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A SIMPLE PROOF OF THE MAXIMAL ERGODIC THEOREM

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1. Introduction. Let X be a σ-finite measure space and let T^k, k any integer, be a group of positive linear transformations in $Lp(X)$ such that

$$(1) \quad \int |T^k f|^p \leqq C \int |f|^p$$

with C independent of f and k. From now on f will be a positive function in $Lp(X)$ and we will use the following notation:

$$(A(n)f)(x) = \frac{1}{n+1} \sum_0^n (T^k f)(x)$$

$$(Mf)_L(x) = \sup_{n<L} (A(n)f)(x)$$

$$(Mf)(x) = \lim (Mf)_L(x) \quad L \to \infty.$$

With the conditions stated above we then have:

$$\int (Mf)^p \leqq C^2 \left(\frac{p}{p-1}\right)^p \int f^p,$$

with C being the same constant as in (1).

In the case $C = 1$, this result is due to A. Ionescu-Tulcea [**4**] and was used by Akcoglu in [**1**] to solve the same problem for a non-invertible positive contraction. The theorem is proved for an arbitrary C since it involves no extra work, but apart from the case of a cyclic group I can not think of an example that is not an isometry.

We will need a few facts about the Hardy-Littlewood maximal operator for functions defined on the integers (i.e. the ergodic maximal function associated with the shift transformation on the integers).

For $G(k)$ a positive function on the integers we define

$$(HG)_L(k) = \sup_{i<L} \frac{1}{i+1} \sum_0^i G(k+1).$$

If $\chi(L+N)$ represents the characteristic function of the interval $(-L-N, L+N)$, then for $0 < k < N$ it is obvious that

$$(HG)_L(k) = (H(G\chi(L+N)))_L(k).$$

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This observation, coupled with the fact that

$$\sum_{-\infty}^{\infty} ((HG)_L(k))^p \leqq \sum_{-\infty}^{\infty} (G(k))^p \cdot \left(\frac{p}{p-1}\right)^p$$

(see [6]) gives us

$$(2) \quad \sum_{-N}^{N} ((HG)_L(k))^p \leqq \sum_{-(N+L)}^{(N+L)} (G(k))^p \cdot \left(\frac{p}{p-1}\right)^p.$$

2. Proof of the theorem. The idea, originally due to Calderon [2] and further developed by Coifman and Weiss in [3] is that one can obtain results in the ergodic case by reducing the problem to the Hardy-Littlewood maximal operator. We claim that for L fixed and any k we have $(M(T^k f))_L(x) \leqq (T^k(Mf)_L)(x)$ for every x. To see this we decompose X into a disjoint union $X = \bigcup B_j$, $j = 1 \ldots L$, such that for x in B_j we have $(M(T^k f))_L(x) = (Aj(T^k f))(x)$.

Since T^k is positive and linear we have $(A_j(T^k f))(x) \leqq (T^k(Mf)_L)(x)$.

Therefore,

$$\int (M(T^k f))_L^p \leqq \int (T^k(Mf)_L)^p.$$

Applying this to $T^{-k}f$ we get

$$\int (Mf)_L^p \leqq \int (T^k(M(T^{-k}f))_L)^p \leqq c \int (M(T^{-k}f))_L^p.$$

Since k is arbitrary we can change k into $-k$ to obtain

$$(3) \quad \int (Mf)_L^p \leqq C \int (M(T^k f))_L^p \quad \text{for fixed } L \text{ and any } k.$$

If N is any positive number, it follows from (3) that

$$\int (Mf)_L^p \leqq \frac{C}{2N+1} \sum_{-N}^{N} \int (M(T^k f))_L^p.$$

Now for x fixed we have a function $G_x(k)$ defined on the integers by $G_x(k) = (T^k f)(x)$. Since it is clear that $(M(T^k f))_L(x) = (HG_x)_L(k)$ we have, using (3) and (2) and (1),

$$\int (Mf)_L^p \leqq \frac{C}{2N+1} \int \sum_{-N}^{N} ((HG_x)_L(k))^p$$

$$\leqq \left(\frac{p}{p-1}\right) \frac{C}{2N+1} \sum_{-N-L}^{N+L} \int (G_x(k))^p$$

$$\leqq \left(\frac{p}{p-1}\right) C^2 \frac{2N+2L+1}{2N+1} \int f^p.$$

Letting N tend to infinity we have

$$\int (Mf)_L{}^p \leqq C^2 \left(\frac{p}{p-1}\right)^p \int f^p$$

and finally

$$\int (Mf)^p \leqq C^2 \left(\frac{p}{p-1}\right)^p \int f^p$$

as we wanted.

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