OF A RING

WITHOUT NIL IDEALS

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We give a new proof of the hypercentre theorem of Herstein.

In [7], Herstein has defined the hypercentre of a ring $R$ as follows:

$$T(R) = \{axR | ax^n = x^n a, \ n = n(x,a) \geq 1, \text{ all } x \in R\}.$$ 

Herstein has proved:

**THEOREM.** If $R$ is a ring without non-zero nil ideals, then $T(R) = Z(R)$.

We show that the theorem can be proved by making use of the method which has been given by Herstein in [2] to circumvent the Köthe conjecture. As it has been shown in [7], it suffices to prove the theorem for prime rings without non-zero nil ideals. First we prove the following:

**LEMMA.** Let $R$ be a prime ring without non-zero nil right ideals. Then $T(R) = Z(R)$.

**Proof.** We prove that $T(R)$ has no non-zero nilpotent elements. If $a \in T(R)$, $a^2 = 0$, then given $x \in R$ there exists $n \geq 1$ such that $0 = a(ax)^n = (ax)^n a$, so $(ax)^{n+1} = 0$. This shows that $aR$ is a nil right ideal, so $a = 0$ by the assumptions on $R$. Following [1, Lemma 4] we show

Received 30 October 1985
that all the elements of $T(R)$ are regular, so $T(R)$ is a domain. Let
$0 \neq a \in T(R)$, and $au = 0$ for some $u \in R$. Then $(uxa)^2 = 0$ for all
$x \in R$, and since $T(R)$ is a subring invariant under quasi inner
automorphisms on $R$, we get that
$$a + uxa^2 = (1+uxa)a(1-uxa) = (1+uxa)a(1+uxa)^{-1} \in T(R).$$
This shows that $uxa^2 \in T(R)$. But $(uxa^2)^2 = 0$, so $uxa^2 = 0$ since $T(R)$
has no non-zero nilpotent elements. We also have $0 \neq a^2$, since
$0 \neq a \in T(R)$, so $u = 0$ since $R$ is prime. Now $T(R)$ is a domain, and
for all $x, y \in T(R)$ there exists $n = n(x,y) \geq 1$ such that $x^ny = yx^n$, so by [3] $T(R)$ is commutative. By a lemma of Herstein [4, p. 378], $T(R)$
centralizes $J(R)$, so if $J(R) \neq 0$ it follows that $T(R) \subseteq Z(R)$, since $R$
is prime. So we have $T(R) = Z(R)$ if $J(R) \neq 0$, and the same result holds
if $J(R) = 0$ by [1, Lemma 2].

Proof of the Theorem. We already know that the result holds if $R$ has
no non-zero nil right ideals. Assume $R$ has a non-zero nil right ideal.
Since $T(R)$ is a subring invariant under quasi inner automorphisms, it
follows by the theorem of Herstein [2], that either $T(R) \subseteq Z(R)$, or $T(R)$
contains a non-zero ideal of $R$. If $T(R) \subseteq Z(R)$ we are done. If $U$
is a non-zero ideal of $R$ contained in $T(R)$, we prove that $R = Z(R) = T(R)$.
For all $x, y \in U$ there exists $n = n(x,y) \geq 1$ such that $x^ny = yx^n$, so by
[3] the commutator ideal $C(U)$ of $U$ is nil. Then $UC(U)U$ is a nil ideal
of $R$, so $UC(U)U = 0$ since $R$ has no non-zero nil ideals. This implies
$C(U) = 0$ since $R$ is prime, so $U$ is commutative. But a prime ring with
a non-zero commutative ideal must be commutative, so $R = Z(R) = T(R)$.

References


Math. 7 (1955), 411-412.

Algebra 44 (1977), 370-388.
Centre-like subset of a ring

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