
SEMIPRIME NEAR-RINGS

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Abstract

Some properties of $\nu$-semiprime ($\nu = 0, 1, 2$) near-rings are pointed out. In particular $\nu$-semiprime near-rings which contain nil non-nilpotent ideals are studied.


1. Preliminaries

Throughout this paper, $N$ will denote a right zerosymmetric near-ring and terminology and notation will agree with those introduced by Pilz in [4]. In particular, for any two sets $A$ and $B$, the product $AB$ will be the set of the products $ab$ with $a$ in $A$ and $b$ in $B$.

Let $I$ be a two-sided ideal of $N$. As Pilz suggests in [4, 2.108], the following definitions can be given:

(a) $I$ is 0-semiprime if every two-sided ideal $A$ of $N$, such that $A^2$ is contained in $I$, is contained in $I$;
(b) $I$ is 1-semiprime if every left ideal $L$ of $N$, such that $L^2$ is contained in $I$, is contained in $I$;
(c) $I$ is 2-semiprime if every $N$-subgroup $S$ of $N$, such that $S^2$ is contained in $I$, is contained in $I$.

Being $N$ zerosymmetric, every 2-semiprime ideal is 1-semiprime too and every 1-semiprime ideal is 0-semiprime too. Moreover, the 0-semiprime ideals are the semiprime ideals in the usual sense [4, Definition 2.82].

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Adapting the proof for 0-semiprime ideals, one proves that for $\nu = 0, 1, 2$ the following conditions are equivalent:

(i) $I$ is a $\nu$-semiprime two-sided ideal;

(ii) if $x$ does not belong to $I$, then $(x)^2\nu$ is not contained in $I$, where $(x)_0, (x)_1, (x)_2$ mean the two-sided ideal, the left ideal and the $N$-subgroup respectively, generated by $x$;

(iii) if $X_\nu$ properly contains $I$, the product $X^2\nu$ is not contained in $I$, where $X_0, X_1$ and $X_2$ respectively denote a two-sided ideal, a left ideal and an $N$-subgroup of $N$.

It is an immediate consequence of condition (ii) that $I$ is a $\nu$-semiprime ideal ($\nu = 0, 1, 2$) if and only if $N/I$ is an $sp\nu$-system, that is, a set $S$ such that if $s \in S$ there exist two elements $s_1, s_2$ of $(s)_\nu$ whose product $s_1s_2$ belongs to $S$.

Observe that, for $\nu = 0, 1, 2$, any intersection of $\nu$-semiprime ideals is $\nu$-semiprime. In particular, this applies to $\nu$-prime ideals [4, 2.108]: so, for every ideal $I$ of $N$, the $\nu$-prime radical $P_\nu(I)$ (that is, the intersection of the $\nu$-prime ideals containing $I$) is $\nu$-semiprime.

2. $\nu$-semiprime near-rings

A near-ring $N$ will be called $\nu$-semiprime if $(0)$ is $\nu$-semiprime ($\nu = 0, 1, 2$).

For instance, for every near-ring $N$ and every $\nu$-semiprime ideal $B$, the near-ring $N' = N/B$ is $\nu$-semiprime: in particular, for each ideal $I$ of $N$, the near-ring $N/P_\nu(I)$ is $\nu$-semiprime. By definition, a $\nu$-semiprime near-ring does not contain any ideal (respectively left ideal or $N$-subgroup) $X_\nu$ such that $X^2\nu = (0)$; moreover,

**Proposition 2.1.** If $N$ is $\nu$-semiprime ($\nu = 0, 1, 2$) and if $X_\nu$ is a two-sided ideal, a left ideal or an $N$-subgroup of $N$ such that there is a positive integer $n$ for which $X^n_\nu = (0)$, then $X$ is zero.

**Proof.** For the sake of brevity, write $X$ instead of $X_\nu$. The statement is true by assumption if $n$ is 2. To obtain a contradiction, suppose now that $X^n = (0)$ with $n > 2$ and $X^{n-1} \neq (0)$. Then there exist $(n-1)$ elements $x_1, x_2, \ldots, x_{n-1}$ of $X$ such that the product $y = x_1 \cdots x_{n-1}$ is different from zero.

If $\nu = 0, 1$ consider the (two-sided or left) ideal $I$ generated by $y$. Since $I$ is contained in the (respectively, two-sided or left) ideal $(X^{n-1})$ generated
by $X^{n-1}$, it follows that

$$I^2 \subseteq (X^{n-1}) \cdot I \subseteq (X^{n-1}) \cdot X.$$  

As right distributivity implies $(X^{n-1}) \cdot X \subseteq (X^n)$ and $X^n$ is zero by assumption, it follows that $I^2 = (0)$, that is, $I = (0)$, since $N$ is $\nu$-semiprime. So $y$ is zero, which is a contradiction.

If $\nu = 2$, consider the $N$-subgroup $I = x_1 \cdot x_2 \cdot x_3 \cdot \ldots \cdot x_{n-1}$. From $I \subseteq X^{n-1}$ it follows $I^2 \subseteq X^{2n-2} \subseteq X^n = (0)$, that is, $I = (0)$, since $N$ is 2-semiprime. Consequently $x_1 \cdot \ldots \cdot x_{n-1} = y$ is zero which contradicts the choice of $y$.

Therefore, if $X^n = (0)$, also $X^{n-1} = (0)$ and, from the inductive assumption, $X$ is zero.

Thus $N$ is $\nu$-semiprime ($\nu = 0, 1, 2$) if and only if $N$ has no nilpotent two-sided ideal, left ideal, $N$-subgroup (respectively).

The 0-semiprime near-rings were studied by many authors (for instance, see [3], [5]), while the 2-semiprime ones with descending chain condition on $N$-subgroups were studied by Blackett in [1]. Here some new properties are pointed out in the case where $N$ is 1-semiprime. In this case $N$ has no nilpotent left ideal; nevertheless we will suppose that $N$ has at least one non-zero nil left ideal. Among other things, this fact implies that in $N$ the descending chain condition on left ideals (and a fortiori on $N$-subgroups) does not hold; so this study is complementary to Blackett’s one. Besides, observe that the nil ideals of $N$ cannot be minimal as $N$-subgroups, because, for every near-ring $N$, the following result holds.

**Proposition 2.2.** If $H$ is a minimal $N$-subgroup of $N$, then $H$ is either nilpotent of index 2 or idempotent.

### 3. 1-semiprime near-rings

From now on $N$ will be a 1-semiprime near-ring with at least one non-zero nil left ideal. By Zorn’s lemma, this assumption forces $N$ to have at least one left ideal which is maximal in the family of nil left ideals: call each of them a *maximal nil left ideal*. Now, for every left ideal $L$, denote by $(0 : L)$ the annihilator of $L$; $(0 : L)$ is a two-sided ideal of $N$ and we prove

**Proposition 3.1.** Let $L$ be a maximal nil left ideal and $L'$ be a nil left ideal of $N$. Then $(0 : L) \subseteq (0 : L')$. 
PROOF. Call \( S \) the left ideal generated by \((0 : L) \cdot L'\); \( S \) is nil since it is contained in \( L' \); actually it will be proved to be zero and hence the result will hold.

First of all observe that the set \((0 : L) \cdot L'\) is contained in \((0 : L)\), so \( SL = (0) \). As a consequence, the left ideal \( L + S \) is nil: in fact, for every \( l \in L \) and every \( s \in S \), let \( h \) and \( k \) be the least positive integers such that \( l^h = 0 = s^k \). It is a routine calculation to verify that, if \( n = \max(h, k) \), the element \((l + s)^n\) belongs to \( S \) and then \( l + s \) is nilpotent. For instance, if \( n = 2 \),

\[
(l + s)^2 = l(l + s) + s(l + s) = (l(l + s) - l^2) + l^2 + (s(l + s) - sl) + sl
\]

and, since by assumption \( l^2 = 0 \) and \( sl \) belongs to \( SL \), which is zero, \((l + s)^2\) is the sum of two elements of \( S \). Now, since \( L \) is a maximal nil left ideal, the nil left ideal \( L + S \) must coincide with \( L \), and therefore \( S \) must be contained in \( L \). As it is also contained in \((0 : L)\), \( S^2 \) is zero, and so \( S \) is zero, for \( N \) is 1-semiprime.

Consequently in a 1-semiprime near-ring \( N \) all the maximal nil left ideals have the same annihilator: it will be called the nil-annihilator of \( N \) and will be denoted by \( \alpha(N) \).

Furthermore, the following statement holds

**Proposition 3.2.** The nil-annihilator of \( N \) coincides with the nil-annihilator of any sum of maximal nil left ideals of \( N \).

**Proof.** Let \( L, L' \) be two maximal nil left ideals and let \( x \) be an element of \( \alpha(N) \). For all \( l \in L \), \( l' \in L' \) we have

\[
x(l + l') = x(l + l') - xl \in L'.
\]

But \( x(l + l') \) belongs also to \( \alpha(N) \) and therefore

\[
x(l + l') \in L' \cap (0 : L') = (0).
\]

This proves that \((0 : L) \subseteq (0 : (L + L'))\). Since the converse is obvious, one sees that \((0 : L) = (0 : (L + L'))\).

By induction the result may be extended to any finite sum of maximal nil left ideals and also to those which are not finite, since every element of such a sum is a finite sum of elements of maximal nil left ideals.

**4. Properties of the nil-annihilator of \( N \)**

The nil-annihilator of \( N \) is a two-sided ideal different from \( N \), because, if \( \alpha(N) \) coincided with \( N \), then for every maximal nil left ideal \( L \) this
would imply \( L^2 \subseteq NL = (0) \), contradicting the fact that \( N \) is 1-semiprime.

**Proposition 4.1.** The nil-annihilator of \( N \) is not nil and does not contain any non-zero nil left ideal.

Indeed if \( L' \) is a nil left ideal contained in \( \alpha(N) \) and \( L \) is a maximal nil left ideal containing \( L' \) it follows that

\[
L' \subseteq L \cap \alpha(N) = L \cap (0 : L) = (0).
\]

**Proposition 4.2.** \( \alpha(N) \) is a 0-semiprime ideal.

**Proof.** Let \( B \) be a two-sided ideal such that \( B^n \) is contained in \( \alpha(N) \). It must be proved that \( B \) is contained in \( \alpha(N) \), that is, for every maximal nil left ideal \( L \) of \( N \), the product \( BL \) is zero. Indeed the left ideal \( K \) generated by \( BL \) is contained in \( B \cap L \) and therefore

\[
K^n \subseteq B^n \cap L \subseteq \alpha(N) \cap L = (0)
\]

which implies \( K = (0) \) (because \( N \) is 1-semiprime) and consequently \( BL = (0) \).

**Proposition 4.3.** The nil-annihilator of \( N \) is zero if and only if every two-sided ideal contains a non-zero nil left ideal.

**Proof.** Let \( \alpha(N) \) be different from zero: then it is a two-sided ideal which contains no non-zero nil left ideal. On the contrary, if \( \alpha(N) = (0) \), for every non-zero two-sided ideal \( B \) and for every maximal nil left ideal \( L \), \( BL \) is different from zero.

Let \( x \) be a non-zero element of \( BL \): the left ideal generated by \( x \) is the required ideal since it is non-zero, is contained in \( B \cap L \) and so nil.

Consider now the factor near-ring \( N' = N/\alpha(N) \) and the canonical epimorphism \( \pi: N \to N' \). If \( L \) is a nil left ideal of \( N \), then by Proposition 4.1, \( \pi(L) \) is a non-zero nil left ideal of \( N' \), so \( N' \) too contains a non-zero nil left ideal. On the other hand, since \( \alpha(N) \) is 0-semiprime, \( N' \) is 0-semiprime (see 4.2): if \( N' \) is also 1-semiprime, its nil-annihilator can be defined and one has

**Theorem 4.4.** If \( N' \) is 1-semiprime, then \( \alpha(N') \) is zero.

**Proof.** In order to prove that \( \alpha(N') \) is zero, it is sufficient to show that if \( B \) is a two-sided ideal of \( N \) with \( \pi(B) = \alpha(N') \), then \( B \) is contained
in \( \alpha(N) \), or, equivalently, that for every maximal nil left ideal \( L \) of \( N \) the product \( BL \) is zero.

Now, the left ideal \( K \) generated by \( BL \) is nil (since it is contained in \( L \)); so also its image \( \pi(K) \) is nil and contained in \( \pi(B) = \alpha(N') \). But \( \alpha(N') \) does not contain any non-zero nil left ideal: thus \( K \) must be contained in \( \alpha(N) \) and, by the same argument, \( K \) and its generating set \( BL \) must be zero.

The assumption of Theorem 4.4 is satisfied when \( N \) is 2-semiprime and has a left identity. Moreover, we have

**Theorem 4.5.** If \( N \) is a 2-semiprime near-ring with a left identity and a non-zero nil left ideal, then \( N' \) is 2-semiprime too (and consequently contains a non-zero nil left ideal and \( \alpha(N') \) is zero.)

**Proof.** In order to prove that \( N' \) has no non-zero nilpotent \( N' \)-subgroups, first of all we remark that if \( S' \) is a nilpotent \( N' \)-subgroup of \( N' \) and \( S \) is its preimage in \( N \), then \( S \) is an \( N \)-subgroup of \( N \).

Since \( S' \) is nilpotent, there is a positive integer \( n \) such that \( S^n \) is contained in \( \alpha(N) \). So, for any maximal nil left ideal \( L \), the product \( S^n L \) is zero and consequently

\[
(SL)^n = (SL) \cdot \ldots \cdot (SL) \subseteq S \cdot S^{n-1} \cdot L = S^n L = (0).
\]

Let now \( sl \) be any element of \( SL \): \( Nsl \) is an \( N \)-subgroup, nilpotent of index at most \( n \), for \( Nsl \) is contained in \( SL \); therefore \( Nsl \) is zero, since \( N \) is 2-semiprime. Since \( N \) has a left identity, this implies \( sl = 0 \), for each \( s \in S \) and \( l \in L \). Thus \( SL \) is zero, so \( S \) is contained in the nil-annihilator of \( N \) and \( S' = \pi(S) \) is zero.

The remaining properties are consequences of the Theorem 4.4 and the preceding remarks.

**References**


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