



The Category of Bratteli Diagrams

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Abstract. A category structure for Bratteli diagrams is proposed and a functor from the category of AF algebras to the category of Bratteli diagrams is constructed. Since isomorphism of Bratteli diagrams in this category coincides with Bratteli's notion of equivalence, we obtain in particular a functorial formulation of Bratteli's classification of AF algebras (and at the same time, of Glimm's classification of UHF algebras). It is shown that the three approaches to classification of AF algebras, namely, through Bratteli diagrams, K-theory, and a certain natural abstract classifying category, are essentially the same from a categorical point of view.

1 Introduction

AF algebras were first introduced and studied by Bratteli in 1972 [2]. An AF algebra is a C^* -algebra which is the closure of the union of an increasing sequence of its finite-dimensional C^* -subalgebras. The class of AF algebras has an interesting variety of examples [2, 4]. AF algebras are generalizations of UHF algebras which were studied by Glimm in 1960 [11] and of matroid C^* -algebras (stably isomorphic to UHF algebras) introduced by Dixmier in 1967 [5]. Glimm gave a classification of UHF algebras. In a brilliant leap, Bratteli generalized Glimm's classification to arbitrary AF algebras (see below—Theorem 3.11 is a reformulation of this).

In 1976, Elliott gave a classification of AF algebras using K-theory [8]. In fact, Elliott showed that the functor $K_0: \mathbf{AF} \rightarrow \mathbf{DG}$, from the category of AF algebras with $*$ -homomorphisms to the category of (scaled countable) dimension groups with order-preserving homomorphisms, is a strong classification functor, in the sense that if $\mathcal{A}_1, \mathcal{A}_2 \in \mathbf{AF}$ and $K_0(\mathcal{A}_1) \cong K_0(\mathcal{A}_2)$, then we have $\mathcal{A}_1 \cong \mathcal{A}_2$, and in fact every isomorphism from $K_0(\mathcal{A}_1)$ onto $K_0(\mathcal{A}_2)$ comes from an isomorphism from \mathcal{A}_1 onto \mathcal{A}_2 (see [8], [16, Section 7.2], and [10, Sections 5.1–5.3] for details). This categorical idea, finding a (strong) classification functor from a given category to another, more accessible category, is useful in the classification of various categories (see [10]).

The classification of AF algebras obtained by Bratteli in [2] used what are now called Bratteli diagrams. Bratteli associated to each AF algebra \mathcal{A} an infinite directed graph $\mathcal{B}(\mathcal{A})$, its Bratteli diagram (see Definition 2.2), and used these very effectively to study AF algebras. Some attributes of an AF algebra can be read directly from its Bratteli diagram, for instance its ideal structure. Bratteli showed that for $\mathcal{A}_1, \mathcal{A}_2 \in \mathbf{AF}$, $\mathcal{A}_1 \cong \mathcal{A}_2$ if \mathcal{A}_1 and \mathcal{A}_2 have the same Bratteli diagram, *i.e.*, $\mathcal{B}(\mathcal{A}_1) = \mathcal{B}(\mathcal{A}_2)$ (see Theorem 3.10). In fact Bratteli determined, in terms of the Bratteli diagrams of \mathcal{A}_1 and \mathcal{A}_2 , exactly when \mathcal{A}_1 and \mathcal{A}_2 are isomorphic.

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Denote by \mathbf{BD} the set of all Bratteli diagrams. Then, Bratteli's theorem asserts that the map $\mathcal{B}: \mathbf{AF} \rightarrow \mathbf{BD}$ has the property that if $\mathcal{A}_1, \mathcal{A}_2 \in \mathbf{AF}$ and $\mathcal{B}(\mathcal{A}_1) = \mathcal{B}(\mathcal{A}_2)$, or even just $\mathcal{B}(\mathcal{A}_1)$ is equivalent in Bratteli's sense to $\mathcal{B}(\mathcal{A}_2)$, then $\mathcal{A}_1 \cong \mathcal{A}_2$. The question that arises naturally here is whether the map $\mathcal{B}: \mathbf{AF} \rightarrow \mathbf{BD}$ can be made into a functor, and if so, whether it is a classification functor. This paper answers these questions.

In Section 2 we define an appropriate notion of morphism in \mathbf{BD} and we show that \mathbf{BD} with these morphisms is a category (Theorem 2.7). In Section 3 we show that $\mathcal{B}: \mathbf{AF} \rightarrow \mathbf{BD}$ is a (strong) classification functor (Theorem 3.11); thus for $\mathcal{A}_1, \mathcal{A}_2 \in \mathbf{AF}$ we have $\mathcal{A}_1 \cong \mathcal{A}_2$ if, and only if, $\mathcal{B}(\mathcal{A}_1) \cong \mathcal{B}(\mathcal{A}_2)$. This is a functorial formulation of Bratteli's theorem and would appear to be a definitive elaboration of the classification of AF algebras from the Bratteli diagram point of view. In particular, just the fact that the map is a functor yields Glimm's classification theorem for UHF algebras (see the proof of Theorem 3.13).

In Section 4, it is shown that the functor $\mathcal{B}: \mathbf{AF} \rightarrow \mathbf{BD}$ is a full functor (Theorem 4.1), which means that homomorphisms in the codomain category can be lifted back to homomorphisms in the domain category (this was done for isomorphisms in Theorem 3.11).

In Section 5, we investigate the relation between the category \mathbf{BD} of Bratteli diagrams and two abstract classifying categories, \mathbf{AF}^{out} and $\overline{\mathbf{AF}^{\text{out}}}$, for AF algebras (cf. [10]). We show that there is a strong classification functor from $\overline{\mathbf{AF}^{\text{out}}}$ to \mathbf{BD} which is faithful and full (Theorem 5.9) and is an equivalence of categories (Theorem 5.11).

In Section 6, we investigate the relation between $\overline{\mathbf{AF}^{\text{out}}}$ and the category \mathbf{DG} of dimension groups. We show that there is a strong classification functor from $\overline{\mathbf{AF}^{\text{out}}}$ to \mathbf{DG} which is faithful and full (Theorem 6.3) and is an equivalence of categories (Theorem 6.4). It is shown that the three strong classification functors $\mathcal{B}: \mathbf{AF} \rightarrow \mathbf{BD}$, $\overline{\mathcal{F}}: \mathbf{AF} \rightarrow \overline{\mathbf{AF}^{\text{out}}}$, and $K_0: \mathbf{AF} \rightarrow \mathbf{DG}$ that classify AF algebras are essentially the same (Theorem 6.7).

2 The Category of Bratteli Diagrams \mathbf{BD}

The notion of a Bratteli diagram was introduced by Bratteli to study AF algebras [2]. There are various formal definitions (just with different formulations) for a Bratteli diagram; for example, see [6] and [13]. What is behind these definitions is the very special structure of a $*$ -homomorphism between finite-dimensional C^* -algebras. The following theorem of Bratteli describes this structure [2]. Let us just quote this theorem, with some slight changes, from [4].

Theorem 2.1 Let $A_1 = \mathcal{M}_{n_1} \oplus \cdots \oplus \mathcal{M}_{n_k}$ and $A_2 = \mathcal{M}_{m_1} \oplus \cdots \oplus \mathcal{M}_{m_l}$ be finite-dimensional C^* -algebras (where the n 's and m 's are non-zero), and let $\phi: A_1 \rightarrow A_2$ be a $*$ -homomorphism. Then there is a unique $l \times k$ matrix $E = (a_{ij})$ of positive (i.e., non-negative) integers with the property that there is a unitary $v = (v_1, \dots, v_l)$ in A_2 such that if we set $\phi_i = \pi_i \circ \phi: A_1 \rightarrow \mathcal{M}_{m_i}$ then

$$v_i \phi_i(u_1, \dots, u_k) v_i^* = \begin{pmatrix} u_1^{(a_{i1})} & & & 0 \\ & \ddots & & \\ & & u_k^{(a_{ik})} & \\ 0 & & & 0^{(s_i)} \end{pmatrix} \text{ for } (u_1, \dots, u_k) \in A_1,$$

where s_i is defined by the equation $\sum_{j=1}^k a_{ij} n_j + s_i = m_i$, for each $1 \leq i \leq l$. Thus, if V_1 and V_2 denote the column matrices such that $V_1^T = (n_1 \cdots n_k)$ and $V_2^T = (m_1 \cdots m_l)$, then $EV_1 \leq V_2$. Moreover, we have

- (i) ϕ is injective if and only if for each j there is an i such that $a_{ij} \neq 0$;
- (ii) ϕ is unital if and only if $EV_1 = V_2$, i.e., $s_i = 0$ for each $1 \leq i \leq l$.

Proof See [2, Proposition 1.7] and [4, Corollary III.2.2]. ■

Let us call the matrix E in the previous theorem the multiplicity matrix of ϕ , and denote it by R_ϕ (this is the notation used in [1]). In general, let V_i be a $k_i \times 1$ matrix of non-zero positive integers for $i = 1, 2$; by a multiplicity matrix $E = (a_{ij})$ from V_1 to V_2 we shall mean a $k_2 \times k_1$ matrix of positive integers such that $EV_1 \leq V_2$. We shall use the notation $E: V_1 \rightarrow V_2$ to mean that E is a multiplicity matrix from V_1 to V_2 . E will be called an embedding matrix if for each j there is an i such that $a_{ij} \neq 0$ (in other words, if the algebra map induced by E , as defined above, is injective).

Let us recall the formulation of the definition of a Bratteli diagram in [10, Sections 2 and 3], which uses the matrix language and is more flexible for our purposes.

Definition 2.2 By a Bratteli diagram let us mean an ordered pair $B = (V, E)$, $V = (V_n)_{n=1}^\infty$ and $E = (E_n)_{n=1}^\infty$, such that the following hold:

- (i) each V_n is a $k_n \times 1$ matrix of non-zero positive integers for some $k_n \geq 1$;
- (ii) each E_n is an embedding matrix from V_n to V_{n+1} .

Let us denote such a B by the diagram

$$V_1 \xrightarrow{E_1} V_2 \xrightarrow{E_2} V_3 \xrightarrow{E_3} \cdots .$$

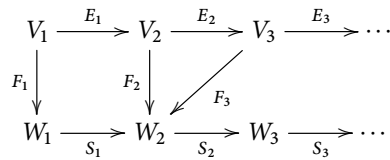
Let us write $E_{nm} = E_{m-1} \cdots E_{n+1} E_n$ for $n < m$ and $E_{nn} = I$, where I is the identity matrix of order k_n . Note that E_{nm} is a multiplicity matrix from V_n to V_m .

Remark In Definition 2.2, we have in fact defined the notion of a “non-zero” Bratteli diagram. This is enough for working with non-zero (in particular, unital) AF algebras. For the zero AF algebra, we get the zero Bratteli diagram, which is nothing but a single zero square matrix of size one, denoted by 0.

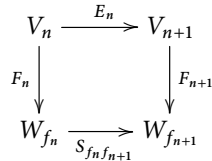
Let **BD** denote the set of all Bratteli diagrams. We wish to define morphisms between objects in **BD** to make it a category. In order to formulate the correct notion of morphism for our purposes, we first need to define the notion of premorphism. Recall that a sequence $(f_n)_{n=1}^\infty$ of positive integers is said to be cofinal in \mathbb{N} if $\sup\{f_n \mid n \in \mathbb{N}\} = +\infty$.

Definition 2.3 Let $B = (V, E)$ and $C = (W, S)$ be Bratteli diagrams. A *premorphisms* $f: B \rightarrow C$ is an ordered pair $((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$ where $(F_n)_{n=1}^\infty$ is a sequence of matrices and $(f_n)_{n=1}^\infty$ a cofinal sequence of positive integers with $f_1 \leq f_2 \leq \dots$ such that the following hold:

- (i) each F_n is a multiplicity matrix from V_n to W_{f_n} ;
- (ii) the diagram of $f: B \rightarrow C$ commutes:



Commutativity of the diagram of course amounts to saying that for any positive integer n we have $F_{n+1}E_n = S_{f_n f_{n+1}} F_n$; that is, the square



commutes. (This implies the general property of commutativity, namely, that any two paths of maps between the same pair of points in the diagram agree, *i.e.*, have the same product.)

Let B, C , and D be objects in **BD** and let $f: B \rightarrow C$ and $g: C \rightarrow D$ be premorphisms, $f = ((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$ and $g = ((G_n)_{n=1}^\infty, (g_n)_{n=1}^\infty)$. The composition of f and g is defined as $gf = ((H_n)_{n=1}^\infty, (h_n)_{n=1}^\infty)$, where $H_n = G_{f_n} F_n$ and $h_n = g_{f_n}$.

Remark In Definition 2.3, it is implicitly assumed that the Bratteli diagrams in question are non-zero. Let us define the zero premorphism as follows. Let B be a Bratteli diagram. The zero premorphism from B to 0 (the zero Bratteli diagram) is the ordered pair $(B, 0)$. Similarly, the zero premorphism from 0 to B is $(0, B)$. (Note that a morphism in a category depends on both the domain and the codomain objects.) The composition of the zero premorphism with any other premorphism is defined to be zero.

Proposition 2.4 *The set **BD**, with premorphisms as defined above as maps, is a (small) category.*

Proof First let us check that if $f: B \rightarrow C$ and $g: C \rightarrow D$ are premorphisms, with $f = ((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$ and $g = ((G_n)_{n=1}^\infty, (g_n)_{n=1}^\infty)$, then gf given as above, by

$H_n = G_{f_n} F_n$ and $h_n = g_{f_n}$, is a premorphism. Write $B = (V, E)$, $C = (W, S)$, and $D = (Z, T)$.

Let n be a positive integer and consider the following diagram:

$$\begin{array}{ccc}
 V_n & \xrightarrow{E_n} & V_{n+1} \\
 F_n \downarrow & & \downarrow F_{n+1} \\
 W_{f_n} & \xrightarrow{S_{f_n f_{n+1}}} & W_{f_{n+1}} \\
 G_{f_n} \downarrow & & \downarrow G_{f_{n+1}} \\
 Z_{h_n} & \xrightarrow{T_{h_n h_{n+1}}} & Z_{h_{n+1}}.
 \end{array}$$

Since f and g are premorphisms, we have

$$\begin{aligned}
 T_{h_{n+1} h_n} H_n &= T_{h_{n+1} h_n} G_{f_n} F_n \\
 &= G_{f_{n+1}} S_{f_n f_{n+1}} F_n \\
 &= G_{f_{n+1}} F_{n+1} E_n \\
 &= H_{n+1} E_n.
 \end{aligned}$$

This shows that $g f$ is indeed a premorphism.

Now for any Bratteli diagram $B = (V, E)$ define the premorphism $\text{id}_B: B \rightarrow B$ by $\text{id}_B = ((I_n)_{n=1}^\infty, (i_n)_{n=1}^\infty)$, where I_n is the identity matrix of order equal to the number of columns of V_n , and $i_n = n$. For any premorphisms $f: B \rightarrow C$ and $h: C \rightarrow B$ we have $\text{id}_B h = h$ and $f \text{id}_B = f$.

One checks easily that associativity holds using the associativity of matrix multiplication. This completes the proof that **BD**, with premorphisms, is a category. ■

It will be clear later that the category **BD** with premorphisms is not suitable for the classification of AF algebras and that we need to consider morphisms—consisting of equivalence classes of premorphisms—for the purposes of classification.

Definition 2.5 Let B, C be Bratteli diagrams, and $f, g: B \rightarrow C$ be premorphisms, i.e., maps in the category **BD** of Proposition 2.4, with $B = (V, E)$, $C = (W, S)$, $f = ((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$, and $g = ((G_n)_{n=1}^\infty, (g_n)_{n=1}^\infty)$. Let us say that f is *equivalent* to g , and write $f \sim g$, if there are sequences $(n_k)_{k=1}^\infty$ and $(m_k)_{k=1}^\infty$ of positive integers such that $n_k < m_k < n_{k+1}$ and $f_{n_k} < g_{m_k} < f_{n_{k+1}}$ for each $k \geq 1$, and the diagram

$$\begin{array}{ccccccc}
 V_{n_1} & \longrightarrow & V_{m_1} & \longrightarrow & V_{n_2} & \longrightarrow & V_{m_2} & \longrightarrow & \dots \\
 F_{n_1} \downarrow & & G_{m_1} \downarrow & & F_{n_2} \downarrow & & G_{m_2} \downarrow & & \\
 W_{f_{n_1}} & \longrightarrow & W_{g_{m_1}} & \longrightarrow & W_{f_{n_2}} & \longrightarrow & W_{g_{m_2}} & \longrightarrow & \dots
 \end{array}$$

commutes, i.e., each minimal square commutes: for each $k \geq 1$,

$$G_{m_k} E_{n_k m_k} = S_{f_{n_k} g_{m_k}} F_{n_k} \quad \text{and} \quad F_{n_{k+1}} E_{m_k n_{k+1}} = S_{g_{m_k} f_{n_{k+1}}} G_{m_k}.$$

At the end of this section we will give two other definitions for equivalence of premorphisms (Definition 2.9 and Definition 2.10). These may be more natural in some

sense (since they do not use subsequences), but we shall show that all three equivalence relations are the same (Proposition 2.11).

Remark In Definition 2.5, we have defined the equivalence of a pair of non-zero premorphisms. This notion extends in an obvious way to encompass the zero premorphism (which is defined in the remark following Definition 2.3), since for each Bratteli diagram B , $\text{Hom}(B, 0)$ and $\text{Hom}(0, B)$ have only one element.

Lemma 2.6 *Let $B, C \in \mathbf{BD}$. Then \sim is an equivalence relation on the set of premorphisms from B to C .*

Proof It is obvious that \sim is symmetric. Reflexivity follows from Definition 2.3, using the cofinality condition. Let $f, g, h: B \rightarrow C$ be premorphisms such that $f \sim g$ and $g \sim h$, and let us show that $f \sim h$. Write $B = (V, E)$, $C = (W, S)$, $f = ((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$, $g = ((G_n)_{n=1}^\infty, (g_n)_{n=1}^\infty)$, and $h = ((H_n)_{n=1}^\infty, (h_n)_{n=1}^\infty)$.

Choose sequences $(n_k)_{k=1}^\infty$ and $(m_k)_{k=1}^\infty$ establishing $f \sim g$, according to Definition 2.5, and sequences $(p_k)_{k=1}^\infty$ and $(q_k)_{k=1}^\infty$ for $g \sim h$. Construct sequences $(r_k)_{k=1}^\infty$ and $(s_k)_{k=1}^\infty$ inductively as follows to show that $f \sim h$. Set $n_1 = r_1$. There is $k_0 \geq 1$ such that $p_{k_0} > m_1$; set $q_{k_0} = s_1$. Each square in the diagram

$$\begin{array}{ccccccc} V_{r_1} & \longrightarrow & V_{m_1} & \longrightarrow & V_{p_{k_0}} & \longrightarrow & V_{s_1} \\ F_{r_1} \downarrow & & G_{m_1} \downarrow & & G_{p_{k_0}} \downarrow & & H_{s_1} \downarrow \\ W_{f_{r_1}} & \longrightarrow & W_{g_{m_1}} & \longrightarrow & W_{g_{p_{k_0}}} & \longrightarrow & W_{h_{s_1}} \end{array}$$

commutes, by the definitions of $f \sim g$ and $g \sim h$, and since g is a premorphism. Thus, $H_{s_1} E_{r_1 s_1} = S_{f_{r_1} h_{s_1}} F_{r_1}$.

There is $k_1 \geq 1$ such that $m_{k_1} > p_{k_0+1}$; set $n_{k_1+1} = r_2$. In the diagram

$$\begin{array}{ccccccc} V_{s_1} & \longrightarrow & V_{p_{k_0+1}} & \longrightarrow & V_{m_{k_1}} & \longrightarrow & V_{r_2} \\ H_{s_1} \downarrow & & G_{p_{k_0+1}} \downarrow & & G_{m_{k_1}} \downarrow & & F_{r_2} \downarrow \\ W_{h_{s_1}} & \longrightarrow & W_{g_{p_{k_0+1}}} & \longrightarrow & W_{g_{m_{k_1}}} & \longrightarrow & W_{f_{r_2}}, \end{array}$$

each square commutes, and so we have $F_{r_2} E_{s_1 r_2} = S_{h_{s_1} f_{r_2}} H_{s_1}$.

Continuing this procedure we obtain sequences $(r_k)_{k=1}^\infty$ and $(s_k)_{k=1}^\infty$ with $r_1 < s_1 < r_2 < s_2 < \dots$ and $f_{r_1} < h_{s_1} < f_{r_2} < h_{s_2} < \dots$ such that commutativity holds as required in Definition 2.5 for $f \sim h$. This shows that \sim is transitive, and so it is an equivalence relation. ■

Let us call an equivalence class of premorphisms between Bratteli diagrams B and C , with respect to the relation \sim , a *morphism* from B to C . Let us denote the equivalence class of a premorphism $f: B \rightarrow C$ by $[f]: B \rightarrow C$, or if there is no confusion, just by f .

The equivalence class of the zero premorphism (which makes sense only when $B = 0$ or $C = 0$) is called the zero morphism (see the remark preceding Lemma 2.6).

The composition of morphisms $[f]: B \rightarrow C$ and $[g]: C \rightarrow D$ should be defined as $[gf]: B \rightarrow D$ where gf is the composition of premorphisms. This composition is well defined, as is shown in the proof of the following theorem.

Theorem 2.7 *The set \mathbf{BD} , with morphisms as defined above, is a category.*

Proof First, we must show that the composition of two morphisms is well defined, i.e., independent of the choice of representatives. Let $f, l: B \rightarrow C$ and $g, h: C \rightarrow D$ be premorphisms such that $f \sim l$ and $g \sim h$, and let us show that $gf \sim hl$.

Write $B = (V, E)$, $C = (W, S)$, and $D = (Z, T)$, and $f = ((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$, $g = ((G_n)_{n=1}^\infty, (g_n)_{n=1}^\infty)$, and $gf = ((U_n)_{n=1}^\infty, (u_n)_{n=1}^\infty)$. Then $u_n = g_{f_n}$ and $U_n = G_{f_n} F_n$, according to Definition 2.3. Also write $h = ((H_n)_{n=1}^\infty, (h_n)_{n=1}^\infty)$, $l = ((L_n)_{n=1}^\infty, (l_n)_{n=1}^\infty)$, and $hl = ((X_n)_{n=1}^\infty, (x_n)_{n=1}^\infty)$. Then $x_n = h_{l_n}$ and $X_n = H_{l_n} L_n$, according to Definition 2.3.

Let $(n_k)_{k=1}^\infty$ and $(m_k)_{k=1}^\infty$ be sequences exhibiting the equivalence $f \sim l$, and let $(p_k)_{k=1}^\infty$ and $(q_k)_{k=1}^\infty$ be sequences exhibiting the equivalence $g \sim h$, according to Definition 2.5. Let us construct sequences $(r_k)_{k=1}^\infty$ and $(s_k)_{k=1}^\infty$ exhibiting the equivalence $gf \sim hl$. Set $n_1 = r_1$. There is $k_0 \geq 1$ such that $p_{k_0} > f_{n_1}$. There is $k_1 \geq 1$ such that $l_{m_{k_1}} \geq q_{k_0}$; set $s_1 = m_{k_1}$. Consider the diagram

$$\begin{array}{ccccccc} W_{f_{r_1}} & \longrightarrow & W_{p_{k_0}} & \longrightarrow & W_{q_{k_0}} & \longrightarrow & W_{l_{s_1}} \\ G_{f_{r_1}} \downarrow & & G_{p_{k_0}} \downarrow & & H_{q_{k_0}} \downarrow & & H_{l_{s_1}} \downarrow \\ Z_{g_{f_{r_1}}} & \longrightarrow & Z_{g_{p_{k_0}}} & \longrightarrow & Z_{h_{q_{k_0}}} & \longrightarrow & Z_{h_{l_{s_1}}} \end{array} .$$

Each square in this diagram commutes, because g and h are premorphisms and $g \sim h$. Thus $H_{l_{s_1}} S_{f_{r_1} l_{s_1}} = T_{u_{r_1} x_{s_1}} G_{f_{r_1}}$ (note that $u_{r_1} = g_{f_{r_1}}$ and $x_{s_1} = h_{l_{s_1}}$). Hence the diagram

$$\begin{array}{ccc} V_{r_1} & \longrightarrow & V_{s_1} \\ F_{r_1} \downarrow & & \downarrow L_{s_1} \\ W_{f_{r_1}} & \longrightarrow & W_{l_{s_1}} \\ G_{f_{r_1}} \downarrow & & \downarrow H_{l_{s_1}} \\ Z_{u_{r_1}} & \longrightarrow & Z_{x_{s_1}} \end{array}$$

commutes, and then we have $r_1 < s_1$, $u_{r_1} < x_{s_1}$, and $X_{s_1} E_{r_1 s_1} = T_{u_{r_1} x_{s_1}} U_{r_1}$.

There is $k_2 \geq 1$ such that $q_{k_2} > l_{s_1}$. There is $k_3 \geq 1$ with $f_{n_{k_3}} \geq p_{k_2+1}$; set $n_{k_3} = r_2$. Each square in the following diagram commutes:

$$\begin{array}{ccccccc} W_{l_{s_1}} & \longrightarrow & W_{q_{k_2}} & \longrightarrow & W_{p_{k_2+1}} & \longrightarrow & W_{f_{r_2}} \\ H_{l_{s_1}} \downarrow & & H_{q_{k_2}} \downarrow & & G_{p_{k_2+1}} \downarrow & & G_{f_{r_2}} \downarrow \\ Z_{h_{l_{s_1}}} & \longrightarrow & Z_{h_{q_{k_2}}} & \longrightarrow & Z_{g_{p_{k_2+1}}} & \longrightarrow & Z_{g_{f_{r_2}}} \end{array} .$$

Thus, we have $G_{f_{r_2}} S_{l_{s_1} f_{r_2}} = T_{x_{s_1} u_{r_2}} H_{l_{s_1}}$. Therefore the diagram

$$\begin{array}{ccc}
 V_{s_1} & \longrightarrow & V_{r_2} \\
 L_{s_1} \downarrow & & \downarrow F_{r_2} \\
 W_{f_{r_1}} & \longrightarrow & W_{l_{s_1}} \\
 H_{l_{s_1}} \downarrow & & \downarrow G_{f_{r_2}} \\
 Z_{x_{s_1}} & \longrightarrow & Z_{u_{r_2}}
 \end{array}$$

commutes, and then we have $s_1 < r_2$, $x_{s_1} < u_{r_2}$, and $U_{r_2} E_{s_1 r_2} = T_{x_{s_1} u_{r_2}} X_{s_1}$.

Continuing this procedure, we obtain sequences $(r_k)_{k=1}^\infty$ and $(s_k)_{k=1}^\infty$ such that $r_1 < s_1 < r_2 < s_2 < \dots$ and $u_{r_1} < x_{s_1} < u_{r_2} < x_{s_2} < \dots$ and the commutativity required in Definition 2.5 for $gf \sim hl$ is valid. Hence, $[gf] = [hl]$.

Finally, since by Proposition 2.4 BD with premorphisms is a category, it follows that composition of morphisms as defined, which we have shown is well defined, makes BD a category. ■

Let us refer to BD, with morphisms as defined above, as the *category of Bratteli diagrams*.

It is tempting to propose an alternative definition for equivalence of two premorphisms (cf. Definition 2.5) as follows: $f \sim g$ if the diagram containing both f and g commutes, in the sense that each triangle and each square in the diagram is commutative (alternatively, any two paths with the same endpoints agree). However, this relation is not transitive, even if we strengthen the definition of a premorphism to consist of only embedding matrices instead of multiplicity matrices. The point is that an embedding matrix is not necessarily injective, as is seen with the following example.

Example 2.8 The following embedding matrices E_1, E_2 , and E_3 are such that $E_3 E_1 = E_3 E_2$, but $E_1 \neq E_2$. Let $V_1 = (1)$, $V_2 = \binom{2}{2}$, $V_3 = (6)$, $E_1 = \binom{2}{2}$, $E_2 = \binom{0}{2}$, and $E_3 = (1\ 2)$. Thus, $E_1, E_2: V_1 \rightarrow V_2$ and $E_3: V_2 \rightarrow V_3$ are embedding matrices. We have $E_3 E_1 = E_3 E_2 = (4)$, but $E_1 \neq E_2$. The only thing we can say is that there is a unitary $u \in C^*(V_3)$ such that $h(E_3)h(E_1) = (\text{Ad } u)h(E_3)h(E_2)$, by Lemmas 3.4 and 3.5, but $h(E_1) \neq h(E_2)$, since $E_1 \neq E_2$. (See the remark following Lemma 3.4 for notation.)

Here are two correct alternative formulations of the definition for equivalence of premorphisms. We shall use the first one in a number of places later.

Definition 2.9 Let $f, g: B \rightarrow C$ be premorphisms in BD such that $B = (V, E)$, $C = (W, S)$, $f = ((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$, and $g = ((G_n)_{n=1}^\infty, (g_n)_{n=1}^\infty)$. Let us say that f is *equivalent* to g , in the second sense, if for each $n \geq 1$ there is an $m \geq f_n, g_n$ such that $S_{f_n m} F_n = S_{g_n m} G_n$.

Definition 2.10 Let $f, g: B \rightarrow C$ be premorphisms in BD such that $B = (V, E)$, $C = (W, S)$, $f = ((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$, and $g = ((G_n)_{n=1}^\infty, (g_n)_{n=1}^\infty)$. Let us say that f is *equivalent* to g , in the third sense, if for each $n \geq 1$ and for each $k \geq n$, there is an $m \geq f_n, g_k$ such that $S_{f_n m} F_n = S_{g_k m} G_k E_{nk}$.

Let us show that these two definitions are equivalent to Definition 2.5.

Proposition 2.11 *Definitions 2.5, 2.9, and 2.10 are equivalent.*

Proof The fact that these definitions are equivalent is based on the following observation. If we assume that a pair of premorphisms are equivalent in the sense of any one of these definitions, then the union of the corresponding diagrams is commutative at infinity, in the sense that any two paths with the same endpoints agree, after going sufficiently further out, *i.e.*, composing with a long enough subsequent path. In fact, this is, in each sense, just a reformulation of the definition. (But let us proceed, more prosaically perhaps, in cyclic order.)

Definition 2.5 implies Definition 2.9. Suppose that $f \sim g$ in the sense of Definition 2.5. Let $n \geq 1$ and assume that $f_n \leq g_n$. There is $k \geq 1$ such that $n_k \geq n$. Thus, $f_n \leq f_{n_k}$ and $g_n \leq g_{m_k}$. Using Definition 2.5 and the fact that f, g are premorphisms we have

$$\begin{aligned} S_{f_n g_{m_k}} F_n &= S_{f_{n_k} g_{m_k}} S_{f_n f_{n_k}} F_n \\ &= S_{f_{n_k} g_{m_k}} F_{n_k} E_{nn_k} \\ &= G_{m_k} E_{n_k m_k} E_{nn_k} \\ &= S_{g_n g_{m_k}} G_n. \end{aligned}$$

Therefore, if $f_n \leq g_n$ we have $S_{f_n g_{m_k}} F_n = S_{g_n g_{m_k}} G_n$. Similarly, if $g_n < f_n$, there is $l \geq 1$ such that $g_n \leq f_n \leq f_{m_1}$ and $S_{f_n f_{m_1}} F_n = S_{g_n f_{m_1}} G_n$. Set $\max(g_{m_k}, f_{m_1}) = m$. Then we have $S_{f_n m} F_n = S_{g_n m} G_n$.

Definition 2.9 implies Definition 2.10. Suppose that k, n are positive integers with $k \geq n$. Applying Definition 2.9, we get $m \geq f_k, g_k$ such that $S_{f_k m} F_k = S_{g_k m} G_k$. Using Definition 2.9 and the fact that f is a premorphism we have

$$\begin{aligned} S_{f_n m} F_n &= S_{f_k m} S_{f_n f_k} F_n \\ &= S_{f_k m} F_k E_{nk} \\ &= S_{g_k m} G_k E_{nk}. \end{aligned}$$

Therefore, $S_{f_n m} F_n = S_{g_k m} G_k E_{nk}$.

Definition 2.10 implies Definition 2.5. Set $n_1 = 1$. Applying Definition 2.10, one obtains $m \geq f_{n_1}, g_{n_1}$ such that $S_{f_{n_1} m} F_{n_1} = S_{g_{n_1} m} G_{n_1}$. Since $\{g_k \mid k \geq 1\}$ is cofinal in \mathbb{N} , there is $m_1 > n_1$ such that $g_{m_1} > m$. Using Definition 2.9 and the fact that g is a premorphism, we have

$$S_{f_{n_1} g_{m_1}} F_{n_1} = S_{g_{n_1} g_{m_1}} G_{n_1} = G_{m_1} E_{n_1 m_1}.$$

Continuing this procedure, we obtain sequences $(n_k)_{k=1}^\infty$ and $(m_k)_{k=1}^\infty$ of positive integers that satisfy Definition 2.5. ■

3 The Category of AF Algebras and the Functor \mathcal{B}

To define the category of AF algebras **AF**, in such a way that we will be able to define a functor \mathcal{B} from **AF** to **BD**, first we need to identify exactly what the Bratteli diagram of an AF algebra depends on.

Let $A = \overline{\bigcup_{n \geq 1} A_n}$ be an AF algebra, where $(A_n)_{n=1}^\infty$ is an increasing sequence of finite-dimensional C^* -subalgebras of A . Since there are infinitely many sequences with this property, we need to fix one of them. Each A_n is $*$ -isomorphic to a finite-dimensional C^* -algebra $A'_n = \mathcal{M}_{m_1} \oplus \cdots \oplus \mathcal{M}_{m_l}$ via the map $\phi_n: A_n \rightarrow A'_n$. We then obtain the $*$ -homomorphism $\phi'_n: A'_n \rightarrow A'_{n+1}$ with $\phi'_n = \phi_{n+1}\phi_n^{-1}$, and the following diagram commutes:

$$\begin{array}{ccc} A_n & \xrightarrow{\subset} & A_{n+1} \\ \phi_n \downarrow & & \downarrow \phi_{n+1} \\ A'_n & \xrightarrow{\phi'_n} & A'_{n+1}. \end{array}$$

By Theorem 2.1, there is a multiplicity matrix E_n corresponding to ϕ'_n ; that is, $E_n = R_{\phi'_n}$. But E_n depends on ϕ_n and ϕ_{n+1} , as a different choice of ϕ_n permuting identical direct summands of A'_n of course results in a different $R_{\phi'_n}$.

Definition 3.1 Let **AF** denote the category whose objects are all triples

$$(A, (A_n)_{n=1}^\infty, (\phi_n)_{n=1}^\infty),$$

where A is an AF algebra, $(A_n)_{n=1}^\infty$ is an increasing sequence of finite-dimensional C^* -subalgebras of A such that $\bigcup_{n=1}^\infty A_n$ is dense in A , and each ϕ_n is a $*$ -isomorphism from A_n onto a C^* -algebra $A'_n = \mathcal{M}_{m_1} \oplus \cdots \oplus \mathcal{M}_{m_l}$ for some m_1, \dots, m_l depending on A_n .

Let $\mathcal{A}_1 = (A, (A_n)_{n=1}^\infty, (\phi_n)_{n=1}^\infty)$ and $\mathcal{A}_2 = (B, (B_n)_{n=1}^\infty, (\psi_n)_{n=1}^\infty)$ be in **AF**. By a *morphism* ϕ from \mathcal{A}_1 to \mathcal{A}_2 let us just mean a $*$ -homomorphism from A to B . Then **AF** with morphisms thus defined is a category. Let us call **AF** with morphisms as defined the *category of AF algebras*.

Remark In Definition 3.1, we have fixed a sequence of $*$ -isomorphisms $(\phi_n)_{n=1}^\infty$ for the AF algebra $(A, (A_n)_{n=1}^\infty, (\phi_n)_{n=1}^\infty)$ to be able to associate a particular Bratteli diagram to the algebra. In [2], Bratteli also fixed a sequence of systems of matrix units for the AF algebra to be able to associate the diagram. These are equivalent procedures.

Next we quote a result of Bratteli with slight changes [2], which will be used to justify Definition 3.3, below, and in a number of places later. Before that, let us fix the following notation which will be used frequently. We need this to avoid restricting the results to just the unital case.

Notation Throughout this note, for a C^* -algebra A , we shall use two (minimal) unitizations \tilde{A} and A^+ as defined in [18]. In fact, when A is not unital, both of them are equal and contain A as a maximal ideal of codimension one. When A is unital,

$A^\sim = A$ but again A^+ contains A as a maximal ideal of codimension one. The units of A^\sim and A^+ will both be denoted by 1.

Lemma 3.2 *Let $A = \overline{\bigcup_{n \geq 1} A_n}$ be an AF algebra where $(A_n)_{n=1}^\infty$ is an increasing sequence of finite-dimensional C^* -subalgebras of A . Let B be a finite-dimensional C^* -subalgebra of A . Then for each $\varepsilon > 0$ there is a unitary $u \in A^\sim$ with $\|u - 1\| < \varepsilon$ and a positive integer n such that $uBu^* \subseteq A_n$.*

Proof In the unital case, the lemma is essentially [2, Lemma 2.3]. To deal with the non-unital case, just add a unit to both algebras and use the same lemma. ■

Definition 3.3 Let $\mathcal{A} = (A, (A_n)_{n=1}^\infty, (\phi_n)_{n=1}^\infty)$ be in the category **AF**, and let us define $\mathcal{B}(\mathcal{A})$ in **BD** as follows. Consider the given isomorphisms $\phi_n: A_n \rightarrow A'_n$ and define $\phi'_n: A'_n \rightarrow A'_{n+1}$ by $\phi'_n = \phi_{n+1}\phi_n^{-1}$, for each $n \geq 1$; then the following diagram commutes, for each n :

$$\begin{array}{ccc} A_n & \hookrightarrow & A_{n+1} \\ \phi_n \downarrow & & \downarrow \phi_{n+1} \\ A'_n & \xrightarrow{\phi'_n} & A'_{n+1} \end{array}$$

Write $A'_n = \mathcal{M}_{m_{n1}} \oplus \cdots \oplus \mathcal{M}_{m_{nk_n}}$, and set

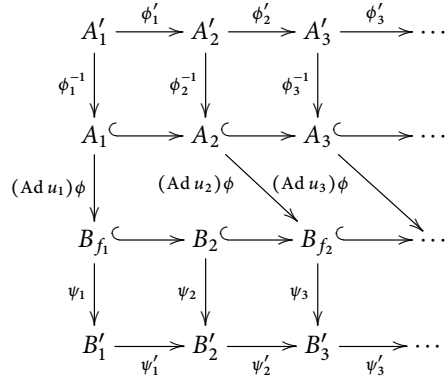
$$V_n = \begin{pmatrix} m_{n1} \\ \vdots \\ m_{nk_n} \end{pmatrix}.$$

By Theorem 2.1, there is a unique embedding matrix E_n corresponding to ϕ'_n ; that is, $E_n = R_{\phi'_n}$. Set $\mathcal{B}(\mathcal{A}) = ((V_n)_{n=1}^\infty, (E_n)_{n=1}^\infty)$.

Let $\phi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ be a morphism in **AF** where $\mathcal{A}_1 = (A, (A_n)_{n=1}^\infty, (\phi_n)_{n=1}^\infty)$ and $\mathcal{A}_2 = (B, (B_n)_{n=1}^\infty, (\psi_n)_{n=1}^\infty)$, in other words, a $*$ -homomorphism from A to B . Define $\mathcal{B}(\phi): \mathcal{B}(\mathcal{A}_1) \rightarrow \mathcal{B}(\mathcal{A}_2)$ as follows. There is an $f_1 \geq 1$ and a unitary $u_1 \in B^\sim$ such that $u_1\phi(A_1)u_1^* \subseteq B_{f_1}$ and $\|u_1 - 1\| < \frac{1}{2}$, by Lemma 3.2. Let $g_1: A_1 \rightarrow B_{f_1}$ be such that $g_1 = (\text{Ad } u_1) \circ \phi \upharpoonright_{A_1}$. Now define $\eta_1: A'_1 \rightarrow B'_{f_1}$ by $\eta_1 = \psi_{f_1}g_1\phi_1^{-1}$. Denote by F_1 the multiplicity matrix corresponding to η_1 , according to Theorem 2.1; that is, $F_1 = R_{\eta_1}$.

Similarly, choose unitaries u_2, u_3, \dots in B^\sim and positive integers f_2, f_3, \dots with $f_1 \leq f_2 \leq \dots$ such that $u_n\phi(A_n)u_n^* \subseteq B_{f_n}$, $\|u_n - 1\| < \frac{1}{2}$, for each $n \geq 1$, and the sequence $(f_n)_{n=1}^\infty$ is cofinal in \mathbb{N} . (The condition $\|u_n - 1\| < \frac{1}{2}$ is important and will be used in the proof of Proposition 3.8.) Let $g_n: A_n \rightarrow B_{f_n}$ be such that $g_n = (\text{Ad } u_n) \circ \phi \upharpoonright_{A_n}$ and define $\eta_n: A'_n \rightarrow B'_{f_n}$ by $\eta_n = \psi_{f_n}g_n\phi_n^{-1}$; set $R_{\eta_n} = F_n$. We will show that $((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$ is a premorphism from $\mathcal{B}(\mathcal{A}_1)$ to $\mathcal{B}(\mathcal{A}_2)$ (see Proposition 3.8). Denote by $\mathcal{B}(\phi)$ the equivalence class of the premorphism $((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$ (as in Definition 2.5). The following (*a priori* non-commutative) diagram illustrates the idea

of this definition:



In an obvious way, the diagram associated to the zero algebra in \mathbf{AF} is the zero Bratteli diagram, and the morphism associated to the zero homomorphism is the zero morphism.

We shall need a number of lemmas to show that $\mathcal{B}: \mathbf{AF} \rightarrow \mathbf{BD}$ is well defined, *i.e.*, that $\mathcal{B}(\phi)$ is independent of the choice of sequences $(f_n)_{n=1}^\infty$ and $(u_n)_{n=1}^\infty$. The first two lemmas are well known in the special case of injective $*$ -homomorphisms. Before that, let us introduce an important notion (or perhaps just notation!) that will be used frequently.

Remark Let $\phi: A \rightarrow B$ be a $*$ -homomorphism between finite-dimensional C^* -algebras. By the multiplicity matrix of ϕ , R_ϕ , we mean that there have been implicitly fixed two $*$ -isomorphisms $\phi_1: A \rightarrow A'$ and $\phi_2: B \rightarrow B'$, where A' and B' are finite-dimensional C^* -algebras $\mathcal{M}_{n_1} \oplus \dots \oplus \mathcal{M}_{n_k}$ and $\mathcal{M}_{m_1} \oplus \dots \oplus \mathcal{M}_{m_l}$, respectively, and R_ϕ is the multiplicity matrix of $\phi_2 \phi \phi_1^{-1}: A' \rightarrow B'$, according to Theorem 2.1.

Lemma 3.4 Let $\phi: A \rightarrow B$ and $\psi: B \rightarrow C$ be $*$ -homomorphisms between finite-dimensional C^* -algebras. Then $R_{\psi\phi} = R_\psi R_\phi$.

Proof One can give a proof for the case of injective $*$ -homomorphisms using the matrix units [14, Lemma 15.3.2], and it is easy to conclude it for the general case using the matrix language and Theorem 2.1. ■

Remark Let V_i be a column matrix of non-zero positive integers for $i = 1, 2$ and let $E: V_1 \rightarrow V_2$ be a multiplicity matrix. Write $V_1^T = (n_1 \dots n_k)$ and $V_2^T = (m_1 \dots m_l)$. Let $C^*(V_1)$ denote the C^* -algebra $\mathcal{M}_{n_1} \oplus \dots \oplus \mathcal{M}_{n_k}$ and similarly for $C^*(V_2)$. Write $E = (a_{ij})$. Then there is a canonical $*$ -homomorphism $h(E): C^*(V_1) \rightarrow C^*(V_2)$ which is defined by

$$\pi_i(h(E)(u_1, \dots, u_k)) = \begin{pmatrix} u_1^{(a_{i1})} & & & 0 \\ & \ddots & & \\ & & u_k^{(a_{ik})} & \\ 0 & & & 0^{(s_i)} \end{pmatrix} \text{ for } (u_1, \dots, u_k) \in C^*(V_1),$$

where s_i is such that $\sum_{j=1}^k a_{ij}n_j + s_i = m_i$, for each $1 \leq i \leq l$. Recall that for a unital C^* -algebra A and a unitary element u in A , the $*$ -isomorphism $\text{Ad } u: A \rightarrow A$ is defined by $(\text{Ad } u)(a) = uau^*$ ($a \in A$). In particular, the conclusion of Theorem 2.1 could be summarized as $\phi = (\text{Ad } u)h(E)$, where $u = v^*$.

We shall need the following lemma in a number of places later. It is proved in [17, Theorem I.11.9], and also is obvious in view of Theorem 2.1 and the remark above.

Lemma 3.5 *Let $\phi, \psi: A \rightarrow B$ be $*$ -homomorphisms between finite-dimensional C^* -algebras. Then $R_\phi = R_\psi$ if and only if there is a unitary u in B such that $\phi = (\text{Ad } u)\psi$.*

The following corollary (used, if not explicitly stated, in [2]) is given in the case of injective $*$ -homomorphisms in [14, Lemma 15.3.2].

Corollary 3.6 *Let V_i be a column matrix of non-zero positive integers for $1 \leq i \leq 4$. Let $E_1: V_1 \rightarrow V_2, E_2: V_3 \rightarrow V_4, E_3: V_1 \rightarrow V_3$, and $E_4: V_2 \rightarrow V_4$ be multiplicity matrices such that the diagram*

$$\begin{array}{ccc} V_1 & \xrightarrow{E_1} & V_2 \\ E_3 \downarrow & & \downarrow E_4 \\ V_3 & \xrightarrow{E_2} & V_4 \end{array}$$

commutes; that is, $E_4E_1 = E_2E_3$. Let $\phi_1: C^*(V_1) \rightarrow C^*(V_2), \phi_2: C^*(V_3) \rightarrow C^*(V_4), \phi_3: C^*(V_1) \rightarrow C^*(V_3)$, and $\phi_4: C^*(V_2) \rightarrow C^*(V_4)$ be $*$ -homomorphisms such that $R_{\phi_i} = E_i$ for $1 \leq i \leq 4$. Then there is a unitary $u \in C^*(V_4)$ such that the following diagram commutes:

$$\begin{array}{ccc} C^*(V_1) & \xrightarrow{\phi_1} & C^*(V_2) \\ \phi_3 \downarrow & & \downarrow (\text{Ad } u)\phi_4 \\ C^*(V_3) & \xrightarrow{\phi_2} & C^*(V_4); \end{array}$$

i.e., $(\text{Ad } u)\phi_4\phi_1 = \phi_2\phi_3$.

Proof By Lemma 3.4 we have $R_{\phi_4\phi_1} = E_4E_1 = E_2E_3 = R_{\phi_2\phi_3}$. Thus by Lemma 3.5, there is a unitary $u \in C^*(V_4)$ such that $(\text{Ad } u)\phi_4\phi_1 = \phi_2\phi_3$. ■

Next we give a slight modification of [2, Lemma 2.4]. Since our $*$ -homomorphisms are not assumed to be unital, we prove a non-unital version which is suitable for our purposes. This lemma gives a criterion to check whether two $*$ -homomorphisms between finite-dimensional C^* -algebras have the same multiplicity matrices. We will use it to show that the functor $\mathcal{B}: \mathbf{AF} \rightarrow \mathbf{BD}$ is well defined (in the proof of Proposition 3.8) and in a number of places later.

Lemma 3.7 *Let $\phi, \psi: A \rightarrow B$ be $*$ -homomorphisms between finite-dimensional C^* -algebras such that $\|\phi - \psi\| < 1$. Then there is a unitary u in B such that $\phi = (\text{Ad } u)\psi$ and hence $R_\phi = R_\psi$.*

Proof Let $\{e_{ij}^l : 1 \leq l \leq k, 1 \leq i, j \leq n_l\}$ be a set of matrix units for A . Set $p_{ij}^l = \phi(e_{ij}^l)$ and $q_{ij}^l = \psi(e_{ij}^l)$, for each $1 \leq l \leq k$ and $1 \leq i, j \leq n_l$. Fix $1 \leq l \leq k$. Since $\|p_{11}^l - q_{11}^l\| < 1$, by [11, Lemma 1.8] there is a partial isometry $w_l \in B$ such that $p_{11}^l = w_l w_l^*$ and $q_{11}^l = w_l^* w_l$. Then $w_l q_{11}^l w_l^* = p_{11}^l$. Set

$$\sum_{l=1}^k \sum_{i=1}^{n_l} p_{i1}^l w_l q_{1i}^l = w.$$

Thus we have $w q_{ij}^l w^* = p_{ij}^l$, for the above values of i, j, l . Also we have $w w^* = \phi(1)$ and $w^* w = \psi(1)$, and so w is a partial isometry from $\psi(1)$ to $\phi(1)$. If ϕ and ψ are unital, the proof is complete at this point. Since B is finite-dimensional, there is a partial isometry $v \in B$ such that $w + v$ is unitary and $w v^* = w^* v = 0$; set $w + v = u$. We have $v q_{ij}^l = v \psi(1) q_{ij}^l = v w^* w q_{ij}^l = 0$, and similarly $q_{ij}^l v^* = 0$, for the above values of i, j, l . Therefore, $u q_{ij}^l u^* = w q_{ij}^l w^* = p_{ij}^l$, and so $\phi = (\text{Ad } u)\psi$. ■

Proposition 3.8 $\mathcal{B}: \mathbf{AF} \rightarrow \mathbf{BD}$ is a functor.

Proof First, let us show that \mathcal{B} is well defined. Following the notation of Definition 3.3, we need to show first that $((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$ is a premorphism from $\mathcal{B}(A_1) = (V, E)$ to $\mathcal{B}(A_2) = (W, S)$. Fix $n \geq 1$. Consider the following (*a priori* non-commutative) diagram:

$$\begin{array}{ccc} A_n & \hookrightarrow & A_{n+1} \\ g_n \downarrow & & \downarrow g_{n+1} \\ B_{f_n} & \hookrightarrow & B_{f_{n+1}} \end{array}$$

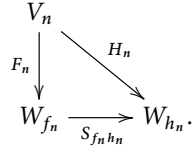
Since $\|u_n - 1\| < \frac{1}{2}$, we have $\|g_n - g_{n+1}\|_{A_n} \leq \|(\text{Ad } u_n)\phi - (\text{Ad } u_{n+1})\phi\| \leq 2\|u_n - u_{n+1}\| < 1$, for each $n \geq 1$. Applying Lemma 3.7 for $g_n: A_n \rightarrow B_{f_{n+1}}$ and $g_{n+1} \upharpoonright_{A_n}: A_n \rightarrow B_{f_{n+1}}$, we get $F_{n+1}E_n = S_{f_n, f_{n+1}}F_n$. This shows that $((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$ is a premorphism.

To check that the morphism $\mathcal{B}(\phi)$ is well defined we also need to show that it is independent of the choice of u_n 's and f_n 's. Therefore, let $(v_n)_{n=1}^\infty$ be another sequence of unitaries in B^\sim and $(h_n)_{n=1}^\infty$ an increasing cofinal sequence of positive integers such that $v_n \phi(A_n) v_n^* \subseteq B_{h_n}$ and $\|v_n - 1\| < \frac{1}{2}$, for each $n \geq 1$. Let $k_n: A_n \rightarrow B_{h_n}$ be such that $k_n = (\text{Ad } v_n) \circ \phi \upharpoonright_{A_n}$ and set $H_n = R_{k_n}$, the multiplicity matrix of k_n . We have to show that the two premorphisms $((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$ and $((H_n)_{n=1}^\infty, (h_n)_{n=1}^\infty)$ are equivalent.

Fix $n \geq 1$. We may assume, without loss of generality, that $f_n \leq h_n$. Then we have the following (*a priori* non-commutative) diagram:

$$\begin{array}{ccc} A_n & & \\ g_n \downarrow & \searrow k_n & \\ B_{f_n} & \hookrightarrow & B_{h_n} \end{array}$$

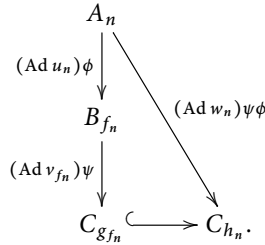
We have $\|g_n - k_n\| \leq \|(\text{Ad } u_n)\phi - (\text{Ad } v_n)\phi\| \leq 2\|u_n - v_n\| < 1$. Applying Lemma 3.7 for $g_n: A_n \rightarrow B_{h_n}$ and $k_n: A_n \rightarrow B_{h_n}$, we conclude that the following diagram is commutative:



Therefore, the premorphisms $((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$ and $((H_n)_{n=1}^\infty, (h_n)_{n=1}^\infty)$ are equivalent, by Proposition 2.11. This completes the proof that the map \mathcal{B} is well defined.

Suppose that $\phi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ and $\psi: \mathcal{A}_2 \rightarrow \mathcal{A}_3$ are morphisms in \mathbf{AF} . Let us show that $\mathcal{B}(\psi\phi) = \mathcal{B}(\psi)\mathcal{B}(\phi)$. Write $\mathcal{A}_1 = (A, (A_n)_{n=1}^\infty, (\phi_n)_{n=1}^\infty)$, $\mathcal{A}_2 = (B, (B_n)_{n=1}^\infty, (\psi_n)_{n=1}^\infty)$, $\mathcal{A}_3 = (C, (C_n)_{n=1}^\infty, (\eta_n)_{n=1}^\infty)$, $\mathcal{B}(\mathcal{A}_1) = (V, E)$, $\mathcal{B}(\mathcal{A}_2) = (W, S)$, and $\mathcal{B}(\mathcal{A}_3) = (Z, T)$. Thus, ϕ and ψ are $*$ -homomorphisms from A to B and from B to C , respectively. Choose a sequence of unitaries $(u_n)_{n=1}^\infty$ in B^\sim with $\|u_n - 1\| < \frac{1}{4}$, $n \in \mathbb{N}$, and a sequence of positive integers $(f_n)_{n=1}^\infty$ which construct a premorphism $((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$ for $\mathcal{B}(\phi)$, according to Definition 3.3. Similarly, choose $(v_n)_{n=1}^\infty$ with $\|v_n - 1\| < \frac{1}{8}$, $n \in \mathbb{N}$, and $(g_n)_{n=1}^\infty$ which give a premorphism $((G_n)_{n=1}^\infty, (g_n)_{n=1}^\infty)$ for $\mathcal{B}(\psi)$ and choose $(w_n)_{n=1}^\infty$ with $\|w_n - 1\| < \frac{1}{8}$, $n \in \mathbb{N}$, and $(h_n)_{n=1}^\infty$ which give a premorphism $((H_n)_{n=1}^\infty, (h_n)_{n=1}^\infty)$ for $\mathcal{B}(\psi\phi)$, according to Definition 3.3.

Fix a positive integer n . Then $(\text{Ad } u_n)\phi(A_n) \subseteq B_{f_n}$, $(\text{Ad } v_{f_n})\psi(B_{f_n}) \subseteq C_{g_{f_n}}$, and $(\text{Ad } w_n)\psi\phi(A_n) \subseteq C_{h_n}$. We may assume, without loss of generality, that $g_{f_n} \leq h_n$. Then we have the following (*a priori* non-commutative) diagram:



Let us estimate the distance between the $*$ -homomorphisms

$$(\text{Ad } v_{f_n})\psi \circ (\text{Ad } u_n)\phi: A_n \rightarrow C_{h_n} \quad \text{and} \quad (\text{Ad } w_n)\psi\phi: A_n \rightarrow C_{h_n}.$$

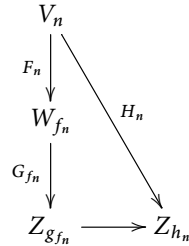
For any $x \in A$ we have

$$\begin{aligned}
 & \|(\text{Ad } v_{f_n})\psi \circ (\text{Ad } u_n)\phi(x) - (\text{Ad } w_n)\psi\phi(x)\| \\
 & \leq \|(\text{Ad } v_{f_n})\psi \circ (\text{Ad } u_n)\phi(x) - (\text{Ad } v_{f_n})\psi\phi(x)\| \\
 & \quad + \|(\text{Ad } v_{f_n})\psi\phi(x) - (\text{Ad } w_n)\psi\phi(x)\| \\
 & \leq \|(\text{Ad } u_n)\phi(x) - \phi(x)\| + 2\|v_{f_n} - w_n\|\|x\| \\
 & \leq 2\|u_n - 1\|\|x\| + 2\|v_{f_n} - w_n\|\|x\|.
 \end{aligned}$$

Thus,

$$\|(\text{Ad } v_{f_n})\psi \circ (\text{Ad } u_n)\phi - (\text{Ad } w_n)\psi\phi\| \leq 2\|u_n - 1\| + 2\|v_{f_n} - w_n\| < \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1.$$

By Lemma 3.7, the homomorphisms $(\text{Ad } v_{f_n})\psi \circ (\text{Ad } u_n)\phi$ and $(\text{Ad } w_n)\psi\phi$ have the same multiplicity matrices. Thus, the diagram

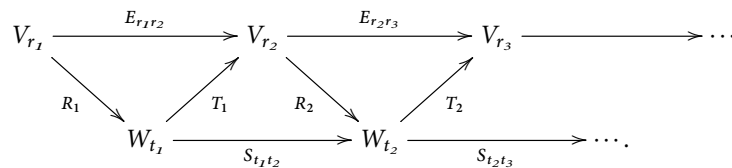


is commutative. By Definition 2.5, the premorphism $((H_n)_{n=1}^\infty, (h_n)_{n=1}^\infty)$ is equivalent to the composition of $((G_n)_{n=1}^\infty, (g_n)_{n=1}^\infty)$ and $((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$. Note that this composition of premorphisms is an admissible premorphism for $\psi\phi$ in the sense required in Definition 3.3, since $\|w_{f_n}u_n - 1\| < \frac{1}{4} + \frac{1}{8} < \frac{1}{2}$. By Definition 3.3 (which is vindicated by the fact, proved above, that $\mathcal{B}(\psi\phi)$ defined in this way is well defined), this composition represents $\mathcal{B}(\psi\phi)$, and so $\mathcal{B}(\psi\phi) = \mathcal{B}(\psi)\mathcal{B}(\phi)$.

The remaining condition, $\mathcal{B}(\text{id}_A) = \text{id}_{\mathcal{B}(A)}$, is clear. ■

The following lemma is used to prove Theorem 3.11. The hypothesis of this lemma is just Bratteli’s notion of equivalence of Bratteli diagrams in his paper [2], which is easily seen to be the same as isomorphism in our category. We include a proof for the convenience of the reader.

Lemma 3.9 *Let $\mathcal{A}_1 = (A, (A_n)_{n=1}^\infty, (\phi_n)_{n=1}^\infty)$, $\mathcal{A}_2 = (B, (B_n)_{n=1}^\infty, (\psi_n)_{n=1}^\infty)$ be in **AF** with $\mathcal{B}(\mathcal{A}_1) = ((V_n)_{n=1}^\infty, (E_n)_{n=1}^\infty)$, and $\mathcal{B}(\mathcal{A}_2) = ((W_n)_{n=1}^\infty, (S_n)_{n=1}^\infty)$. Then $\mathcal{A}_1 \cong \mathcal{A}_2$ in **AF** if and only if there are sequences $(r_k)_{k=1}^\infty$ and $(t_k)_{k=1}^\infty$ of positive integers with $r_1 < t_1 < r_2 < t_2 < \dots$, and there are multiplicity matrices $R_k: V_{r_k} \rightarrow W_{t_k}$ and $T_k: W_{t_k} \rightarrow V_{r_{k+1}}$, for each $k \geq 1$, such that the following diagram commutes:*



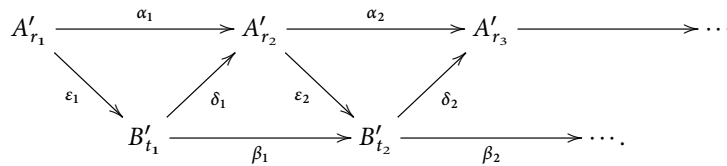
Proof First suppose that $\mathcal{A}_1 \cong \mathcal{A}_2$. There is a $*$ -isomorphism $\phi: A \rightarrow B$ such that $\phi(\bigcup_{n=1}^\infty A_n) = \bigcup_{n=1}^\infty B_n$, by [2, Lemma 2.6]. Set $r_1 = 1$. Since $\phi(\bigcup_{n=1}^\infty A_n) = \bigcup_{n=1}^\infty B_n$, there is $t_1 > 1$ with $\phi(A_{r_1}) \subseteq B_{t_1}$. Similarly there is $r_2 > t_1$ such that $\phi^{-1}(B_{t_1}) \subseteq A_{r_2}$. Continuing this procedure, we obtain sequences $(r_k)_{k=1}^\infty$ and $(t_k)_{k=1}^\infty$ with $r_1 < t_1 < r_2 < t_2 < \dots$ such that $\phi(A_{r_k}) \subseteq B_{t_k}$ and $\phi^{-1}(B_{t_k}) \subseteq A_{r_{k+1}}$, for each $k \geq 1$. Note that $C^*(V_n) = A'_n$ and $C^*(W_n) = B'_n$, for $n \geq 1$. Fix $k \geq 1$. Define $\varepsilon_k: A'_{r_k} \rightarrow$

B'_{t_k} and $\delta_k: B'_{t_k} \rightarrow A'_{r_{k+1}}$ with $\varepsilon_k(x) = \psi_{t_k}(\phi(\phi_{r_k}^{-1}(x)))$, $x \in A'_{r_k}$, and $\delta_k(x) = \phi_{r_{k+1}}(\phi^{-1}(\psi_{t_k}^{-1}(x)))$, $x \in B'_{t_k}$. Set

$$\alpha_k = \phi'_{r_{k+1}-1} \phi'_{r_{k+1}-2} \cdots \phi'_{r_k} \quad \text{and} \quad \beta_k = \psi'_{t_{k+1}-1} \psi'_{t_{k+1}-2} \cdots \psi'_{t_k}.$$

Then $\delta_k \varepsilon_k = \alpha_k$ and $\varepsilon_{k+1} \delta_k = \beta_k$. Set $R_k = R_{\varepsilon_k}$ and $T_k = R_{\delta_k}$. By Lemma 3.4 we have $T_k R_k = E_{r_k r_{k+1}}$ and $R_{k+1} T_k = S_{t_k t_{k+1}}$, i.e., the above diagram commutes.

Now let us prove the converse. Define α_k and β_k as above. Let $\varepsilon_1 = h(R_1)$ and $\delta'_1 = h(T_1)$ (see the remark following Lemma 3.4 for the notation $h(\cdot)$). By Corollary 3.6, there is a unitary $u \in A'_{r_2}$ such that $(\text{Ad } u)\delta'_1 \varepsilon_1 = \alpha_1$; set $\delta_1 = (\text{Ad } u)\delta'_1$, thus $\delta_1 \varepsilon_1 = \alpha_1$. Set $\varepsilon'_2 = h(R_2)$. Again by Corollary 3.6, there is a unitary $v \in B'_{t_2}$ such that $(\text{Ad } v)\varepsilon'_2 \delta_1 = \beta_1$; set $\varepsilon_2 = (\text{Ad } v)\varepsilon'_2$, thus $\varepsilon_2 \delta_1 = \beta_1$. Continuing this procedure, we obtain injective $*$ -homomorphisms $\varepsilon_k: A'_{r_k} \rightarrow B'_{t_k}$ and $\delta_k: B'_{t_k} \rightarrow A'_{r_{k+1}}$, for each $k \geq 1$, such that the following diagram commutes:



Let $A' = \varinjlim(A'_{r_k}, \alpha_k)$ and $B' = \varinjlim(B'_{t_k}, \beta_k)$. Thus there is an injective $*$ -homomorphism $\varepsilon: A' \rightarrow B'$. Let $\alpha^k: A'_{r_k} \rightarrow A'$ and $\beta^k: B'_{t_k} \rightarrow B'$ be the $*$ -homomorphisms that come from the construction of the direct limit; thus $\alpha^{k+1}\alpha_k = \alpha^k$, $\beta^{k+1}\beta_k = \beta^k$, and $\varepsilon\alpha^k = \beta^k\varepsilon_k$, for each $k \geq 1$. We have $\beta^k = \beta^{k+1}\beta_k = \beta^{k+1}\varepsilon_{k+1}\delta_k = \varepsilon\alpha^{k+1}\delta_k$, hence $\beta^k(B'_{t_k}) \subseteq \varepsilon(A')$, for each $k \geq 1$; thus ε is also onto and hence is a $*$ -isomorphism. Moreover,

$$A \cong \varinjlim(A'_k, \alpha_k) \cong A' \cong B' \cong \varinjlim(B'_k, \alpha_k) \cong B.$$

Thus there is a $*$ -isomorphism $\phi: A \rightarrow B$ such that $\phi(\bigcup_{n=1}^\infty A_n) \subseteq \bigcup_{n=1}^\infty B_n$. Therefore, $\mathcal{A}_1 \cong \mathcal{A}_2$ in **AF**. ■

The following theorem is due to Bratteli ([2], [4, Proposition III.2.7]).

Theorem 3.10 (Bratteli) *If $A = \overline{\bigcup_{n \geq 1} A_n}$ and $B = \overline{\bigcup_{n \geq 1} B_n}$ have the same Bratteli diagrams, then they are isomorphic.*

In fact, as indicated in Lemma 3.9 above, Bratteli proved more. In the setting of Theorem 3.10, he showed that if the Bratteli diagram of A is equivalent, in his sense (which is exactly the same as being isomorphic, in our sense, i.e., in the category of Bratteli diagrams of Theorem 2.7) to the Bratteli diagram of B , then A is isomorphic to B .

Recall that a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ was called in [10] a *classification functor* if $F(a) \cong F(b)$ implies $a \cong b$, for each $a, b \in \mathcal{C}$, and a *strong classification functor* if each isomorphism from $F(a)$ onto $F(b)$ is the image of an isomorphism from a to b . With these concepts, one has a functorial formulation of Bratteli's theorem.

Theorem 3.11 *The functor $\mathcal{B}: \mathbf{AF} \rightarrow \mathbf{BD}$ is a strong classification functor.*

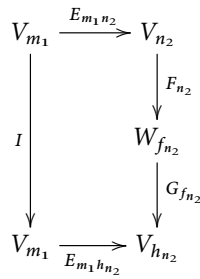
Proof Let $\mathcal{A}_1 = (A, (A_n)_{n=1}^\infty, (\phi_n)_{n=1}^\infty)$ and $\mathcal{A}_2 = (B, (B_n)_{n=1}^\infty, (\psi_n)_{n=1}^\infty)$ in **AF** be such that $\mathcal{B}(\mathcal{A}_1) \cong \mathcal{B}(\mathcal{A}_2)$. Write $\mathcal{B}(\mathcal{A}_1) = ((V_n)_{n=1}^\infty, (E_n)_{n=1}^\infty)$ and $\mathcal{B}(\mathcal{A}_2) = ((W_n)_{n=1}^\infty, (S_n)_{n=1}^\infty)$. There are premorphisms $f: \mathcal{B}(\mathcal{A}_1) \rightarrow \mathcal{B}(\mathcal{A}_2)$ and $g: \mathcal{B}(\mathcal{A}_2) \rightarrow \mathcal{B}(\mathcal{A}_1)$ such that $[gf] = [\text{id}_{\mathcal{B}(\mathcal{A}_1)}]$ and $[fg] = [\text{id}_{\mathcal{B}(\mathcal{A}_2)}]$.

Suppose that $f = ((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$, $g = ((G_n)_{n=1}^\infty, (g_n)_{n=1}^\infty)$, $h = gf$, and $l = fg$ where $h = ((H_n)_{n=1}^\infty, (h_n)_{n=1}^\infty)$ and $l = ((L_n)_{n=1}^\infty, (l_n)_{n=1}^\infty)$. Choose sequences $(n_k)_{k=1}^\infty$ and $(m_k)_{k=1}^\infty$ for $gf \sim \text{id}_{\mathcal{B}(\mathcal{A}_1)}$ and choose $(p_k)_{k=1}^\infty$ and $(q_k)_{k=1}^\infty$ for $fg \sim \text{id}_{\mathcal{B}(\mathcal{A}_2)}$, according to Definition 2.5. Thus we have

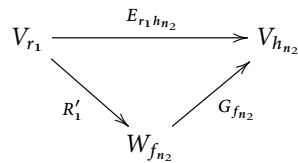
$$(3.1) \quad E_{n_k m_j} = E_{h_{n_k} m_j} H_{n_k}, \quad H_{n_j} E_{m_k n_j} = E_{m_k h_{n_j}},$$

$$(3.2) \quad S_{p_k q_j} = S_{l_{p_k} q_j} L_{p_k}, \quad L_{p_j} S_{q_k p_j} = S_{q_k l_{p_j}},$$

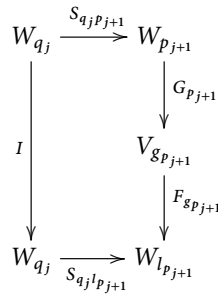
for any positive integers k and j with $k \leq j$. We construct sequences $(r_k)_{k=1}^\infty$ and $(t_k)_{k=1}^\infty$ to apply Lemma 3.9. Set $r_1 = m_1$. The diagram



commutes by Equation (3.1), where I is the identity matrix with suitable size. Set $R'_1 = F_{n_2} E_{m_1 n_2}$. Thus the diagram



commutes. There is $j \geq 1$ such that $q_j > f_{n_2}$. The diagram



commutes, by Equation (3.2). Set $T'_1 = G_{p_{j+1}} S_{q_j l_{p_{j+1}}}$. Then the diagram

$$\begin{array}{ccc} & V_{g_{p_{j+1}}} & \\ T'_1 \nearrow & & \searrow F_{g_{p_{j+1}}} \\ W_{q_j} & \xrightarrow{S_{q_j l_{p_{j+1}}}} & W_{l_{p_{j+1}}} \end{array}$$

commutes. Since g is a premorphism we have

$$T'_1 S_{f_{n_2} q_j} = G_{p_{j+1}} S_{q_j l_{p_{j+1}}} S_{f_{n_2} q_j} = G_{p_{j+1}} S_{f_{n_2} p_{j+1}} = E_{h_{n_2} g_{p_{j+1}}} G_{f_{n_2}}.$$

Hence, the following diagram is commutative:

$$\begin{array}{ccccccc} V_{r_1} & \xrightarrow{E_{r_1 h_{n_2}}} & V_{h_{n_2}} & \xrightarrow{E_{h_{n_2} g_{p_{j+1}}}} & V_{g_{p_{j+1}}} & & \\ & \searrow R'_1 & \nearrow G_{f_{n_2}} & & \nearrow T'_1 & \searrow F_{g_{p_{j+1}}} & \\ & & W_{f_{n_2}} & \xrightarrow{S_{f_{n_2} q_j}} & W_{q_j} & \xrightarrow{S_{q_j l_{p_{j+1}}}} & W_{l_{p_{j+1}}} \end{array}$$

Set $t_1 = q_j$ and $R_1 = S_{f_{n_2} q_j} R'_1$. Then the diagram

$$\begin{array}{ccc} V_{r_1} & \xrightarrow{E_{r_1 h_{n_2}}} & V_{g_{p_{j+1}}} \\ & \searrow R_1 & \nearrow T'_1 \\ & & W_{t_1} \xrightarrow{S_{t_1 l_{p_{j+1}}}} W_{l_{p_{j+1}}} \end{array}$$

is commutative. Continuing this procedure, we obtain sequences $(r_k)_{k=1}^\infty$ and $(t_k)_{k=1}^\infty$ of positive integers with $r_1 < t_1 < r_2 < t_2 < \dots$, and multiplicity matrices $R_k: V_{r_k} \rightarrow W_{t_k}$ and $T_k: W_{t_k} \rightarrow V_{r_{k+1}}$, for each $k \geq 1$, such that all the diagrams in Lemma 3.9 commute. In fact, by the construction, for each $k \geq 1$ there are positive integers x_k and y_k with $r_k \leq x_k \leq t_k$ and $t_k \leq y_k \leq r_{k+1}$ such that $R_k = S_{f_{x_k} t_k} F_{x_k} E_{r_k x_k}$ and $T_k = E_{g_{y_k} r_{k+1}} G_{y_k} S_{t_k y_k}$. Therefore, by Lemma 3.9, $\mathcal{A}_1 \cong \mathcal{A}_2$.

To show that the classification functor $\mathcal{B}: \mathbf{AF} \rightarrow \mathbf{BD}$ is strong, note that the $*$ -isomorphism $\phi: A \rightarrow B$ given by Lemma 3.9 satisfies $\phi(\cup_{n=1}^\infty A_n) \subseteq \cup_{n=1}^\infty B_n$, and therefore f above is admissible in Definition 3.3, so that $\mathcal{B}(\phi) = [f]$.

An alternative proof can be given using [10, Theorem 3] as follows. By Theorem 5.9 we have $\mathcal{B} = \overline{\mathcal{B}} \overline{\mathcal{F}}$ and $\overline{\mathcal{B}}$ is a strong classification functor, and by [10, Theorem 3] so also is $\overline{\mathcal{F}}$. This shows that \mathcal{B} is the composition of two strong classification functors and so it is also a strong classification functor. ■

Corollary 3.12 *Let $\mathcal{A}_1, \mathcal{A}_2 \in \mathbf{AF}$. Then $\mathcal{A}_1 \cong \mathcal{A}_2$ in \mathbf{AF} if and only if $\mathcal{B}(\mathcal{A}_1) \cong \mathcal{B}(\mathcal{A}_2)$ in \mathbf{BD} .*

Proof Since $\mathcal{B}: \mathbf{AF} \rightarrow \mathbf{BD}$ is a functor, $\mathcal{A}_1 \cong \mathcal{A}_2$ implies $\mathcal{B}(\mathcal{A}_1) \cong \mathcal{B}(\mathcal{A}_2)$. The converse follows from Theorem 3.11. ■

As an application of Theorem 3.11, we can give a proof of Glimm’s theorem. Recall that, with the notation of Definition 3.1, an AF algebra $\mathcal{A} = (A, (A_n)_{n=1}^\infty, (\phi_n)_{n=1}^\infty)$ in **AF** is called a *UHF algebra* if A is unital, each A_n contains the unit of A , and each A_n is a simple C^* -algebra; thus $A_n \cong \mathcal{M}_{k_n}$, for some $k_n \geq 1$. Let $\mathcal{B}(\mathcal{A}) = (V, E)$; thus $V_n = (k_n)$. According to Theorem 2.1, since the $*$ -homomorphism $\phi_{n+1}\phi_n^{-1}: \mathcal{M}_{k_n} \rightarrow \mathcal{M}_{k_{n+1}}$ is unital, we have $k_n|k_{n+1}$ and $E_n = R_{\phi_{n+1}\phi_n^{-1}} = k_{n+1}/k_n$. Therefore, the Bratteli diagram of \mathcal{A} is independent of the choice of ϕ_n ’s.

Denote by \mathbb{P} the set of all prime numbers. Define $\varepsilon_{\mathcal{A}}: \mathbb{P} \rightarrow \mathbb{N} \cup \{0, \infty\}$ with

$$\varepsilon_{\mathcal{A}}(p) = \sup\{m \geq 0 : p^m | k_n, \text{ for some } n \geq 1\}, \quad p \in \mathbb{P}.$$

The following famous result of Glimm is an easy consequence of Proposition 3.8 and Theorem 3.11.

Theorem 3.13 (Glimm, [11], Theorem 1.12) *Let $\mathcal{A}_1, \mathcal{A}_2 \in \mathbf{AF}$ be two UHF algebras. Then $\mathcal{A}_1 \cong \mathcal{A}_2$ if and only if $\varepsilon_{\mathcal{A}_1} = \varepsilon_{\mathcal{A}_2}$.*

Proof Let $\mathcal{A}_1 = (A, (A_n)_{n=1}^\infty, (\phi_n)_{n=1}^\infty)$ and $\mathcal{A}_2 = (B, (B_n)_{n=1}^\infty, (\psi_n)_{n=1}^\infty)$. Write $\mathcal{B}(\mathcal{A}_1) = (V, E)$ and $\mathcal{B}(\mathcal{A}_2) = (W, S)$. According to the remarks preceding this theorem, there are sequences $(k_n)_{n=1}^\infty$ and $(m_n)_{n=1}^\infty$ of natural numbers such that $V_n = (k_n)$ and $W_n = (m_n)$, for each $n \geq 1$, and $k_1|k_2|\dots$ and $m_1|m_2|\dots$. Note that $\varepsilon_{\mathcal{A}_1} = \varepsilon_{\mathcal{A}_2}$ if and only if

$$(3.3) \quad \forall n \geq 1 \exists l \geq 1 \quad k_n | m_l \quad \text{and} \quad \forall l \geq 1 \exists n \geq 1 \quad m_l | k_n.$$

Suppose that $\mathcal{A}_1 \cong \mathcal{A}_2$. Then by Proposition 3.8, $\mathcal{B}(\mathcal{A}_1) \cong \mathcal{B}(\mathcal{A}_2)$, i.e., there are premorphisms $f: \mathcal{B}(\mathcal{A}_1) \rightarrow \mathcal{B}(\mathcal{A}_2)$ and $g: \mathcal{B}(\mathcal{A}_2) \rightarrow \mathcal{B}(\mathcal{A}_1)$ such that $gf \sim \text{id}_{\mathcal{B}(\mathcal{A}_1)}$ and $fg \sim \text{id}_{\mathcal{B}(\mathcal{A}_2)}$, and Condition (3.3) follows.

Now suppose that $\varepsilon_{\mathcal{A}_1} = \varepsilon_{\mathcal{A}_2}$; thus, Condition (3.3) is satisfied. In other words, there are strictly increasing sequences $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ of non-zero positive integers such that $k_n | m_{f_n}$ and $m_n | k_{g_n}$, for each $n \geq 1$. Define premorphisms $f: \mathcal{B}(\mathcal{A}_1) \rightarrow \mathcal{B}(\mathcal{A}_2)$ and $g: \mathcal{B}(\mathcal{A}_2) \rightarrow \mathcal{B}(\mathcal{A}_1)$ with $f = ((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$ and $g = ((G_n)_{n=1}^\infty, (g_n)_{n=1}^\infty)$, where $F_n = (m_{f_n}/k_n)$ and $G_n = (k_{g_n}/m_n)$. It is easy to see that $gf \sim \text{id}_{\mathcal{B}(\mathcal{A}_1)}$ and $fg \sim \text{id}_{\mathcal{B}(\mathcal{A}_2)}$; thus, $\mathcal{B}(\mathcal{A}_1) \cong \mathcal{B}(\mathcal{A}_2)$. Therefore by Theorem 3.11 we have $\mathcal{A}_1 \cong \mathcal{A}_2$.

Alternatively (not using Theorem 3.11, but the functorial property of Proposition 3.8 is still used in the first half of the theorem), A and B can be seen each to have the structure of an infinite tensor product of matrix algebras of prime order, with the multiplicities of the primes determined by the Bratteli diagram data, from which isomorphism is immediate if the data is the same. ■

4 A Homomorphism Theorem

Let us consider further the properties of the functor $\mathcal{B}: \mathbf{AF} \rightarrow \mathbf{BD}$. The following result may be considered as a generalization of part of Theorem 3.11. Theorem 3.11 says that isomorphisms in the codomain category can be lifted back to isomorphisms in the domain category, and in particular to homomorphisms. The following theorem states this for arbitrary homomorphisms. (Theorem 3.11 cannot be deduced immediately

from Theorem 4.1; but see [10].) (The proofs of the two theorems are similar: roughly speaking, a two-sided and a one-sided intertwining argument.)

Theorem 4.1 *The functor $\mathcal{B}: \mathbf{AF} \rightarrow \mathbf{BD}$ is a full functor in the sense that if $\mathcal{A}_1, \mathcal{A}_2 \in \mathbf{AF}$ and $f: \mathcal{B}(\mathcal{A}_1) \rightarrow \mathcal{B}(\mathcal{A}_2)$ is a morphism in \mathbf{BD} , then there is a morphism $\phi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ in \mathbf{AF} such that $\mathcal{B}(\phi) = f$.*

Proof Let $\mathcal{A}_1, \mathcal{A}_2 \in \mathbf{AF}$ and $f: \mathcal{B}(\mathcal{A}_1) \rightarrow \mathcal{B}(\mathcal{A}_2)$ be a morphism in \mathbf{BD} . Write $\mathcal{A}_1 = (A, (A_n)_{n=1}^\infty, (\phi_n)_{n=1}^\infty)$, $\mathcal{A}_2 = (B, (B_n)_{n=1}^\infty, (\psi_n)_{n=1}^\infty)$, $\mathcal{B}(\mathcal{A}_1) = ((V_n)_{n=1}^\infty, (E_n)_{n=1}^\infty)$, and $\mathcal{B}(\mathcal{A}_2) = ((W_n)_{n=1}^\infty, (S_n)_{n=1}^\infty)$. Suppose that f is the equivalence class of the pre-morphism $((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$, according to Definition 2.3. Thus, each F_n is a multiplicity matrix from V_n to W_{f_n} and the following diagram commutes:

$$\begin{array}{ccccccc} V_1 & \xrightarrow{E_1} & V_2 & \xrightarrow{E_2} & V_3 & \xrightarrow{E_3} & \dots \\ \downarrow F_1 & & \downarrow F_2 & \swarrow F_3 & & & \\ W_1 & \xrightarrow{s_1} & W_2 & \xrightarrow{s_2} & W_3 & \xrightarrow{s_3} & \dots \end{array}$$

By Theorem 2.1, there is a $*$ -homomorphism $g_n: A_n \rightarrow B_{f_n}$ with multiplicity matrix F_n , i.e., $R_{g_n} = F_n$, $n \geq 1$, and we have the following (a priori non-commutative) diagram:

$$\begin{array}{ccccccc} A_1 & \hookrightarrow & A_2 & \hookrightarrow & A_3 & \hookrightarrow & \dots \\ \downarrow g_1 & & \downarrow g_2 & \swarrow g_3 & & & \\ B_1 & \hookrightarrow & B_2 & \hookrightarrow & B_3 & \hookrightarrow & \dots \end{array}$$

Using Corollary 3.6, we can replace g_2 with $(\text{Ad } u_2)g_2$, for some unitary $u_2 \in B_2$, such that the first left square is commutative. Since unitaries do not change multiplicity matrices (Lemma 3.5), one can continue this procedure to obtain unitaries $(u_n)_{n \geq 2}$ such that the above diagram is commutative when each g_n is replaced by $(\text{Ad } u_n)g_n$ ($n \geq 2$). Therefore there is a $*$ -homomorphism $\phi: A \rightarrow B$ such that $\phi \upharpoonright_{A_1} = g_1$ and $\phi \upharpoonright_{A_n} = (\text{Ad } u_n)g_n$, for each $n \geq 2$. Using Lemma 3.5, we have $R_{(\text{Ad } u_n)g_n} = F_n$, $n \geq 2$. Therefore, $\mathcal{B}(\phi) = f$. ■

Proposition 4.2 *Let $B = ((V_n)_{n=1}^\infty, (E_n)_{n=1}^\infty)$ be a Bratteli diagram. Then there is an $\mathcal{A} \in \mathbf{AF}$ such that $\mathcal{B}(\mathcal{A}) = B$.*

Proof By Definition 2.2, each E_n is an embedding matrix. Set $A'_n = C^*(V_n)$ and $h_n = h(E_n)$, for each $n \geq 1$. (See the remark following Lemma 3.4 for the notations $C^*(V_n)$ and $h(E_n)$.) By Theorem 2.1, $h_n: A'_n \rightarrow A'_{n+1}$ is injective. Now set $A = \varinjlim (A'_n, h_n)$ and let $\alpha^n: A'_n \rightarrow A$ denote the $*$ -homomorphism that comes from the construction of the direct limit, $n \geq 1$. Set $A_n = \alpha^n(A'_n)$ and denote by $\phi_n: A_n \rightarrow A'_n$ the inverse of $\alpha^n: A'_n \rightarrow A_n$ (which exists, since each h_n is injective). Now $\mathcal{A} = (A, (A_n)_{n=1}^\infty, (\phi_n)_{n=1}^\infty) \in \mathbf{AF}$ and $\mathcal{B}(\mathcal{A}) = B$. ■

Let \mathbf{AF}_1 be the subcategory of \mathbf{AF} whose objects are unital AF algebras and whose morphisms are unital homomorphisms; more precisely, $(A, (A_n)_{n=1}^\infty, (\phi_n)_{n=1}^\infty) \in \mathbf{AF}$

is an object of \mathbf{AF}_1 if A is unital and each A_n contains the unit of A . The next proposition follows from part (2) of Theorem 2.1.

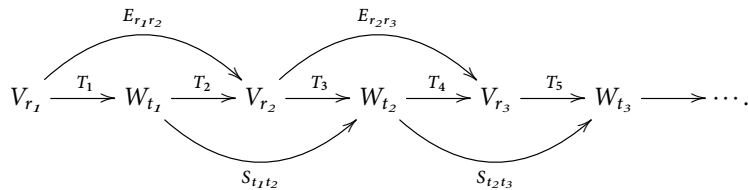
Proposition 4.3 *Let $\mathcal{A} \in \mathbf{AF}$ and $\mathcal{B}(\mathcal{A}) = (V, E)$. Then \mathcal{A} is in \mathbf{AF}_1 if and only if $E_n V_n = V_{n+1}$, for each $n \in \mathbb{N}$.*

Next we give a criterion to check if the functor \mathcal{B} sends two AF algebras to the same diagram. The proof is straightforward.

Proposition 4.4 *Let \mathcal{A}_1 and \mathcal{A}_2 be in \mathbf{AF} with $\mathcal{A}_1 = (A, (A_n)_{n=1}^\infty, (\phi_n)_{n=1}^\infty)$ and $\mathcal{A}_2 = (B, (B_n)_{n=1}^\infty, (\psi_n)_{n=1}^\infty)$. Then $\mathcal{B}(\mathcal{A}_1) = \mathcal{B}(\mathcal{A}_2)$ if and only if there is a $*$ -isomorphism $\phi: A \rightarrow B$ such that $\phi(A_n) = B_n$, and the multiplicity matrix of $\phi \upharpoonright_{A_n}: A_n \rightarrow B_n$ is the identity, for each $n \geq 1$.*

The following theorem (essentially due to Bratteli) gives a combinatorial criterion for isomorphism of Bratteli diagrams.

Theorem 4.5 *Let $B = (V, E)$ and $C = (W, S)$ be two Bratteli diagrams. Then $B \cong C$ in \mathbf{BD} if and only if there is a third Bratteli diagram $D = (Z, T)$ that is constructed from two subsequences of B and C as follows. There are positive integers $(r_k)_{k=1}^\infty$ and $(t_k)_{k=1}^\infty$ with $r_1 < t_1 < r_2 < t_2 < \dots$ such that $Z_{2k-1} = V_{r_k}$, $Z_{2k} = W_{t_k}$, $T_{2k-1, 2k+1} = E_{r_k r_{k+1}}$, and $T_{2k, 2k+2} = S_{t_k t_{k+1}}$, for each $k \geq 1$, i.e., the following diagram is commutative:*



Proof Choose \mathcal{A}_1 and \mathcal{A}_2 in \mathbf{AF} such that $\mathcal{B}(\mathcal{A}_1) = B$ and $\mathcal{B}(\mathcal{A}_2) = C$, as in Proposition 4.2. We have $B \cong C$ if and only if $\mathcal{A}_1 \cong \mathcal{A}_2$, by Corollary 3.12. Now the statement follows from Lemma 3.9, on observing that in the proof of this lemma, the multiplicity matrices R_k and T_k are indeed embedding matrices. ■

Proposition 4.6 *Assume $\phi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is a morphism in \mathbf{AF} and $((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$ is an arbitrary premorphism whose equivalence class is $\mathcal{B}(\phi)$. Write $\mathcal{B}(\mathcal{A}_1) = (V, E)$ and $\mathcal{B}(\mathcal{A}_2) = (W, S)$.*

- (i) ϕ is injective if and only if each F_n is an embedding matrix.
- (ii) If $\mathcal{A}_1, \mathcal{A}_2 \in \mathbf{AF}_1$, then ϕ is unital if and only if $F_n V_n = W_{f_n}$, $n \geq 1$.

Proof First suppose that $f = ((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$ is the premorphism associated to ϕ , as in Definition 3.3. In this case, the statements (i) and (ii) follow from the parts (i) and (ii) in Theorem 2.1. Now suppose that $f = ((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$ is an arbitrary premorphism, the equivalence class of which is $\mathcal{B}(\phi)$. Let $g = ((G_n)_{n=1}^\infty, (g_n)_{n=1}^\infty)$ be a premorphism associated to ϕ , as in Definition 3.3. Applying

Definition 2.9 and Proposition 2.11, we see that the statements (i) and (ii) hold for F_n if and only if they hold for G_n , which they do as shown. ■

5 Relations with Abstract Classifying Categories

In this section, let us investigate the relation between the following three classifying categories for AF algebras: the category of Bratteli diagrams **BD**, the abstract category \mathbf{AF}^{out} , and the abstract category $\overline{\mathbf{AF}}^{\text{out}}$ introduced in [10]. In the next section we will also consider the category of dimension groups **DG** introduced in [9].

In particular, now and in the next section we shall show that the three categories **BD**, \mathbf{AF}^{out} , and **DG** are all equivalent, and hence are classifying categories for each other.

Let us also investigate the relation between the strong classification functors $\mathcal{B}: \mathbf{AF} \rightarrow \mathbf{BD}$, $\mathcal{F}: \mathbf{AF} \rightarrow \mathbf{AF}^{\text{out}}$, and $\overline{\mathcal{F}}: \mathbf{AF} \rightarrow \overline{\mathbf{AF}}^{\text{out}}$, and in the next section, the relation to the functor $K_0: \mathbf{AF} \rightarrow \mathbf{DG}$.

The following lemma may be considered as part of the literature (basically due to Glimm, see below); we give a proof anyway (cf. Lemma 4.2 and Theorem 4.3 of [9]).

Lemma 5.1 *For each $\varepsilon > 0$ there is a $\delta > 0$ such that if A is a unital C^* -algebra, B is a C^* -subalgebra of A containing the unit of A , and u is a unitary of A with $d(u, B) < \delta$, then there is a unitary $v \in B$ such that $\|u - v\| < \varepsilon$.*

Proof The statement follows from [11, Lemma 1.9]. In fact, in the proof of [11, Lemma 1.9], Glimm does not use the assumption of orthogonality of projections. Thus putting $E_1 = E_2 = F_1 = F_2 = 1$ in that lemma, the statement follows. There is also a direct proof as follows. Set $\delta = \min\{\varepsilon, \frac{1}{8}\}$. Let A be a unital C^* -algebra, B a C^* -subalgebra of A containing the unit of A , and u a unitary of A with $d(u, B) < \delta$. Thus there is $a \in B$ such that $\|u - a\| < \delta$. Thus a is invertible. Set $v = (aa^*)^{-\frac{1}{2}}a$; hence $vv^* = v^*v = 1$ and v is a unitary in B . We have

$$\begin{aligned} \|aa^* - 1\| &\leq \|aa^* - au^*\| + \|au^* - uu^*\| \\ &\leq \|a\| \|a^* - u^*\| + \|a - u\| \\ &\leq (\|a\| + 1)\|a - u\| \\ &\leq 3\|a - u\| < \frac{1}{2}. \end{aligned}$$

Thus $\|(aa^*)^{-1}\| < 2$. Using functional calculus we have

$$\begin{aligned} \|(aa^*)^{-\frac{1}{2}} - 1\| &\leq \|(aa^*)^{-1} - 1\| \\ &\leq \|(aa^*)^{-1}\| \|aa^* - 1\| \\ &\leq 2\|aa^* - 1\| \\ &\leq 6\|a - u\| < 1. \end{aligned}$$

Therefore we have

$$\begin{aligned} \|u - v\| &\leq \|u - (aa^*)^{-\frac{1}{2}}u\| + \|(aa^*)^{-\frac{1}{2}}u - v\| \\ &\leq \|1 - (aa^*)^{-\frac{1}{2}}\| + \|(aa^*)^{-\frac{1}{2}}\| \|u - a\| \\ &\leq 6\|a - u\| + 2\|a - u\| \\ &\leq 8\|a - u\| < \varepsilon. \end{aligned}$$

■

The following immediate consequence of Lemma 5.1 enables us to approximate the unitaries of an AF algebra by unitaries of an increasing sequence of C^* -subalgebras with dense union. We will use this statement in the proof of Lemma 5.3.

Corollary 5.2 Let $A = \overline{\bigcup_{n \geq 1} A_n}$ be a unital AF algebra where $(A_n)_{n=1}^\infty$ is an increasing sequence of finite-dimensional C^* -subalgebras of A each containing the unit of A . Then $u \in A$ is a unitary if and only if there are unitaries $(u_n)_{n=1}^\infty$ such that $u_n \in A_n$, $n \geq 1$, and $u_n \rightarrow u$.

We shall need the following technical lemma in the proof of Lemma 5.4 and Corollary 5.7.

Lemma 5.3 Let $B = \overline{\bigcup_{n \geq 1} B_n}$ be an AF algebra where $(B_n)_{n=1}^\infty$ is an increasing sequence of finite-dimensional C^* -subalgebras of B . Let A be a finite-dimensional C^* -algebra and let $\phi, \psi: A \rightarrow B$ be $*$ -homomorphisms such that $\|\phi - \psi\| < 1$. Let u, v be unitaries in B^\sim such that there are positive integers n and m such that $u\phi(A)u^* \subseteq B_n$ and $v\psi(A)v^* \subseteq B_m$. Then there is a positive integer $k \geq n, m$ such that the $*$ -homomorphisms $(\text{Ad } u)\phi, (\text{Ad } v)\psi: A \rightarrow B_k$ have the same multiplicity matrices, i.e., $R_{(\text{Ad } u)\phi} = R_{(\text{Ad } v)\psi}$.

Proof Let 1 denote the unit of B^\sim . Set $B_n^\sim = B_n + \mathbb{C}1$, so that $B^\sim = \overline{\bigcup_{n \geq 1} B_n^\sim}$. By Corollary 5.2, there is a positive integer $k \geq n, m$ and a unitary $w \in B_k^\sim$ such that $\|uv^* - w\| < \frac{1}{2}(1 - \|\phi - \psi\|)$. We have the following (a priori non-commutative) diagram:

$$\begin{array}{ccc} A & \xrightarrow{(\text{Ad } v)\psi} & B_m \\ (\text{Ad } u)\phi \downarrow & & \downarrow \text{Ad } w \\ B_n & \xrightarrow{\subset} & B_k. \end{array}$$

Consider the two $*$ -homomorphisms $(\text{Ad } w)(\text{Ad } v)\psi, (\text{Ad } u)\phi: A \rightarrow B_k$. We have

$$\|(\text{Ad } w)(\text{Ad } v)\psi - (\text{Ad } u)\phi\| = \|(\text{Ad } wv)\psi - (\text{Ad } u)\phi\| \leq 2\|wv - u\| + \|\phi - \psi\| < 1.$$

Hence by Lemma 3.7, $R_{(\text{Ad } w)(\text{Ad } v)\psi} = R_{(\text{Ad } u)\phi}$. Now define $\eta: B_k \rightarrow B_k$ with $\eta(x) = (\text{Ad } w)(x)$, $x \in B_k$. Then there is a unitary $w' \in B_k$ such that $\eta = \text{Ad } w'$. In fact, $w \in B_k^\sim$ and thus $w = a + \lambda 1$, for some $a \in B_k$ and $\lambda \in \mathbb{C}$. Set $w' = a + \lambda 1_{B_k}$. Since w is a unitary, so is w' , and we have $(\text{Ad } w)(x) = (\text{Ad } w')(x)$, $x \in B_k$. Thus $\eta = \text{Ad } w'$ and hence R_η is the identity matrix, by Theorem 2.1. On the other hand, $(\text{Ad } w)(\text{Ad } v)\psi = \eta \circ (\text{Ad } v)\psi$, and so $R_{(\text{Ad } w)(\text{Ad } v)\psi} = R_{\eta \circ (\text{Ad } v)\psi} = R_\eta R_{(\text{Ad } v)\psi} = R_{(\text{Ad } v)\psi}$. Therefore we have $R_{(\text{Ad } u)\phi} = R_{(\text{Ad } v)\psi}$. ■

The functor $\mathcal{B}: \mathbf{AF} \rightarrow \mathbf{BD}$ is of course not faithful (we follow [12, 15] for categorical definitions). The following gives useful criteria to check whether the images under \mathcal{B} of two morphisms of \mathbf{AF} are equal in \mathbf{BD} . This enables us to make connections between morphisms of \mathbf{BD} and morphisms of the categories \mathbf{AF}^{out} and $\overline{\mathbf{AF}^{\text{out}}}$ (see Theorems 5.8 and 5.9 below).

Lemma 5.4 *Let $\mathcal{A}_1, \mathcal{A}_2 \in \mathbf{AF}$ where $\mathcal{A}_1 = (A, (A_n)_{n=1}^\infty, (\phi_n)_{n=1}^\infty)$ and $\mathcal{A}_2 = (B, (B_n)_{n=1}^\infty, (\psi_n)_{n=1}^\infty)$. Let $\phi, \psi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ be morphisms in \mathbf{AF} . The following statements are equivalent:*

- (i) $\mathcal{B}(\phi) = \mathcal{B}(\psi)$;
- (ii) *there is a sequence of unitaries $(u_n)_{n=1}^\infty$ in B^\sim such that $\phi = (\text{Ad } u_n)\psi$ on A_n , for $n \geq 1$;*
- (iii) *there is a sequence of unitaries $(u_n)_{n=1}^\infty$ in B^\sim such that*

$$\phi(a) = \lim_{n \rightarrow \infty} (\text{Ad } u_n)\psi(a),$$

for $a \in A$.

Proof Choose a sequence of unitaries $(v_n)_{n=1}^\infty$ in B^\sim , and a sequence of positive integers $(f_n)_{n=1}^\infty$, as in Definition 3.3, giving rise to a premorphism $((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$ with equivalence class $\mathcal{B}(\phi)$. Similarly, choose a sequence of unitaries $(w_n)_{n=1}^\infty$ in B^\sim , and a sequence of positive integers $(g_n)_{n=1}^\infty$, giving rise to a premorphism $((G_n)_{n=1}^\infty, (g_n)_{n=1}^\infty)$ for $\mathcal{B}(\psi)$.

(i) \Rightarrow (ii): Suppose that $\mathcal{B}(\phi) = \mathcal{B}(\psi)$. Hence $((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$ is equivalent to $((G_n)_{n=1}^\infty, (g_n)_{n=1}^\infty)$. Fix $n \geq 1$. By Proposition 2.11, there is an $m \geq f_n, g_n$ such that $S_{f_n m} F_n = S_{g_n m} G_n$, where $S_{f_n m}$ and $S_{g_n m}$ are the multiplicity matrices of the injections $j_1: B_{f_n} \hookrightarrow B_m$ and $j_2: B_{g_n} \hookrightarrow B_m$, respectively. On the other hand, F_n and G_n are the multiplicity matrices of $(\text{Ad } v_n)\phi: A_n \rightarrow B_{f_n}$ and $(\text{Ad } w_n)\psi: A_n \rightarrow B_{g_n}$, respectively, by Definition 3.3. Thus $R_{j_1(\text{Ad } v_n)\phi} = S_{f_n m} F_n = S_{g_n m} G_n = R_{j_2(\text{Ad } w_n)\psi}$. By Lemma 3.5, there is a unitary $u \in B_m$ such that $j_1(\text{Ad } v_n)\phi = (\text{Ad } u)j_2(\text{Ad } w_n)\psi$ on A_n . Set $w = u - 1_{B_m} + 1$, where 1 is the unit of B^\sim . One can easily see that w is a unitary in B^\sim and again we have $j_1(\text{Ad } v_n)\phi = (\text{Ad } w)j_2(\text{Ad } w_n)\psi$ on A_n . Set $u_n = v_n^* w w_n$. Therefore $\phi = (\text{Ad } u_n)\psi$ on A_n .

(ii) \Rightarrow (iii): This holds as $(A_n)_{n=1}^\infty$ is increasing with union dense in A .

(iii) \Rightarrow (i): Suppose that there is a sequence of unitaries $(u_n)_{n=1}^\infty$ in B^\sim such that ϕ is the pointwise limit of the sequence $((\text{Ad } u_n)\psi)_{n=1}^\infty$ on A . Fix $n \geq 1$. Since $((\text{Ad } u_m)\psi)_{m=1}^\infty$ converges ϕ on compact subsets of A and the unit ball of A_n is compact, $\|(\text{Ad } u_m)\psi - \phi\|_{A_n} \rightarrow 0$, as m tends to infinity. Thus there is an $n' \geq 1$ such that $\|(\text{Ad } u_{n'})\psi - \phi\|_{A_n} < 1$. Set $u = v_n$ and $v = w_n u_{n'}^*$. Hence $u\phi(A_n)u^* \subseteq B_{f_n}$ and $v(\text{Ad } u_{n'})\psi(A_n)v^* = w_n\psi(A_n)w_n^* \subseteq B_{g_n}$. Applying Lemma 5.3, there is an $m \geq f_n, g_n$ such that $(\text{Ad } u)\phi, (\text{Ad } v) \circ ((\text{Ad } u_{n'})\psi): A_n \rightarrow B_m$ have the same multiplicity matrices; that is, $(\text{Ad } v_n)\phi, (\text{Ad } w_n)\psi: A_n \rightarrow B_m$ have the same multiplicity matrices. By Proposition 2.11, the premorphisms $((F_n)_{n=1}^\infty, (f_n)_{n=1}^\infty)$ and $((G_n)_{n=1}^\infty, (g_n)_{n=1}^\infty)$ are equivalent and therefore $\mathcal{B}(\phi) = \mathcal{B}(\psi)$. ■

Remark Lemma 5.4 remains valid if we replace B^\sim with B^+ . Since, if B is non-unital, we have $B^\sim = B^+$, and if B is unital, according to our convention, $B^\sim = B$; thus by the techniques applied in the proof of Lemma 5.3 and Lemma 5.4 for interchanging the unitaries of B and B^+ , the statement is also true for B^+ instead of B^\sim .

In general, the sequence of unitaries in Lemma 5.4 cannot be replaced by a single unitary. In other words, in the setting of that lemma, the condition $\mathcal{B}(\phi) = \mathcal{B}(\psi)$ does not necessarily imply that there is a unitary $u \in B^\sim$ with $\phi = (\text{Ad } u)\psi$. (See the following example.)

Example 5.5 Consider the C^* -algebra $A = \mathcal{K}(l^2)$ and let $(e_n)_{n=1}^\infty$ be an orthonormal basis for l^2 . Consider the C^* -subalgebra A_n generated by the rank one operators $\{e_i \otimes e_j^* \mid 1 \leq i, j \leq n\}$ for $n \geq 1$. Then $A_1 \subseteq A_2 \subseteq \dots$, and $A = \overline{\bigcup_{n \geq 1} A_n}$. Define $\phi, \psi: A \rightarrow A$ as follows. Set $\psi = \text{id}_A$. For each $n \in \mathbb{N}$, let u_n denote the unitary in $A^\sim = \mathcal{K}(l^2) \oplus I$ defined by $u_n(e_k) = e_{k+1}$, for $1 \leq k \leq n$, $u_n(e_{n+1}) = e_1$, and $u_n(e_k) = e_k$, for $k \geq n + 2$. Then $\text{Ad } u_n$ and $\text{Ad } u_m$ agree on A_n when $n \leq m$. Set $\phi = \text{Ad } u_n$ on A_n , $n \geq 1$. Then $\phi: A \rightarrow A$ is a $*$ -homomorphism and $\phi = (\text{Ad } u_n)\psi$ on A_n , $n \geq 1$. Suppose that there were a unitary $u \in A^\sim$ such that $\phi = \text{Ad } u$. Then $u(e_n) \otimes u(e_n)^* = \phi(e_n \otimes e_n^*) = e_{n+1} \otimes e_{n+1}^*$, $n \geq 1$. Thus, $u(e_n) = \lambda_n e_{n+1}$ for some complex number λ_n with absolute value one. Set $f_n = \lambda_1 \lambda_2 \dots \lambda_{n-1} e_n$, $n \geq 1$. Then $(f_n)_{n=1}^\infty$ is an orthonormal basis for l^2 and $u(f_n) = f_{n+1}$, $n \geq 1$; in other words, u is the unilateral shift, which is not a unitary.

Corollary 5.6 Let $\mathcal{A}_1, \mathcal{A}_2 \in \mathbf{AF}$ and $\phi, \psi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ be morphisms in \mathbf{AF} such that $\phi = (\text{Ad } u)\psi$ for some unitary u in \mathcal{A}_2^\sim . Then $\mathcal{B}(\phi) = \mathcal{B}(\psi)$.

Corollary 5.7 Let $\phi, \psi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ morphisms in \mathbf{AF} with $\|\phi - \psi\| < 1$. Then we have $\mathcal{B}(\phi) = \mathcal{B}(\psi)$.

Proof Following the notation of Lemma 5.4 and the first paragraph of its proof, we have $v_n \phi(A_n) v_n^* \subseteq B_{f_n}$ and $w_n \psi(A_n) w_n^* \subseteq B_{g_n}$, $n \geq 1$, by Definition 3.3. Fix $n \geq 1$. By Lemma 5.3, the $*$ -homomorphisms $(\text{Ad } v_n)\phi, (\text{Ad } w_n)\psi: A_n \rightarrow B_{k_n}$ have the same multiplicity matrices, for some positive integer $k_n \geq f_n, g_n$. By Lemma 3.5, there is a unitary $u \in B_{k_n}$ such that $(\text{Ad } v_n)\phi = (\text{Ad } u)(\text{Ad } w_n)\psi$ on A_n . Set $w = u - 1_{B_{k_n}} + 1$, where 1 is the unit of B^\sim . Then w is a unitary in B^\sim and again we have $(\text{Ad } v_n)\phi = (\text{Ad } w)(\text{Ad } w_n)\psi$ on A_n . Setting $u_n = v_n^* w w_n$, we have $\phi = (\text{Ad } u_n)\psi$ on A_n , and so $\mathcal{B}(\phi) = \mathcal{B}(\psi)$ by Lemma 5.4. ■

Consider the category \mathbf{AF}^{out} associated to \mathbf{AF} as described in [10]; its objects are the same as those of \mathbf{AF} and its morphisms are as follows. An inner automorphism for an object $(A, (A_n)_{n=1}^\infty, (\phi_n)_{n=1}^\infty)$ of \mathbf{AF} is a $*$ -isomorphism $\text{Ad } u: A \rightarrow A$, for some unitary $u \in A^+$. Two morphisms $\phi, \psi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ are equivalent if $\phi = (\text{Ad } u)\psi$ for some inner automorphism $\text{Ad } u$ of \mathcal{A}_2 . Let $\mathcal{F}(\phi)$ denote the equivalence class of ϕ . These equivalence classes are the morphisms of \mathbf{AF}^{out} . Denote by $\mathcal{F}: \mathbf{AF} \rightarrow \mathbf{AF}^{\text{out}}$ the functor which assigns to each object of \mathbf{AF} itself, and maps morphisms as above. Now [10, Theorem 1] states that $\mathcal{F}: \mathbf{AF} \rightarrow \mathbf{AF}^{\text{out}}$ is a strong classification functor. Obviously, it is also a full functor.

Theorem 5.8 *There is a unique functor $\tilde{\mathcal{B}}: \mathbf{AF}^{\text{out}} \rightarrow \mathbf{BD}$ such that $\mathcal{B} = \tilde{\mathcal{B}}\mathcal{F}$:*

$$\begin{array}{ccc}
 \mathbf{AF} & \xrightarrow{\mathcal{F}} & \mathbf{AF}^{\text{out}} \\
 & \searrow \mathcal{B} & \downarrow \tilde{\mathcal{B}} \\
 & & \mathbf{BD}.
 \end{array}$$

Moreover, it is a strong classification functor and a full functor.

Proof Define $\tilde{\mathcal{B}}: \mathbf{AF}^{\text{out}} \rightarrow \mathbf{BD}$ as follows. For $\mathcal{A} \in \mathbf{AF}$ set $\tilde{\mathcal{B}}(\mathcal{A}) = \mathcal{B}(\mathcal{A})$. Let $\phi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ be a morphism in \mathbf{AF} . By Lemma 5.4 (and the remark following that), $\mathcal{B}((\text{Ad } u)\phi) = \mathcal{B}(\phi)$, for each inner automorphism $\text{Ad } u$ of \mathcal{A}_2 . It therefore makes sense to set $\tilde{\mathcal{B}}(\mathcal{F}(\phi)) = \mathcal{B}(\phi)$. It is immediate that $\tilde{\mathcal{B}}: \mathbf{AF}^{\text{out}} \rightarrow \mathbf{BD}$ is a functor and $\mathcal{B} = \tilde{\mathcal{B}}\mathcal{F}$. Hence $\tilde{\mathcal{B}}$ is full, since \mathcal{B} is (by Theorem 4.1). Uniqueness follows from $\mathcal{B} = \tilde{\mathcal{B}}\mathcal{F}$ and the fact that \mathcal{F} is surjective on both objects and (since it is full) on maps. That $\tilde{\mathcal{B}}: \mathbf{AF}^{\text{out}} \rightarrow \mathbf{BD}$ is a strong classification functor follows from the fact that \mathcal{B} is (Theorem 3.11). ■

Note that the functor $\tilde{\mathcal{B}}: \mathbf{AF}^{\text{out}} \rightarrow \mathbf{BD}$ is not faithful. For example, let ϕ and ψ be the morphisms in \mathbf{AF} defined in Example 5.5. Then $\tilde{\mathcal{B}}(\mathcal{F}(\phi)) = \mathcal{B}(\phi) = \mathcal{B}(\psi) = \tilde{\mathcal{B}}(\mathcal{F}(\psi))$, but $\mathcal{F}(\phi) \neq \mathcal{F}(\psi)$, by Example 5.5; cf. Theorem 5.9.

Now let us examine the classifying category $\overline{\mathbf{AF}^{\text{out}}}$ for \mathbf{AF} , as described in [10]. It is better than \mathbf{AF}^{out} (in some sense) for the purposes of classification, because $\overline{\mathbf{AF}^{\text{out}}}$ is a classifying category not only for \mathbf{AF} , but also for \mathbf{AF}^{out} (and it has even fewer automorphisms); however, \mathbf{BD} is even better than (although, by Theorem 5.11, it is just equivalent to) $\overline{\mathbf{AF}^{\text{out}}}$, since it is a classifying category for $\overline{\mathbf{AF}^{\text{out}}}$ and so for all three of these categories (by Theorem 5.9), but is in some sense more explicit. (For one thing, it is a small category.)

Consider the category $\overline{\mathbf{AF}^{\text{out}}}$ as a subcategory of $\overline{\mathcal{S}^{\text{out}}}$ which is defined in [10, Example 4.3], where \mathcal{S} denotes the category of separable C^* -algebras (not necessarily unital). More precisely, the objects of $\overline{\mathbf{AF}^{\text{out}}}$ are the same as of \mathbf{AF} and its morphisms are as follows. For each pair of objects \mathcal{A}_1 and \mathcal{A}_2 in \mathbf{AF} , and for each ϕ in $\text{Hom}(\mathcal{A}_1, \mathcal{A}_2)$, denote by $\overline{\mathcal{F}}(\phi)$ the closure of the equivalence class $\mathcal{F}(\phi)$ in $\text{Hom}(\mathcal{A}_1, \mathcal{A}_2)$, in the topology of pointwise convergence. These are the morphisms of $\overline{\mathbf{AF}^{\text{out}}}$. By [10, Theorem 3] and [10, Example 4.3], $\overline{\mathbf{AF}^{\text{out}}}$ is a category. Now define the functor $\overline{\mathcal{F}}: \mathbf{AF} \rightarrow \overline{\mathbf{AF}^{\text{out}}}$ as follows. $\overline{\mathcal{F}}$ assigns to each object of \mathbf{AF} itself and maps morphisms as above. By [10, Theorem 3] and [10, Example 4.3], $\overline{\mathcal{F}}: \mathbf{AF} \rightarrow \overline{\mathbf{AF}^{\text{out}}}$ is a strong classification functor. (It follows immediately that the quotient map from \mathbf{AF}^{out} to $\overline{\mathbf{AF}^{\text{out}}}$ is also a strong classification functor, but this is not of interest to us here.)

Theorem 5.9 There is a unique functor $\overline{\mathcal{B}}: \overline{\mathbf{AF}^{\text{out}}} \rightarrow \mathbf{BD}$ such that $\mathcal{B} = \overline{\mathcal{B}}\overline{\mathcal{F}}$:

$$\begin{array}{ccc} \mathbf{AF} & \xrightarrow{\overline{\mathcal{F}}} & \overline{\mathbf{AF}^{\text{out}}} \\ & \searrow \mathcal{B} & \downarrow \overline{\mathcal{B}} \\ & & \mathbf{BD} \end{array}$$

It is a strong classification functor, surjective on objects, and a full functor. Moreover, for each pair of morphisms $\phi, \psi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ in \mathbf{AF} we have $\mathcal{B}(\phi) = \mathcal{B}(\psi)$ if and only if $\overline{\mathcal{F}}(\phi) = \overline{\mathcal{F}}(\psi)$; in other words, $\overline{\mathcal{B}}$ is a faithful functor.

Proof First let us show that for each pair of morphisms $\phi, \psi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ in \mathbf{AF} we have $\mathcal{B}(\phi) = \mathcal{B}(\psi)$ if and only if $\overline{\mathcal{F}}(\phi) = \overline{\mathcal{F}}(\psi)$. Suppose that $\mathcal{B}(\phi) = \mathcal{B}(\psi)$. By Lemma 5.4 and the remark following that, there is a sequence of unitaries $(u_n)_{n=1}^\infty$ in B^+ such that ϕ is the pointwise limit of the sequence $((\text{Ad } u_n)\psi)_{n=1}^\infty$ where B is the algebra (i.e., the first component) of \mathcal{A}_2 . Thus, for each unitary $u \in B^+$, $(\text{Ad } u)\phi$ is the pointwise limit of the sequence $((\text{Ad } uu_n)\psi)_{n=1}^\infty$. Therefore, $\mathcal{F}(\phi) \subseteq \mathcal{F}(\psi)$. Hence $\overline{\mathcal{F}}(\phi) \subseteq \overline{\mathcal{F}}(\psi)$ and by symmetry $\overline{\mathcal{F}}(\phi) = \overline{\mathcal{F}}(\psi)$. Now suppose that $\overline{\mathcal{F}}(\phi) = \overline{\mathcal{F}}(\psi)$. Then, ϕ is the pointwise limit of a sequence $((\text{Ad } u_n)\psi)_{n=1}^\infty$, for some sequence of unitaries $(u_n)_{n=1}^\infty$ in B^+ . By Lemma 5.4, $\mathcal{B}(\phi) = \mathcal{B}(\psi)$.

Now define $\overline{\mathcal{B}}: \overline{\mathbf{AF}^{\text{out}}} \rightarrow \mathbf{BD}$ as follows. For $\mathcal{A} \in \mathbf{AF}$ set $\overline{\mathcal{B}}(\mathcal{A}) = \mathcal{B}(\mathcal{A})$. Let $\phi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ be a morphism in \mathbf{AF} . Set $\overline{\mathcal{B}}(\overline{\mathcal{F}}(\phi)) = \mathcal{B}(\phi)$. By the preceding paragraph, $\overline{\mathcal{B}}$ is well defined, and faithful. Also, we have $\mathcal{B} = \overline{\mathcal{B}}\overline{\mathcal{F}}$. That $\overline{\mathcal{B}}$ is a functor, and uniqueness of $\overline{\mathcal{B}}$, follow from the fact that $\overline{\mathcal{F}}$ is a full functor, or, rather, even surjective. Since \mathcal{B} is a strong classification functor and a full functor, so also is $\overline{\mathcal{B}}$. (That $\overline{\mathcal{B}}$ is a strong classification functor also follows from the fact that it is full and faithful and applying Lemma 5.10, below.) ■

As we shall see, the functor $\overline{\mathcal{B}}: \overline{\mathbf{AF}^{\text{out}}} \rightarrow \mathbf{BD}$ is an equivalence of categories (see Theorem 5.11 below). This is mainly based on the categorical properties of this functor. Therefore, let us first state this result in a categorical setting, in Lemma 5.10. We shall use this lemma to show that the functor $\overline{\mathcal{B}}: \overline{\mathbf{AF}^{\text{out}}} \rightarrow \mathbf{BD}$ is an equivalence of categories (Theorems 5.11).

Recall that a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is called an equivalence of categories if there is a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that $FG \cong \text{id}_{\mathcal{C}}$ and $GF \cong \text{id}_{\mathcal{D}}$ [12, 15]. If $H: \mathcal{D} \rightarrow \mathcal{C}$ is another functor with this property, then it is easy to see that H is naturally isomorphic to G . Therefore, G is unique up to natural isomorphism. It is well known that a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence of categories if and only if it is full, faithful, and essentially surjective, i.e., for each $d \in \mathcal{D}$ there is a $c \in \mathcal{C}$ such that $d \cong F(c)$ [15, Theorem IV.4.1]. In the case that F is surjective on objects, a right inverse for F can be constructed, i.e., a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that $FG = \text{id}_{\mathcal{C}}$ and $GF \cong \text{id}_{\mathcal{D}}$. (A remark on the use of the axiom of choice in this context is given in the proof.)

Lemma 5.10 Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a full and faithful functor. Then F is a strong classification functor. If F is also surjective on objects, then it is an equivalence of categories, and,

furthermore, there is a unique (up to natural isomorphism) functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that $FG = \text{id}_{\mathcal{C}}$ and $GF \cong \text{id}_{\mathcal{D}}$. The functor G is full, faithful, injective on objects, essentially surjective, and (hence) a strong classification functor.

Proof That a full and faithful functor is a strong classification functor is straightforward. Since $F: \mathcal{C} \rightarrow \mathcal{D}$ is surjective on objects, it has a right inverse $G: \mathcal{D} \rightarrow \mathcal{C}$ (just as a map on objects). Here we have used the ‘‘axiom of choice’’ for sets or classes: when the objects of \mathcal{C} form a set we use the axiom of choice for sets, and when the objects of \mathcal{C} form a proper class we use the global axiom of choice [3] (if for each object c of \mathcal{D} there is a canonical object a in \mathcal{C} such that $F(a) = c$, one could avoid the axiom of choice).

The definition of G on the morphisms of \mathcal{D} was described in the proof of [15, Theorem IV.4.1], and so we have a functor G such that $FG = \text{id}_{\mathcal{C}}$ and $GF \cong \text{id}_{\mathcal{D}}$. The rest follows from the fact that each functor which is an equivalence of categories is full, faithful, and essentially surjective [15, Theorem IV.4.1]. ■

The following theorem states that the categories $\overline{\mathbf{AF}^{\text{out}}}$ and \mathbf{BD} are equivalent, and this equivalence, given by the functor $\overline{\mathcal{B}}: \overline{\mathbf{AF}^{\text{out}}} \rightarrow \mathbf{BD}$, is compatible with the classification of AF algebras via the functors $\mathcal{B}: \mathbf{AF} \rightarrow \mathbf{BD}$ and $\overline{\mathcal{F}}: \mathbf{AF} \rightarrow \overline{\mathbf{AF}^{\text{out}}}$, i.e., the related diagrams commute.

Theorem 5.11 *The functor $\overline{\mathcal{B}}: \overline{\mathbf{AF}^{\text{out}}} \rightarrow \mathbf{BD}$ is an equivalence of categories. More precisely, there is a unique (up to natural isomorphism) functor $\mathcal{G}: \mathbf{BD} \rightarrow \overline{\mathbf{AF}^{\text{out}}}$ such that $\overline{\mathcal{B}}\mathcal{G} = \text{id}_{\mathbf{BD}}$ and $\mathcal{G}\overline{\mathcal{B}} \cong \text{id}_{\overline{\mathbf{AF}^{\text{out}}}}$. The functor \mathcal{G} is full, faithful, injective on objects, essentially surjective, and a strong classification functor. Moreover, for each $B, C \in \mathbf{BD}$ and each morphism $\phi: \mathcal{G}(B) \rightarrow \mathcal{G}(C)$ in \mathbf{AF} , we have $\mathcal{G}\overline{\mathcal{B}}(\phi) = \overline{\mathcal{F}}(\phi)$, i.e., the following diagram commutes:*

$$\begin{array}{ccc}
 \mathbf{AF} & \xrightarrow{\overline{\mathcal{F}}} & \overline{\mathbf{AF}^{\text{out}}} \\
 \mathcal{B} \downarrow & \nearrow \mathcal{G} & \downarrow \overline{\mathcal{B}} \\
 \mathbf{BD} & \xrightarrow{\text{id}} & \mathbf{BD}.
 \end{array}$$

Proof By Theorem 5.9, the functor $\overline{\mathcal{B}}: \overline{\mathbf{AF}^{\text{out}}} \rightarrow \mathbf{BD}$ is full, faithful, surjective on objects, and a strong classification functor. By Lemma 5.10, it is also an equivalence of categories and the functor $\mathcal{G}: \mathbf{BD} \rightarrow \overline{\mathbf{AF}^{\text{out}}}$ with the desired properties exists. As indicated in the proof of Lemma 5.10, here the use of the axiom of choice is justified as follows. Since the collection of the objects $\overline{\mathbf{AF}^{\text{out}}}$ is a proper class and \mathbf{BD} is a small category, we can use the global axiom of choice [3]. Alternatively, one could use the fact that each AF algebra is (isomorphic to) a C^* -subalgebra of $\mathbb{B}(l^2)$, and essentially the axiom of choice for sets is enough. Finally, one can choose $\mathcal{G}(B)$ to be the AF algebra constructed as in Proposition 4.2.

For the last statement, let $B, C \in \mathbf{BD}$ and $\phi: \mathcal{G}(B) \rightarrow \mathcal{G}(C)$ be a morphism in \mathbf{AF} . Note that the objects of \mathbf{AF} and $\overline{\mathbf{AF}^{\text{out}}}$ are the same, and so $\mathcal{G}(B)$ and $\mathcal{G}(C)$ are

also in \mathbf{AF} . We have $\mathcal{G}\mathcal{B}(\mathcal{G}(B)) = \mathcal{G}\overline{\mathcal{B}}(\mathcal{G}(B)) = \mathcal{G}(B) = \overline{\mathcal{F}}(\mathcal{G}(B))$, and similarly for C . Thus, $\mathcal{G}\mathcal{B}(\phi)$ and $\overline{\mathcal{F}}(\phi)$ have the same domains and the same ranges. We have $\overline{\mathcal{B}}(\mathcal{G}\mathcal{B}(\phi)) = \mathcal{B}(\phi) = \overline{\mathcal{B}}(\overline{\mathcal{F}}(\phi))$, since $\mathcal{B} = \overline{\mathcal{B}}\overline{\mathcal{F}}$. By Theorem 5.9, $\overline{\mathcal{B}}$ is faithful, and so $\mathcal{G}\mathcal{B}(\phi) = \overline{\mathcal{F}}(\phi)$. ■

6 Relations with K-Theory

Consider the category \mathbf{DG} of dimension groups, *i.e.*, the set of all scaled countable ordered groups which are unperforated and have the Riesz decomposition property, with order and scale preserving homomorphisms (see [1,7,18]), and consider the well-known K_0 functor $K_0: \mathbf{AF} \rightarrow \mathbf{DG}$. The following statement summarizes the main properties of the functor $K_0: \mathbf{AF} \rightarrow \mathbf{DG}$.

Theorem 6.1 *The functor $K_0: \mathbf{AF} \rightarrow \mathbf{DG}$ is a strong classification functor and a full functor. Moreover, it is essentially surjective on objects.*

Proof That the functor $K_0: \mathbf{AF} \rightarrow \mathbf{DG}$ is a strong classification functor is Elliott’s theorem [8]. That the functor $K_0: \mathbf{AF} \rightarrow \mathbf{DG}$ is full is known, and the proof is similar to the proof that it is a strong classification functor—one uses a one-sided intertwining argument rather than a two-sided one, just as in Theorem 4.1. In fact, one can deduce it from Theorem 4.1 together with the factorization of $K_0: \mathbf{AF} \rightarrow \mathbf{DG}$ through \mathbf{BD} by means of the inductive limit functor described in the alternative proof of Corollary 6.5 below, which is easily seen to be full, and so K_0 is expressed as the composition of two full functors. The last statement follows from the Effros–Handelman–Shen theorem [7, Theorem 2.2], and the result of Elliott, [8, Theorem 5.5] characterizing K_0 groups of AF algebras as inductive limits. ■

The following lemma is surely part of the literature; we give a proof for the sake of completeness. We follow [18] for K-theory notation.

Lemma 6.2 *Let $\mathcal{A}_1, \mathcal{A}_2 \in \mathbf{AF}$ where $\mathcal{A}_1 = (A, (A_n)_{n=1}^\infty, (\phi_n)_{n=1}^\infty)$ and $\mathcal{A}_2 = (B, (B_n)_{n=1}^\infty, (\psi_n)_{n=1}^\infty)$. Let $\phi, \psi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ be morphisms in \mathbf{AF} . The following statements are equivalent:*

- (i) $K_0(\phi) = K_0(\psi)$;
- (ii) *there is a sequence of unitaries $(u_n)_{n=1}^\infty$ in B^\sim such that $\phi = (\text{Ad } u_n)\psi$ on A_n , $n \geq 1$.*

Proof Let $\phi^+, \psi^+: A^+ \rightarrow B^+$ denote the unital extensions of $\phi, \psi: A \rightarrow B$. Note that $A^+ = \overline{\bigcup_{n=1}^\infty A_n^+}$ and $B^+ = \overline{\bigcup_{n=1}^\infty B_n^+}$. The proof is similar to the finite-dimensional case [16, Theorem 7.2.6].

(i) \Rightarrow (ii): The proof really should be thought of as three separate steps—first reducing to the case that the domain is a single finite-dimensional algebra, and then to the case that the codomain is a single finite-dimensional algebra (using for the second step that K_0 of the limit is the limit of the K_0 ’s). The third step, that both algebras are finite-dimensional, follows immediately from an argument due to Bratteli. The details are as follows.

Fix $n \geq 1$. By Lemma 3.2, there are unitaries $u, v \in B^+$ and a positive integer m such that $u\phi^+(A_n^+)u^* \subseteq B_m^+$ and $v\psi^+(A_n^+)v^* \subseteq B_m^+$. Define $*$ -homomorphisms $F, G: A_n^+ \rightarrow B_m^+$ with $F(a) = u\phi^+(a)u^*$ and $G(a) = v\psi^+(a)v^*$, $a \in A_n^+$. Let $\{e_{ij}^l : 1 \leq l \leq k, 1 \leq i, j \leq n_l\}$ be a set of matrix units for A_n and set $e_{11}^{k+1} = 1 - 1_{A_n}$ (where 1 is the unit element of A_n^+) and $n_{k+1} = 1$. Then $\{e_{ij}^l : 1 \leq l \leq k+1, 1 \leq i, j \leq n_l\}$ is a set of matrix units for A_n^+ .

Set $p_{ij}^l = F(e_{ij}^l)$ and $q_{ij}^l = G(e_{ij}^l)$, for the above values of i, j, l . Fix $1 \leq l \leq k+1$. Let $[p_{11}^l]$ be the equivalence class of the projection p_{11}^l in $V(B^+)$. Since $e_{11}^l \in A$, for $1 \leq l \leq k$, the formal difference $x = [e_{11}^l] - [0]$ is in $K_0(A)$. We have $[p_{11}^l] - [0] = [\phi(e_{11}^l)] - [0] = K_0(\phi)(x) = K_0(\psi)(x) = [q_{11}^l] - [0]$. Thus there is $r \in \mathbb{M}_\infty(B^+)$ such that $[p_{11}^l] + [r] = [q_{11}^l] + [r]$ in $V(B^+)$. For $l = k+1$ we have $[e_{11}^{k+1}] - [1] \in K_0(A)$ and similarly there is $r \in \mathbb{M}_\infty(B^+)$ such that $[p_{11}^{k+1}] + [r] = [q_{11}^{k+1}] + [r]$ in $V(B^+)$. Since $B^+ = \overline{\bigcup_{n=1}^\infty B_n^+}$, there is $m' \geq m$ such that r is equivalent to some projection in $\mathbb{M}_\infty(B_{m'}^+)$, and we may assume that $r \in \mathbb{M}_\infty(B_{m'}^+)$. Hence the projections $\text{diag}(p_{11}^l, r), \text{diag}(q_{11}^l, r) \in \mathbb{M}_\infty(B_{m'}^+)$ are equivalent in B^+ . By [16, Lemma 7.2.8] (or, rather, by its proof), there is $m_l \geq m'$ such that $\text{diag}(p_{11}^l, r)$ and $\text{diag}(q_{11}^l, r)$ are equivalent in $B_{m_l}^+$. Since $B_{m_l}^+$ is finite-dimensional, p_{11}^l is equivalent to q_{11}^l , and so there is a partial isometry $w_l \in B_{m_l}^+$ such that $p_{11}^l = w_l w_l^*$ and $q_{11}^l = w_l^* w_l$. Now set

$$\sum_{l=1}^{k+1} \sum_{i=1}^{n_l} p_{i1}^l w_l q_{1i}^l = w.$$

An easy calculation shows that w is unitary in B^+ and $wq_{ij}^l = p_{ij}^l w$, for the above values of i, j, l . Thus $F = (\text{Ad } w)G$. Set $u_n = u^* w v$. Then $\phi = (\text{Ad } u_n)\psi$ on A_n . We have found a sequence of unitaries $(u_n)_{n=1}^\infty$ in B^+ with the property stated in part (ii), but this is equivalent to the existence of a sequence of unitaries in B^\sim with the same property, by the remark following Lemma 5.4.

(ii) \Rightarrow (i): Suppose that (ii) holds. As stated above, we may assume that the unitaries $(u_n)_{n=1}^\infty$ are in B^+ . Let p be in $\mathbb{M}_\infty(A^+)$. Then there is a positive integer n and a projection $q \in \mathbb{M}_\infty(A_n^+)$ such that $[p] = [q]$. Since $\phi = (\text{Ad } u_n)\psi$ on A_n , and we have $\phi^+ = (\text{Ad } u_n)\psi^+$ on A_n^+ , hence $\phi_*([p]) = \phi_*([q]) = \psi_*([q]) = \psi_*([p])$. Therefore $K_0(\phi) = K_0(\psi)$. ■

Theorem 6.3 *There is a unique functor $\overline{K}_0: \overline{\mathbf{AF}}^{\text{out}} \rightarrow \mathbf{DG}$ such that $K_0 = \overline{K}_0 \overline{\mathcal{F}}$:*

$$\begin{array}{ccc} \mathbf{AF} & \xrightarrow{\overline{\mathcal{F}}} & \overline{\mathbf{AF}}^{\text{out}} \\ & \searrow K_0 & \downarrow \overline{K}_0 \\ & & \mathbf{DG}. \end{array}$$

It is a strong classification functor, essentially surjective on objects, and a full functor. Moreover, for each pair of morphisms $\phi, \psi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ in \mathbf{AF} , we have $K_0(\phi) = K_0(\psi)$ if and only if $\overline{\mathcal{F}}(\phi) = \overline{\mathcal{F}}(\psi)$; in other words, \overline{K}_0 is a faithful functor.

Proof Define $\overline{K}_0: \overline{\mathbf{AF}}^{\text{out}} \rightarrow \mathbf{DG}$ as follows. For $\mathcal{A} \in \overline{\mathbf{AF}}^{\text{out}}$ set $\overline{K}_0(\mathcal{A}) = K_0(\mathcal{A})$. Let $\phi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ be a morphism in \mathbf{AF} . Set $\overline{K}_0(\overline{\mathcal{F}}(\phi)) = K_0(\phi)$. For each pair of morphisms $\phi, \psi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ in \mathbf{AF} we have $K_0(\phi) = K_0(\psi)$ if and only if $\overline{\mathcal{F}}(\phi) = \overline{\mathcal{F}}(\psi)$, by Lemma 6.2. This proves that \overline{K}_0 is well defined. The other properties of \overline{K}_0 are easy to prove, using Theorem 6.1 and the same properties of $\overline{\mathcal{F}}$ as used in the proof of Theorem 5.9.

Alternatively, the whole theorem follows immediately from Theorem 5.9 together with the equivalence of the categories \mathbf{BD} and \mathbf{DG} , an elementary proof of which is given below as an alternative proof of Corollary 6.5; one uses that the functor $\mathbf{BD} \rightarrow \mathbf{DG}$ in question is just the inductive limit functor, which acts as a natural isomorphism between the functors \mathcal{B} and K_0 , and therefore also between $\overline{\mathcal{B}}$ and \overline{K}_0 . ■

Next let us show that the categories $\overline{\mathbf{AF}}^{\text{out}}$ and \mathbf{DG} are equivalent, and this equivalence, given by the functor $\overline{K}_0: \overline{\mathbf{AF}}^{\text{out}} \rightarrow \mathbf{DG}$, is compatible with the classification of AF algebras via the functors $\mathcal{B}: \mathbf{AF} \rightarrow \mathbf{BD}$ and $\overline{\mathcal{F}}: \mathbf{AF} \rightarrow \overline{\mathbf{AF}}^{\text{out}}$, i.e., the related diagrams commute.

Theorem 6.4 *The functor $\overline{K}_0: \overline{\mathbf{AF}}^{\text{out}} \rightarrow \mathbf{DG}$ is an equivalence of categories. More precisely, there is a unique (up to natural isomorphism) functor $\mathcal{G}_0: \mathbf{DG} \rightarrow \overline{\mathbf{AF}}^{\text{out}}$ such that $\overline{K}_0 \mathcal{G}_0 \cong \text{id}_{\mathbf{DG}}$ and $\mathcal{G}_0 \overline{K}_0 \cong \text{id}_{\overline{\mathbf{AF}}^{\text{out}}}$. The functor \mathcal{G}_0 is full, faithful, essentially surjective, and a strong classification functor. Moreover, we have $\overline{K}_0 \mathcal{G}_0 K_0 = K_0$:*

$$\begin{array}{ccc}
 \mathbf{AF} & \xrightarrow{K_0} & \mathbf{DG} \\
 \downarrow K_0 & & \uparrow \overline{K}_0 \\
 \mathbf{DG} & \xrightarrow{\mathcal{G}_0} & \overline{\mathbf{AF}}^{\text{out}}.
 \end{array}$$

Proof The first part of the statement follows from Theorem 6.3 and [15, Theorem IV.4.1]. The functorial properties of \mathcal{G}_0 follow from [15, Theorem IV.4.1] and Lemma 5.10. For the proof of the last part of the statement note that we can construct \mathcal{G}_0 to have the property that $\overline{K}_0(\mathcal{G}_0(\overline{K}_0(\mathcal{A}))) = \overline{K}_0(\mathcal{A})$, for each $\mathcal{A} \in \overline{\mathbf{AF}}^{\text{out}}$, and that $\overline{K}_0(\mathcal{G}_0(\overline{K}_0(\psi))) = \overline{K}_0(\psi)$, for each morphism $\psi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ in $\overline{\mathbf{AF}}^{\text{out}}$ (see the proof of [15, Theorem IV.4.1]). Therefore, for each morphism $\phi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ in \mathbf{AF} , setting $\psi = \overline{\mathcal{F}}(\phi)$ we get $\overline{K}_0(\mathcal{G}_0(K_0(\phi))) = K_0(\phi)$.

Alternatively, use the remarks at the end of the proof of Theorem 6.3. ■

The category $\overline{\mathbf{AF}}^{\text{out}}$ can be used to relate the categories \mathbf{BD} and \mathbf{DG} . Let us collect all the functors in one (commutative) diagram; see Figure 1.

Corollary 6.5 *There is a full, faithful, essentially surjective, and (hence) strong classification functor from \mathbf{BD} to \mathbf{DG} . Also there is such a functor from \mathbf{DG} to \mathbf{BD} .*

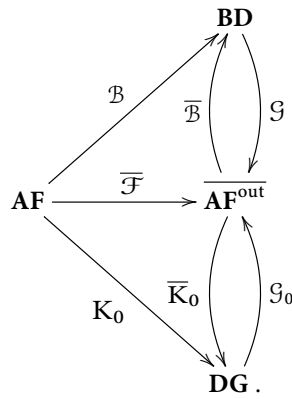


Figure 1

Proof Consider the functors $\mathcal{G}: \mathbf{BD} \rightarrow \overline{\mathbf{AF}^{\text{out}}}$ and $\overline{\mathcal{K}}_0: \overline{\mathbf{AF}^{\text{out}}} \rightarrow \mathbf{DG}$. Both of these are strong classification functors and also full, faithful and essentially surjective, by Theorems 5.11 and 6.3, and so also is $\overline{\mathcal{K}}_0\mathcal{G}: \mathbf{BD} \rightarrow \mathbf{DG}$. Similarly, the functor $\overline{\mathcal{B}}\mathcal{G}_0: \mathbf{DG} \rightarrow \mathbf{BD}$ has the above properties, by Theorems 5.9 and 6.4.

Alternatively, and in a much more elementary way, the obvious inductive limit functor $\mathbf{BD} \rightarrow \mathbf{DG}$, obtained by interpreting a Bratteli diagram as a sequence of finite ordered group direct sums of copies of \mathbb{Z} , as in [10, Section 2], is full, faithful, and essentially surjective by [7, Theorem 2.2], and so by [15, Theorem IV.4.1] an equivalence of categories. ■

Corollary 6.6 *The categories \mathbf{BD} , $\overline{\mathbf{AF}^{\text{out}}}$, and \mathbf{DG} are equivalent.*

Finally let us show that the three strong classification functors $\mathcal{B}: \mathbf{AF} \rightarrow \mathbf{BD}$, $\overline{\mathcal{F}}: \mathbf{AF} \rightarrow \overline{\mathbf{AF}^{\text{out}}}$, and $\mathcal{K}_0: \mathbf{AF} \rightarrow \mathbf{DG}$ for classification of AF algebras are essentially the same.

Theorem 6.7 *Consider the three strong classification functors $\mathcal{B}: \mathbf{AF} \rightarrow \mathbf{BD}$, $\overline{\mathcal{F}}: \mathbf{AF} \rightarrow \overline{\mathbf{AF}^{\text{out}}}$, and $\mathcal{K}_0: \mathbf{AF} \rightarrow \mathbf{DG}$ for AF algebras. For each objects $\mathcal{A}_1, \mathcal{A}_2$ in \mathbf{AF} , the following are equivalent:*

- (i) $\mathcal{A}_1 \cong \mathcal{A}_2$ in \mathbf{AF} ;
- (ii) $\mathcal{B}(\mathcal{A}_1) \cong \mathcal{B}(\mathcal{A}_2)$ in \mathbf{BD} ;
- (iii) $\overline{\mathcal{F}}(\mathcal{A}_1) \cong \overline{\mathcal{F}}(\mathcal{A}_2)$ in $\overline{\mathbf{AF}^{\text{out}}}$;
- (iv) $\mathcal{K}_0(\mathcal{A}_1) \cong \mathcal{K}_0(\mathcal{A}_2)$ in \mathbf{DG} .

For each pair of morphisms $\phi, \psi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ in \mathbf{AF} , the following statements are equivalent:

- (i) $\mathcal{B}(\phi) = \mathcal{B}(\psi)$;
- (ii) $\overline{\mathcal{F}}(\phi) = \overline{\mathcal{F}}(\psi)$;

- (iii) $K_0(\phi) = K_0(\psi)$;
 (iv) there is a sequence of unitaries $(u_n)_{n=1}^\infty$ in B^\sim such that $\phi = (\text{Ad } u_n)\psi$ on A_n , for $n \geq 1$;
 (v) there is a sequence of unitaries $(u_n)_{n=1}^\infty$ in B^\sim such that

$$\phi(a) = \lim_{n \rightarrow \infty} (\text{Ad } u_n)\psi(a),$$

for $a \in A$.

Proof The first part follows from the fact that we are considering classification functors. The second part follows from Lemma 5.4, Theorem 5.9, and Lemma 6.2. ■

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