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A NOTE ON A SQUARE TYPE FUNCTIONAL EQUATION

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The following square functional equation

$$(1)\quad f(x+\nu,\ y+\nu)+f(x+\nu,\ y-\nu)+f(x-\nu,\ y+\nu)+f(x-\nu,\ y-\nu) = 4f(x,\ y)$$

was considered (for example [1]–[9]) previously.

It is known [4] that (a) the square functional equation (1), or alternatively

$$(X^\nu Y^\nu+X^\nu Y^{-\nu}+X^{-\nu}Y^\nu+X^{-\nu}Y^{-\nu})f(x,\ y) = 4f(x,\ y)$$

has the harmonic polynomials

$$(2)\qquad f(x,\ y) = \mathrm{Re}(i\alpha_4 z^4+a_3z^3+a_2z^2+a_1z+a_0)$$

as the only measurably bounded solutions (bounded on a set of positive measure), where $X^\nu f(x,y)=f(x+\nu,y)$, $Y^\nu f(x,y)=f(x,y+\nu)$; $f(x,\ y)$ is a real-valued function of two real variables x, y in the plane R^2, ν is real, α_4 is real, and $a_j, j=0, 1, 2, 3$ are complex constants.

Further, (b) (1) and

$$(3)\qquad (X^\nu+X^{-\nu}+Y^\nu+Y^{-\nu})f(x,\ y) = 4f(x,\ y)$$

are equivalent without any regularity assumptions ([2], [4]).

We shall consider the following functional equation

$$(4)\qquad \begin{aligned}(X^{-\nu}Y^{-\nu}+Y^{-p\nu}+X^\nu Y^{-\nu}+X^{SC\nu}+X^\nu Y^\nu+Y^{SC\nu}+X^{-\nu}Y^\nu\\ +X^{-SC\nu})f(x,\ y) = 8f(x,\ y),\end{aligned}$$

for some arbitrary real number SC.

THEOREM. *The only measurably bounded solutions of equation* (4) *for arbitrary fixed* p ($|p|\neq\sqrt{2}$) *are the harmonic polynomials of the form* (2).

Proof. We may assume that $f(x,y)$ is of class C^∞ by the results in [1]. The equation (4) in the plane R^2 yields, by repeatedly differentiating both sides with respect to ν for $\nu=0$,

$$(5)\qquad f_{xx}+f_{yy} = 0,$$

$$(6)\qquad (2+p^4)f_{xxxx}+(2+p^4)f_{yyyy}+12f_{xxyy} = 0.$$

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From equation (5) it follows that

$$f_{xxxx}+f_{xxyy} = 0, \qquad f_{xxyy}+f_{yyyy} = 0, \tag{7}$$

and substituting (7) into (6) yields $(4-p^4)f_{xxyy}=0$. Since $|p|\neq\sqrt{2}$ we obtain

$$f_{xxyy} = 0, \tag{8}$$

which with (7) implies

$$f_{xxxx} = 0, \qquad f_{yyyy} = 0. \tag{9}$$

The equations (5), (8), (9) yield the form (2). Conversely, by substituting (2) into (4), one verifies that (2) satisfies equation (4). Q.E.D.

REMARK 1. For the case $|p|=\sqrt{2}$, the equation (4) implies a regular octagonal functional equation whose solutions are known [4] to be polynomials of degree 8; in particular, (2) is such a solution.

COROLLARY. *If* (4) *is satisfied, for fixed* $|p|\neq\sqrt{2}$, *by a measurably bounded function* $f(x, y)$, *then this function satisfies* (4) *for all* p.

REMARK 2. By the corollary, for measurably bounded solutions, the equation (4), for various values of $|p|\neq\sqrt{2}$ are equivalent. This does not seem to be true for the general solutions of (4).

Similar to equation (1) (c.f. [2], [4], [8], [9], [10]), equation (4) also has some geometric interpretation. For example, the case $p=0$ yields the square functional equation.

REMARK 3. The square functional equation implies the equation (4) for all p. This may readily be verified in view of (b).

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