Canad. J. Math. Vol. 67 (4), 2015 pp. 827–847
http://dx.doi.org/10.4153/CJM-2014-028-4

The Bochner–Schoenberg–Eberlein Property and Spectral Synthesis for Certain Banach Algebra Products

Eberhard Kaniuth

Abstract. Associated with two commutative Banach algebras A and B and a character θ of B is a certain Banach algebra product $A \times_\theta B$, which is a splitting extension of B by A. We investigate two topics for the algebra $A \times_\theta B$ in relation to the corresponding ones of A and B. The first one is the Bochner–Schoenberg–Eberlein property and the algebra of Bochner–Schoenberg–Eberlein functions on the spectrum, whereas the second one concerns the wide range of spectral synthesis problems for $A \times_\theta B$.

Introduction

Let A and B be commutative Banach algebras with Gelfand spectrum $\Delta(B) \neq \varnothing$ and let $\theta \in \Delta(B)$. Then the θ-product $A \times_\theta B$ is the Cartesian product $A \times B$ equipped with the multiplication

$$(a,b)\cdot(a',b') = \big(aa' + \theta(b)a' + \theta(b')a, bb'\big)$$

and the norm $\|(a,b)\| = \|a\| + \|b\|$. Then $A \times_\theta B$ is a commutative Banach algebra. The algebras $A \times_\theta B$ are sometimes referred to as *Lau products* because in [18] they have been introduced for the special case where B is the predual of a von Neumann algebra M and the identity of M is a multiplicative linear functional on B. The general definition and the first intensive study of these algebras are due to Monfared [19]. These algebras can serve as examples or counterexamples to several questions in abstract harmonic analysis. Apart from dealing with several other properties, such as amenability (see also [20]), [19] was mainly concerned with spectral synthesis problems for $A \times_\theta B$, building on an explicit description of the topology on the spectrum $\Delta(A \times_\theta B)$ of $A \times_\theta B$.

One purpose of this paper is a fairly comprehensive investigation of spectral synthesis and weak spectral synthesis for $A \times_\theta B$, thereby considerably extending results of [19]. In particular, we obtain characterizations of sets of synthesis and of weak spectral sets in $\Delta(A \times_\theta B)$ in terms of such sets in $\Delta(A)$ and $\Delta(B)$. The second concern is the study of the Bochner–Schoenberg–Eberlein property for $A \times_\theta B$ and of the algebra $C_{\mathrm{BSE}}(\Delta(A \times_\theta B))$ of BSE-functions on $\Delta(A \times_\theta B)$. We establish necessary and sufficient conditions, in terms of A and B, for $A \times_\theta B$ to be a BSE-algebra. Moreover,

Received by the editors February 5, 2014; revised July 7, 2014.
Published electronically January 29, 2015.
AMS subject classification: **46J10**, 46J25, 43A30, 43A45.
Keywords: commutative Banach algebra, splitting extension, Gelfand spectrum, set of synthesis, weak spectral set, multiplier algebra, BSE-algebra, BSE-function.

we describe BSE-functions on $\Delta(A \times_\theta B)$ by means of such functions on $\Delta(A)$ and $\Delta(B)$ and prove a criterion for $C_{\mathrm{BSE}}(\Delta(A \times_\theta B)) \cap C_0(\Delta(A \times_\theta B))$ to coincide with the Gelfand image $\widehat{A \times_\theta B}$ of $A \times_\theta B$.

Definitions, basic facts and references will be given at the outset of each section.

1 Preliminaries

Let A, B and θ be as in the introduction. Identifying A with $A \times \{0\}$ and B with $\{0\} \times B$, A is a closed ideal of $A \times_\theta B$ and B is a closed subalgebra, which is isometrically isomorphic with $A \times_\theta B/A$. Thus $A \times_\theta B$ may be viewed as a strongly splitting extension of B by A or twisted product of A and B. If $B = \mathbb{C}$ and θ is the identity map of $\mathbb{C}$, then $A \times_\theta \mathbb{C}$ coincides with the unitization of A.

The Banach space dual $(A \times_\theta B)^*$ can be identified with $A^* \times B^*$ through $\langle(\phi, \psi), (a, b)\rangle = \langle\phi, a\rangle + \langle\psi, b\rangle$, $\phi \in A^*$, $\psi \in B^*$. The dual norm on $A^* \times B^*$ is the maximum norm $\|(\phi, \psi)\| = \max\{\|\phi\|, \|\psi\|\}$.

Proposition 1.1 ([19, Theorem 3.1]) *Let A and B be commutative Banach algebras and let $\theta \in \Delta(B)$.*

(i) *$A \times_\theta B$ is semisimple if and only if A and B are semisimple.*
(ii) *If A and B are semisimple, then $A \times_\theta B$ is regular if and only both A and B are regular.*

Concerning regularity, it is worth mentioning the following very general result. If C is a commutative Banach algebra and I is a closed ideal of C, then C is regular if (and only if) both I and A/I are regular (see [12, Theorem 4.3.8]). The following explicit description of the topology on $\Delta(A \times_\theta B)$ will be substantially used throughout this paper. Note that the topology of $\Delta(A \times_\theta B)$ is the induced w^*-topology from $A^* \times B^*$, which in turn equals the product of the w^*-topologies of A^* and B^*.

Proposition 1.2 ([19, Propostion 2.4]) *Define subsets Φ and Ψ of $\Delta(A \times_\theta B)$ by*

$$\Phi = \{(\phi, \theta) : \phi \in \Delta(A)\} \quad \textit{and} \quad \Psi = \{(0, \psi) : \psi \in \Delta(B)\}.$$

Then $\Delta(A \times_\theta B) = \Phi \cup \Psi$, and the following hold:

(i) *Ψ is closed in $\Delta(A \times_\theta B)$ and $\Phi \cup \{(0, \theta)\}$ is compact.*
(ii) *The sets $U \times \{\theta\}$, where U is a neighbourhood of ϕ in $\Delta(A)$, form a neighbourhood base at (ϕ, θ).*
(iii) *For any $\psi \in \Delta(B)$, $\psi \neq \theta$, the sets $\{0\} \times V$, where V is a neighbourhood of ψ not containing θ, form a neighbourhood base at $(0, \psi)$.*
(iv) *The sets $(U \cap \Delta(A)) \times \{\theta\} \cup \{0\} \times W$, where W is a neighbourhood of θ in $\Delta(B)$ and U is a neighbourhood of 0 in A^*, form a neighbourhood base at $(0, \theta)$.*

Note that the codimension one ideal $k((0, \theta))$ of $A \times_\theta B$ is isometrically isomorphic to the direct product of A and $k(\theta)$. This simple fact will be used later.

A bounded net $(e_\alpha)_\alpha$ in A is called a *bounded Δ-weak approximate identity* if it satisfies $\langle e_\alpha, \gamma\rangle \to 1$ for every $\gamma \in \Delta(A)$. Such approximate identities were introduced in [8], where it was shown that they need not be bounded approximate identities.

The question of when $A \times_\theta B$ possesses a bounded approximate identity was already clarified in [19, Proposition 2.3].

Lemma 1.3 *$A \times_\theta B$ has an approximate identity (a bounded approximate identity, a bounded Δ-weak approximate identity) if and only if B has the respective kind of approximate identity.*

Proof Recall that, for $(a,b),(u,v) \in A \times_\theta B$,

$$\|(a,b)(u,v) - (a,b)\| = \|au + \theta(b)u + \theta(v)a - a\| + \|bv - b\|.$$

It follows that if $(u_\alpha, v_\alpha)_\alpha$ is an approximate identity of any of the three types for $A \times_\theta B$, then $(v_\alpha)_\alpha$ is an approximate identity of the corresponding type for B (note that $\theta \neq 0$).

Conversely, let $(v_\alpha)_\alpha$ be an approximate identity for B of any of the three types. Then, taking into account that, for $\phi \in \Delta(A)$ and $\psi \in \Delta(B)$,

$$(\phi,\theta)(0,v_\alpha) = \theta(v_\alpha) \to 1 \quad \text{and} \quad (0,\psi)(0,v_\alpha) = \psi(v_\alpha) \to 1,$$

it is immediate that $(0, v_\alpha)_\alpha$ forms an approximate identity of the respective type for $A \times_\theta B$. ∎

2 BSE-functions on $\Delta(A \times_\theta B)$

Let A be a commutative Banach algebra with Gelfand spectrum $\Delta(A)$. A bounded continuous function σ on $\Delta(A)$ is called a BSE-*function* if there exists a constant $C > 0$ such that for any finitely many elements $\phi_1, \dots, \phi_n$ of $\Delta(A)$ and complex numbers $c_1, \dots, c_n$ the inequality

$$\Big|\sum_{j=1}^{n} c_j \sigma(\phi_j)\Big| \leq C \cdot \Big\|\sum_{j=1}^{n} c_j \phi_j\Big\|_{A^*}$$

holds. The BSE-norm of σ, $\|\sigma\|_{\mathrm{BSE}}$, is defined to be the infimum of all such constants C. Let $C_{\mathrm{BSE}}(\Delta(A))$ denote the set of all BSE-functions. With the norm $\|\cdot\|_{\mathrm{BSE}}$, $C_{\mathrm{BSE}}(\Delta(A))$ forms a semisimple Banach algebra [25, Lemma 1]. A bounded continuous function σ on $\Delta(A)$ is a BSE-function if and only if there exists a bounded net $(a_\alpha)_\alpha$ in A such that $\widehat{a}_\alpha(\phi) \to \sigma(\phi)$ for every $\phi \in \Delta(A)$. This extremely useful criterion was shown in [25, Theorem 4]. Clearly, for every $a \in A$, the Gelfand transform $\widehat{a}$ belongs to $C_{\mathrm{BSE}}(\Delta(A)) \cap C_0(\Delta(A))$ and satisfies $\|\widehat{a}\|_\infty \leq \|\widehat{a}\|_{\mathrm{BSE}} \leq \|a\|$.

Recall that a bounded linear map $T\colon A \to A$ is called a *multiplier* of A if it satisfies $T(ab) = aT(b)$ for all $a, b \in A$. The set $M(A)$ of all multipliers of A is a closed subalgebra of the algebra of all bounded linear operators on A. For any $T \in M(A)$, there exists a unique continuous function $\widehat{T}$ on $\Delta(A)$ such that $\widehat{T}(\gamma)\widehat{a}(\gamma) = \widehat{T(a)}(\gamma)$ for all $a \in A$ and $\gamma \in \Delta(A)$ [17, Theorem 1.2.2]. For any subset M of $M(A)$, let $\widehat{M}$ or $M^\wedge$ denote the set of all $\widehat{T}$, $T \in M$. A Banach algebra A without order (that is, $aA = \{0\}$ implies $a = 0$) is called a BSE-*algebra* (or is said to have the BSE-*property*) if $C_{\mathrm{BSE}}(\Delta(A)) = \widehat{M(A)}$.

The notion of BSE-algebra and the algebra of BSE-functions were first introduced and studied by Takahasi and Hatori [25] and subsequently by several authors for various kinds of Banach algebras ([9], [10], [14], [26], and [27]). BSE-algebras also played a role in [15] and [28].

The contraction BSE stands for Bochner–Schoenberg–Eberlein and refers to the classical theorem, proved by Bochner [3] and Schoenberg [24] for the additive group of real numbers and by Eberlein [5] for general locally compact Abelian groups G, stating that, in the above terminology, the group algebra $L^1(G)$ is a BSE-algebra (see [23]). We remind the reader that $\Delta(L^1(G))$ is canonically homeomorphic to the dual group $\widehat{G}$ of G and the multiplier algebra $M(L^1(G))$ is isometrically isomorphic to the measure algebra $M(G)$.

Now let A and B be semisimple commutative Banach algebras and $\theta \in \Delta(B)$. Our first aim in this section is to describe explicitly the BSE-functions on $\Delta(A \times_\theta B)$ in terms of those functions on $\Delta(A)$ and $\Delta(B)$.

Lemma 2.1 *Let $\tau \in C_{\mathrm{BSE}}(\Delta(A)) \cap C_0(\Delta(A))$ and let $\rho \in C_{\mathrm{BSE}}(\Delta(B))$. Define a function σ on $\Delta(A \times_\theta B)$ by*

$$\sigma(\phi, \theta) = \tau(\phi) + \rho(\theta) \quad \textit{and} \quad \sigma(0, \psi) = \rho(\psi)$$

for $\phi \in \Delta(A)$ and $\psi \in \Delta(B)$. Then $\sigma \in C_{\mathrm{BSE}}(\Delta(A \times_\theta B))$.

Proof Note that σ is continuous since τ vanishes at infinity and $(0, \theta)$ is the only possible accumulation point of $\Delta(A) \times \{\theta\}$ in $\Delta(A \times_\theta B) \setminus (\Delta(A) \times \{\theta\})$.

Since $\tau \in C_{\mathrm{BSE}}(\Delta(A))$ and $\rho \in C_{\mathrm{BSE}}(\Delta(B))$, there exist bounded nets $(a_\alpha)_\alpha$ in A and $(b_\beta)_\beta$ in B such that $\widehat{a}_\alpha(\phi) \to \tau(\phi)$ for all $\phi \in \Delta(A)$ and $\widehat{b}_\beta(\psi) \to \rho(\psi)$ for all $\psi \in \Delta(B)$. Then the bounded product net $(a_\alpha, b_\beta)_{(\alpha,\beta)}$ in $A \times_\theta B$ satisfies $(\widehat{a_\alpha, b_\beta})(0, \psi) \to \rho(\psi)$ and

$$(\widehat{a_\alpha, b_\beta})(\phi, \theta) = \widehat{a}_\alpha(\phi) + \widehat{b}_\beta(\theta) \to \tau(\phi) + \rho(\theta) = \sigma(\phi, \theta).$$

This shows that $\sigma \in C_{\mathrm{BSE}}(\Delta(A \times_\theta B))$. ∎

Lemma 2.2 *Let $\sigma \in C_{\mathrm{BSE}}(\Delta(A \times_\theta B))$ and define functions τ on $\Delta(A)$ and ρ on $\Delta(B)$ by*

$$\tau(\phi) = \sigma(\phi, \theta) - \sigma(0, \theta) \quad \textit{and} \quad \rho(\psi) = \sigma(0, \psi)$$

for $\phi \in \Delta(A)$ and $\psi \in \Delta(B)$. Then $\tau \in C_{\mathrm{BSE}}(\Delta(A)) \cap C_0(\Delta(A))$ and $\rho \in C_{\mathrm{BSE}}(\Delta(B))$.

Proof Of course, the functions τ and ρ are both continuous. Moreover, the structure of the topology on $\Delta(A \times_\theta B)$ shows that τ vanishes at infinity on $\Delta(A)$. There exists a bounded net $(a_\alpha, b_\alpha)_\alpha$ in $A \times_\theta B$ such that $(\widehat{a_\alpha, b_\alpha})(\omega) \to \sigma(\omega)$ for every $\omega \in \Delta(A \times_\theta B)$. Thus

$$\widehat{b}_\alpha(\psi) = (\widehat{a_\alpha, b_\alpha})(0, \psi) \to \sigma(0, \psi)$$

for each $\psi \in \Delta(B)$, and therefore also for every $\phi \in \Delta(A)$,

$$\widehat{a}_\alpha(\phi) = (\widehat{a_\alpha, b_\alpha})(\phi, \theta) - \widehat{b}_\alpha(\theta) \to \sigma(\phi, \theta) - \sigma(0, \theta) = \tau(\phi).$$

It follows that $\tau \in C_{\mathrm{BSE}}(\Delta(A))$ and $\rho \in C_{\mathrm{BSE}}(\Delta(B))$. ∎

Combining Lemma 2.1 and Lemma 2.2, we obtain the following theorem.

Theorem 2.3 *The* BSE*-functions on* $\Delta(A \times_\theta B)$ *are precisely the functions* σ *of the form*

$$\sigma(\phi,\theta) = \tau(\phi) + \rho(\theta) \quad \textit{and} \quad \sigma(0,\psi) = \rho(\psi)$$

for $\phi \in \Delta(A)$ *and* $\psi \in \Delta(B)$*, where* $\tau \in C_{\mathrm{BSE}}(\Delta(A)) \cap C_0(\Delta(A))$ *and* $\rho \in C_{\mathrm{BSE}}(\Delta(B))$*.*

In passing we observe that the algebra of BSE-functions on $\Delta(A \times_\theta B)$ is also a twisted product type algebra.

Corollary 2.4 *Let* $\delta_\theta \in \Delta\big(C_{\mathrm{BSE}}(\Delta(B))\big)$ *denote the point evaluation at* $\theta \in \Delta(B)$*. Then the map* $\Sigma\colon (\tau,\rho) \to \sigma$ *of Theorem 2.3 is an isometric isomorphism from*

$$\big(C_{\mathrm{BSE}}(\Delta(A)) \cap C_0(\Delta(A))\big) \times_{\delta_\theta} C_{\mathrm{BSE}}(\Delta(B))$$

onto $C_{\mathrm{BSE}}(\Delta(A \times_\theta B))$*.*

Proof It is obvious that Σ is a bijective linear map. Let $\tau_j \in C_{\mathrm{BSE}}\big(\Delta(A)) \cap C_0(\Delta(A)\big)$ and $\rho_j \in C_{\mathrm{BSE}}(\Delta(B))$ and set $\sigma_j = \Sigma((\tau_j,\rho_j))$, $j = 1,2$, and $\sigma = \Sigma\big((\tau_1,\rho_1)(\tau_2,\rho_2)\big)$. Then, for any $\phi \in \Delta(A)$,

$$\begin{aligned}\sigma(\phi,\theta) &= \Sigma\Big(\big(\tau_1\tau_2 + \rho_1(\theta)\tau_2 + \rho_2(\theta)\tau_1, \rho_1\rho_2\big)\Big)\\ &= [\tau_1(\phi) + \rho_1(\theta)][\tau_2(\phi) + \rho_2(\theta)]\\ &= \sigma_1(\phi,\theta)\sigma_2(\phi.\theta).\end{aligned}$$

Similarly, $\sigma(0,\psi) = \sigma_1(0,\theta)\sigma_2(0,\theta)$ for every $\psi \in \Delta(B)$. Thus Σ is an algebra isomorphism. Finally, Σ is isometric. Indeed, using the characterization of BSE-functions in terms of bounded nets [25, Theorem 4] and the description of the BSE-norm in [25, Remark, p. 154], it is clear that the assignment $\sigma \to (\tau,\rho)$ has the property that $\|\sigma\|_{\mathrm{BSE}} = \|\tau\|_{\mathrm{BSE}} + \|\rho\|_{\mathrm{BSE}}$. ∎

In [27] the authors were interested in the class of commutative Banach algebras A for which the given norm coincides with the BSE-norm and they provided a long list of examples for this to happen. We note here that $A \times_\theta B$ belongs to this class if and only if both A and B do so. In fact, if $\|(a,b)\| = \|\widehat{(a,b)}\|_{\mathrm{BSE}}$ for $a \in A$ and $b \in B$, then

$$\|a\| + \|b\| = \|(a,b)\| = \|\widehat{(a,b)}\|_{\mathrm{BSE}} = \|\widehat{a}\|_{\mathrm{BSE}} + \|\widehat{b}\|_{\mathrm{BSE}} \le \|a\| + \|b\|,$$

and hence $\|a\| = \|\widehat{a}\|_{\mathrm{BSE}}$ and $\|b\| = \|\widehat{b}\|_{\mathrm{BSE}}$. The converse follows in the same manner.

The following lemma will be used twice in Section 3.

Lemma 2.5 *Let* $\sigma \in C_{\mathrm{BSE}}(\Delta(A \times_\theta B))$*, and let* $\tau \in C_{\mathrm{BSE}}(\Delta(A))$ *and* $\rho \in C_{\mathrm{BSE}}(\Delta(B))$ *associated with* σ *as in Lemma 2.2. If* $\sigma = \widehat{S}$ *for some multiplier* S *of* $A \times_\theta B$*, then* $\tau = \widehat{T}$ *for some* $T \in M(A)$ *and* $\rho = \widehat{R}$ *for some* $R \in M(B)$*.*

Proof Note first that the multiplier S maps the ideal $I = A \times \{0\}$ into itself. In fact, for $a \in A$ and $\psi \in \Delta(B)$, we have $\widehat{S(a,0)}(0,\psi) = \sigma(0,\psi)\widehat{(a,0)}(0,\psi) = 0$, and hence

https://doi.org/10.4153/CJM-2014-028-4 Published online by Cambridge University Press

$S(a,0) \in k(\{0\} \times \Delta(B)) = A \times \{0\}$ since B is semisimple. Define $T: I \to I$ by

$$T(a,0) = [S - \rho(\theta)](a,0), \quad a \in A.$$

It is clear that T is a multiplier. Now, identifying A with I,

$$\begin{aligned}\widehat{T}(\phi)\widehat{a}(\phi) &= \widehat{T(a)}(\phi) = \widehat{S(a,0)}(\phi,\theta) - \rho(\theta)\widehat{(a,0)}(\phi,\theta)\\ &= \widehat{S}(\phi,\theta)\widehat{(a,0)}(\phi,\theta) - \rho(\theta)\widehat{(a,0)}(\phi,\theta)\\ &= [\sigma(\phi,\theta) - \rho(\theta)]\widehat{a}(\phi)\end{aligned}$$

for all $a \in A$ and $\phi \in \Delta(A)$, whence $\widehat{T} = \tau$.

For the second assertion, for any $b \in B$ write $S(0,b) = (u_b, v_b)$ and define $R: B \to B$ by $R(b) = v_b$. Then R is a multiplier of B since, for $b, c \in B$,

$$\begin{aligned}(u_{bc}, v_{bc}) &= S(0,bc) = (0,b)S(0,c) = (0,b)(u_c,v_c)\\ &= (\theta(b)u_c, bv_c),\end{aligned}$$

and hence $R(bc) = v_{bc} = bv_c = bR(c)$. Moreover,

$$\begin{aligned}\widehat{R}(\psi)\widehat{b}(\psi) &= \widehat{R(b)}(\psi) = \widehat{v_b}(\psi) = \widehat{(u_b,v_b)}(0,\psi)\\ &= \widehat{S(0,b)}(0,\psi) = \widehat{S}(0,\psi)\widehat{(0,b)}(0,\psi)\\ &= \sigma(0,\psi)\widehat{b}(\psi)\end{aligned}$$

for all $b \in B$ and $\psi \in \Delta(B)$. This implies that $\widehat{R} = \rho$. ■

We now turn to the second purpose of this section, the question of when the algebra $\widehat{A \times_\theta B}$ coincides with $C_{\mathrm{BSE}}(\Delta(A \times_\theta B)) \cap C_0(\Delta(A \times_\theta B))$.

Proposition 2.6 *Let A be a semisimple commutative Banach algebra such that $C_{\mathrm{BSE}}(\Delta(A)) \cap C_0(\Delta(A)) = \widehat{A}$, and let I be a closed ideal of A such that A/I is semisimple. Then*

$$C_{\mathrm{BSE}}(\Delta(I)) \cap C_0(\Delta(I)) = \widehat{I}.$$

Proof Let $\tau \in C_{\mathrm{BSE}}(\Delta(I)) \cap C_0(\Delta(I))$ and define a function σ on $\Delta(A)$ by $\sigma(\phi) = \tau(\phi)$ for $\phi \in \Delta(I)$ and $\sigma = 0$ on $\Delta(A) \setminus \Delta(I)$. Then σ is continuous, since τ vanishes at infinity on the open subset $\Delta(I)$ of $\Delta(A)$, and hence $\sigma \in C_0(\Delta(A))$. Moreover, σ is a BSE-function. In fact, there exists a bounded net $(x_\alpha)_\alpha$ in I such that $\widehat{x}_\alpha(\phi) \to \tau(\phi)$ for all $\phi \in \Delta(I)$, and since $\widehat{x}_\alpha = 0$ on $\Delta(A) \setminus \Delta(I)$, we have $\widehat{x}_\alpha(\rho) \to \sigma(\rho)$ for every $\rho \in \Delta(A)$. Now, by hypothesis, $\sigma = \widehat{x}$ for some $x \in A$. It suffices to observe that $x \in I$. However, this follows from the facts that A/I is semisimple and $\widehat{x}(\psi) = \sigma(\psi) = 0$ for all $\psi \in \Delta(A) \setminus \Delta(I)$. ■

Theorem 2.7 *For $A \times_\theta B$ the following two conditions are equivalent.*

(i) $C_{\mathrm{BSE}}(\Delta(A \times_\theta B)) \cap C_0(\Delta(A \times_\theta B)) = \widehat{A \times_\theta B}$.
(ii) $C_{\mathrm{BSE}}(\Delta(A)) \cap C_0(\Delta(A)) = \widehat{A}$ *and* $C_{\mathrm{BSE}}(\Delta(B)) \cap C_0(\Delta(B)) = \widehat{B}$.

Proof (i) ⇒ (ii) Since $(A \times_\theta B)/(A \times \{0\}) = B$ is semisimple, Proposition 2.6 implies that $C_{\mathrm{BSE}}(\Delta(A)) \cap C_0(\Delta(A)) = \widehat{A}$.

https://doi.org/10.4153/CJM-2014-028-4 Published online by Cambridge University Press

Let $\rho \in C_{\mathrm{BSE}}(\Delta(B)) \cap C_0(\Delta(B))$ and define the function σ on $\Delta(A \times_\theta B)$ as in Lemma 2.1, taking $\tau = 0$. Then $\sigma \in C_{\mathrm{BSE}}(\Delta(A \times_\theta B))$. Moreover, σ vanishes at infinity on $\Delta(A \times_\theta B)$ since $\rho \in C_0(\Delta(B))$ and the subset $(\Delta(A) \times \{\theta\}) \cup \{(0,\theta)\}$ of $\Delta(A \times_\theta B)$ is compact (Proposition 1.2). Then, by hypothesis, $\sigma = \widehat{(a,b)}$ for some $(a,b) \in A \times_\theta B$ and hence, for every $\psi \in \Delta(B)$,

$$\rho(\psi) = \sigma(0,\psi) = \widehat{(a,b)}(0,\psi) = \widehat{b}(\psi).$$

(ii) $\Rightarrow$ (i) Let $\sigma \in C_{\mathrm{BSE}}(\Delta(A \times_\theta B)) \cap C_0(\Delta(A \times_\theta B))$ and define functions τ on $\Delta(A)$ and ρ on $\Delta(B)$ as in Lemma 2.2. Then $\tau \in C_{\mathrm{BSE}}(\Delta(A)) \cap C_0(\Delta(A))$ and $\rho \in C_{\mathrm{BSE}}(\Delta(B)) \cap C_0(\Delta(B))$. Therefore, $\tau = \widehat{a}$ and $\rho = \widehat{b}$ for certain $a \in A$ and $b \in B$, respectively. Now this implies that $\sigma = \widehat{(a,b)}$. In fact, for $\phi \in \Delta(A)$,

$$\begin{aligned}\widehat{(a,b)}(\phi,\theta) &= \widehat{a}(\phi) + \widehat{b}(\theta) = \tau(\phi) + \rho(\theta)\\ &= \sigma(\phi,\theta) - \sigma(0,\theta) + \rho(\theta) = \sigma(\phi,\theta),\end{aligned}$$

and for every $\psi \in \Delta(B)$, $\widehat{(a,b)}(0,\psi) = \widehat{b}(\psi) = \sigma(0,\psi)$. ∎

3 The BSE Property for $A \times_\theta B$

In this section, we address the problem of finding necessary and sufficient conditions, in terms of A and B, for $A \times_\theta B$ to be a BSE-algebra. We start with a simple lemma which was shown in [10, Theorem 3.4] for unital B.

Lemma 3.1 *If $A \times_\theta B$ is a* BSE-*algebra, then so is B.*

Proof Recall that by [25, Corollary 5], for any commutative Banach algebra C, we have $\widehat{M(C)} \subseteq C_{\mathrm{BSE}}(\Delta(C))$ if and only if C has a bounded Δ-weak approximate identity. Moreover, if $A \times_\theta B$ has such an approximate identity, then so does B (Lemma 1.3). Since $A \times_\theta B$ is a BSE-algebra, it follows that $\widehat{M(B)} \subseteq C_{\mathrm{BSE}}(\Delta(B))$. To prove the reverse inclusion, let $\rho \in C_{\mathrm{BSE}}(\Delta(B))$ and define σ on $\Delta(A \times_\theta B)$ by $\sigma(\phi,\theta) = \rho(\theta)$ for $\phi \in \Delta(A)$ and $\sigma(0,\psi) = \rho(\psi)$ for $\psi \in \Delta(B)$. By Lemma 2.1 (taking $\tau = 0$), $\sigma \in C_{\mathrm{BSE}}(\Delta(A \times_\theta B))$, and hence $\sigma = \widehat{S}$ for some multiplier S of $A \times_\theta B$. Lemma 2.5 now yields that $\rho = \widehat{R}$ for some $R \in M(B)$, as was to be shown. ∎

Let A_u denote the unitization of A, and let $I = \{0\} \times k(\theta)$, which is a closed ideal of $A \times_\theta B$. Fix $e \in B$ such that $\theta(e) = 1$. Using that $e^2 - e \in k(\theta)$, it is easily verified that the map

$$F\colon A_u \to (A \times_\theta B)/I, \quad a + \lambda u \to (a, \lambda e) + I, \quad a \in A, \lambda \in \mathbb{C},$$

is an algebra isomorphism. Moreover, F is a topological isomorphism since both algebras are semisimple and

$$\begin{aligned}\|(a,\lambda e) + I\| &\le \|(a,\lambda e)\| = \|a\| + |\lambda| \cdot \|e\|\\ &\le \|e\|(\|a\| + |\lambda|) = \|e\| \cdot \|a + \lambda u\|.\end{aligned}$$

It follows that there is a homeomorphism Γ between $\Delta(A_u) = \Delta(A) \cup \{\phi_\infty\}$, the one point compactification of $\Delta(A)$, and $\Delta((A \times_\theta B)/I) = (\Delta(A) \cup \{0\}) \times \{\theta\}$ given by

https://doi.org/10.4153/CJM-2014-028-4 Published online by Cambridge University Press

$\Gamma(\phi) = (\phi, \theta)$ for $\phi \in \Delta(A)$ and $\Gamma(\phi_\infty) = (0, \theta)$. In the sequel, we shall identify these two Gelfand spaces accordingly.

Proposition 3.2 *Let A be nonunital and suppose that $A \times_\theta B$ is a* BSE*-algebra. Then the unitization A_u is a* BSE*-algebra.*

Proof Since the Banach algebras A_u and $(A \times_\theta B)/I$ are topologically isomorphic, we can consider whichever is convenient. Since for any commutative Banach algebra C, $\widehat{M(C)} \subseteq C_{\mathrm{BSE}}(\Delta(C))$ if and only if C has a bounded Δ-weak approximate identity [25, Corollary 5], it is clear that $\widehat{M(A_u)} \subseteq C_{\mathrm{BSE}}(\Delta(A_u))$.

Conversely, let $\tau \in C_{\mathrm{BSE}}(\Delta(A_u))$. Then, by [14, Lemma 4.6(i)],

$$\tau|_{\Delta(A)} - \tau(\phi_\infty) 1_{\Delta(A)} \in C_{\mathrm{BSE}}(\Delta(A)) \cap C_0(\Delta(A)).$$

Let ρ be constantly equal to $\tau(\phi_\infty)$ on $\Delta(B)$, and define σ on $\Delta(A \times_\theta B)$ by

$$\sigma(\phi, \theta) = \tau(\phi) - \tau(\phi_\infty) + \rho(\theta) \quad \text{and} \quad \sigma(0, \psi) = \rho(\theta),$$

for $\phi \in \Delta(A)$, $\psi \in \Delta(B)$. Since B has a bounded Δ-weak approximate identity (Lemma 1.3), $\rho \in C_{\mathrm{BSE}}(\Delta(B))$, and hence $\sigma \in C_{\mathrm{BSE}}(\Delta(A \times_\theta B))$ by Lemma 2.1. Since $A \times_\theta B$ is a BSE-algebra, $\sigma = \widehat{S}$ for some $S \in M(A \times_\theta B)$. It is easily verified that $S(I) \subseteq I$ and hence S defines a multiplier $\widetilde{S}$ of $(A \times_\theta B)/I$ by $\widetilde{S}(x + I) = S(x) + I$ for $x \in A \times_\theta B$. Let $T = F^{-1} \widetilde{S} F \colon A_u \to A_u$, where F is the above isomorphism between A_u and $(A \times_\theta B)/I$. Clearly, T is a multiplier of A_u.

It remains to show that $\widehat{T} = \tau$. Now, for any $a \in A$, $\lambda \in \mathbb{C}$ and $\phi \in \Delta(A_u)$ such that $\phi \neq \phi_\infty$,

$$\begin{aligned}
\widehat{T}(\phi)\langle a + \lambda u, \phi\rangle &= \langle T(a + \lambda u), \phi\rangle = \langle F^{-1}\widetilde{S}F(a + \lambda u), \phi\rangle \\
&= \Big\langle \widetilde{S}\big((a, \lambda e) + I\big), (F^{-1})^*(\phi)\Big\rangle \\
&= \Big\langle S\big((a, \lambda e)\big), (\phi, \theta)\Big\rangle = \sigma(\phi, \theta)\big(\widehat{a}(\phi) + \lambda\big) \\
&= \tau(\phi)\langle a + \lambda u, \phi\rangle.
\end{aligned}$$

This shows that $\widehat{T}|_{\Delta(A)} = \tau|_{\Delta(A)}$. Similarly, it is verified that

$$\widehat{T}(\phi_\infty)\langle a + \lambda u, \phi_\infty\rangle = \tau(\phi_\infty)\langle a + \lambda u, \phi_\infty\rangle,$$

thus completing the proof. ∎

Theorem 3.3 *For $A \times_\theta B$ the following two conditions are equivalent.*

(i) *$A \times_\theta B$ is a* BSE*-algebra.*

(ii) *$C_{\mathrm{BSE}}(\Delta(A)) \cap C_0(\Delta(A)) = \widehat{A}$ and B is a* BSE*-algebra.*

Proof (i) ⇒ (ii) By Lemma 3.1, B is a BSE-algebra. The same is true for the unitization of A by Proposition 3.2. Then Theorem 4.8 of [14] shows that $C_{\mathrm{BSE}}(\Delta(A)) \cap C_0(\Delta(A)) = \widehat{A}$ provided that A is nonunital.

Now assume that A is unital and let $\tau \in C_{\mathrm{BSE}}(\Delta(A))$. Since then $\Delta(A) \times \{\theta\}$ is open and closed in $\Delta(A \times_\theta B)$, we can define a continuous function σ on $\Delta(A \times_\theta B)$ by $\sigma(\phi, \theta) = \tau(\phi)$ for $\phi \in \Delta(A)$ and $\sigma(0, \psi) = 0$ for $\psi \in \Delta(B)$. Then $\sigma \in C_{\mathrm{BSE}}(\Delta(A \times_\theta B))$ by Lemma 2.2 and hence $\sigma = \widehat{S}$ for some multiplier S of $A \times_\theta B$.

Lemma 2.5 shows that $T \in M(A) = A$. Thus $C_{\mathrm{BSE}}(\Delta(A)) \subseteq \widehat{A}$ and hence $C_{\mathrm{BSE}}(\Delta(A)) \cap C_0(\Delta(A)) = \widehat{A}$.

(ii) ⇒ (i) Since B is a BSE-algebra, B has a bounded Δ-weak approximate identity. By Lemma 1.3, the same is true for $A \times_\theta B$, and hence

$$M(A \times_\theta B)^\wedge \subseteq C_{\mathrm{BSE}}(\Delta(A \times_\theta B)).$$

To prove the reverse inclusion, let $\sigma \in C_{\mathrm{BSE}}(\Delta(A \times_\theta B))$ be given and define $\tau \in C_{\mathrm{BSE}}(\Delta(A)) \cap C_0(\Delta(A))$ and $\rho \in C_{\mathrm{BSE}}(\Delta(B))$ as in Lemma 2.2. Then by hypothesis, $\rho = \widehat{R}$ for some $R \in M(B)$ and $\tau = \widehat{a}_0$ for some $a_0 \in A$. Now define $S \colon A \times_\theta B \to A \times_\theta B$ by

$$S((a,b)) = \big(a_0 a + \theta(b) a_0 + \rho(\theta) a, R(b) \big).$$

Clearly, S is a bounded linear map. For $a \in A$ and $b \in B$ we have

$$\begin{aligned} S\big((a,0)(0,b)\big) &= \big(\theta(b) a_0 a + \rho(\theta)\widehat{b}(\theta) a, 0\big) \\ &= \big(\theta(b) a_0 a + \widehat{R(b)}(\theta) a, 0\big) \\ &= (a,0)\big(\theta(b) a_0, R(b)\big) = (a,0) S((0,b)) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} S\big((0,b)(0,c)\big) &= \big(\theta(bc) a_0, R(bc)\big) = \big(\theta(b)\theta(c) a_0, bR(c)\big) \\ &= (0,b)\big(\theta(c) a_0, R(c)\big) = (0,b) S((0,c)). \end{aligned} \tag{3.2}$$

Equations (3.1) and (3.2) together with the fact that the restriction of S to the ideal $I = A \times \{0\}$ is a multiplier of I, imply that S is a multiplier of $A \times_\theta B$.

It remains to verify that $\sigma = \widehat{S}$. For all $(a,b) \in A \times_\theta B$ and $\phi \in \Delta(A)$,

$$\begin{aligned} \widehat{S}(\phi,\theta)\widehat{(a,b)}(\phi,\theta) &= \widehat{a_0 a}(\phi) + \theta(b)\widehat{a}_0(\phi) + \rho(\theta)\widehat{a}(\phi) + \widehat{R(b)}(\theta) \\ &= \tau(\phi)[\widehat{a}(\phi) + \theta(b)] + \rho(\theta)[\widehat{a}(\phi) + \widehat{b}(\theta)] \\ &= [\tau(\phi) + \rho(\theta)][\widehat{a}(\phi) + \widehat{b}(\theta)] \\ &= \sigma(\phi,\theta)\widehat{(a,b)}(\phi,\theta). \end{aligned}$$

Similarly, we see that $\widehat{S}(0,\psi)\widehat{(a,b)}(0,\psi) = \sigma(0,\psi)\widehat{(a,b)}(0,\psi)$ for every $\psi \in \Delta(B)$. Since $A \times_\theta B$ is semisimple, it follows that $\widehat{S} = \sigma$. ■

In passing we remind the reader that a commutative Banach algebra is called *Tauberian* if the ideal consisting of all $x \in C$ such that $\widehat{x}$ has compact support in $\Delta(C)$ is dense in C. Note that $A \times_\theta B$ is Tauberian if and only if B is Tauberian because if $(a,b) \in A \times_\theta B$ and $\widehat{b}$ has compact support, then $\widehat{(a,b)}$ has compact support. The next lemma generalizes Corollary 4.9 of [14]. The proof, however, is similar.

Lemma 3.4 *Let A be a semisimple commutative Banach algebra such that*

$$C_{\mathrm{BSE}}(\Delta(A)) \cap C_0(\Delta(A)) = \widehat{A}.$$

In addition, suppose that A is Tauberian and has a bounded Δ-weak approximate identity. Then A is a BSE-*algebra.*

https://doi.org/10.4153/CJM-2014-028-4 Published online by Cambridge University Press

Proof Since A has a bounded Δ-weak approximate identity, $\widehat{M(A)} \subseteq C_{\mathrm{BSE}}(\Delta(A))$. To verify the reverse inclusion, let $C^0_{BE}(\Delta(A))$ denote the subalgebra of $C_{\mathrm{BSE}}(\Delta(A))$ introduced in [9, Definition 3.5] and note that $C^0_{BE}(\Delta(A)) \subseteq C_0(\Delta(A))$. Then, by hypothesis and since A is Tauberian,

$$\widehat{A} \subseteq C^0_{BE}(\Delta(A)) \subseteq C_{\mathrm{BSE}}(\Delta(A)) \cap C_0(\Delta(A)) = \widehat{A}$$

by [9, Proposition 4.1]. Thus $\widehat{A} = C^0_{BE}(\Delta(A))$ and hence $C_{\mathrm{BSE}}(\Delta(A)) \subseteq \widehat{M(A)}$ by [9, Proposition 4.3]. ∎

As an immediate consequence of Theorem 3.3 and Lemma 3.4 we obtain the following.

Corollary 3.5 *Suppose that A is Tauberian and has a bounded Δ-weak approximate identity. If $A \times_\theta B$ is a* BSE*-algebra, then both A and B are* BSE*-algebras. Conversely, if A and B are* BSE*-algebras and A is unital, then $A \times_\theta B$ is a* BSE*-algebra.*

For any semisimple commutative Banach algebra C, consider the following two properties, which we have studied for $C = A \times_\theta B$.

(1) C is a BSE-algebra;
(2) $C_{\mathrm{BSE}}(\Delta(C)) \cap C_0(\Delta(C)) = \widehat{C}$.

In concluding this section we point out that neither (1) nor (2) implies the other. Actually, examples can even be found within the class of Fourier and Fourier–Stieltjes algebras of locally compact groups [6] (Example 3.6 below). Further examples are provided by certain algebras of Lipschitz functions (compare [14, Example 6.2] and [9]).

Example 3.6 (a) We first give an example satisfying (2), but not (1). Let G be a connected, noncompact simple Lie group with finite centre. Let $B_0(G) = B(G) \cap C_0(G)$, the so-called Rajchman algebra associated with G. Then, as shown by Cowling [4, Theorem], $\Delta(B_0(G)) = G = \Delta(A(G))$ (equivalently, $B_0(G)/A(G)$ is a radical Banach algebra).

We claim that this fact implies that $B_0(G)$ satisfies condition (2). To see this, let $\sigma: G \to \mathbb{C}$ be a BSE-function for $B_0(G)$. Then, by [25, Theorem 4], there exists a bounded net $(u_\alpha)_\alpha$ in $B_0(G)$ such that $u_\alpha(x) \to \sigma(x)$ for all $x \in G$. The following argument is also used in [16]. Consider the functions u_α as functions on G_d, the group G endowed with the discrete topology. Then $(u_\alpha)_\alpha$ is a bounded net in $B(G_d) = C^*(G_d)^*$. Let u be a w^*-cluster point of this net in $B(G_d)$. Of course, then $u(x) = \sigma(x)$ for all $x \in G$. Thus $\sigma \in B(G_d)$ and hence $\sigma \in B(G)$ since σ is continuous [6, Corollaire 2.24]. This shows that

$$C_{\mathrm{BSE}}\big(\Delta(B_0(G))\big) \cap C_0\big(\Delta(B_0(G))\big) \subseteq \widehat{B_0(G)},$$

the converse inclusion being trivial. On the other hand, it is shown in [16] that for any locally compact group G, $B_0(G)$ cannot be a BSE-algebra unless G is amenable.

(b) Let G be an amenable, second countable, locally compact group whose regular representation is not completely reducible, that is, not unitarily equivalent to the direct sum of irreducible representations. Then, as shown in [7] and [2], $B(G) \cap C_0(G) \neq A(G)$. Moreover, the Fourier algebra $A(G)$ is a BSE-algebra, and $M(A(G)) = B(G)$,

https://doi.org/10.4153/CJM-2014-028-4 Published online by Cambridge University Press

the Fourier–Stieltjes algebra of G (see [14, Theorem 5.1] and[22, Corollary 19.2]). Thus, identifying $\Delta(A(G))$ with G [6],

$$C_{\mathrm{BSE}}(G) \cap C_0(G) = M(A(G)) \cap C_0(G) = B(G) \cap C_0(G) \neq A(G).$$

Example 3.6 (b) also shows that the converse of Corollary 3.5 may fail to be true even when A has a bounded approximate identity, but A is not unital.

4 Sets of Synthesis and Ditkin Sets for $A \times_\theta B$

Let A be a regular and semisimple commutative Banach algebra. For any subset M of A, the *hull* $h(M)$ of M is defined by

$$h(M) = \{\phi \in \Delta(A) : \phi(M) = \{0\}\}.$$

Associated with each closed subset E of $\Delta(A)$ are two distinguished ideals with hull equal to E, namely

$$k(E) = \{a \in A : \widehat{a}(\phi) = 0 \text{ for all } \phi \in E\}$$

and

$$j(E) = \{a \in A : \widehat{a} \text{ has compact support disjoint from } E\}.$$

If E is a singleton, say $\{\phi\}$, we simply write $k(\phi)$ and $j(\phi)$ rather than $k(\{\phi\})$ and $j(\{\phi\})$.

Then $k(E)$ is the largest ideal with hull E and $j(E)$ is the smallest such ideal. Recall that E is a *spectral set* (or *set of synthesis*) if $k(E) = \overline{j(E)}$ (equivalently, $k(E)$ is the only closed ideal with hull equal to E). Note that A is Tauberian if and only if $\varnothing$ is a set of synthesis. We say that *spectral synthesis* holds for A if every closed subset of $\Delta(A)$ is a spectral set. Moreover, E is a *Ditkin set* if $a \in \overline{a\,j(E)}$ for every $a \in k(E)$. Finally, A is said to *satisfy Ditkin's condition at infinity* if $\varnothing$ is a Ditkin set.

Let A and B be regular and semisimple commutative Banach algebras and $\theta \in \Delta(B)$. Our main purpose is to describe sets of synthesis for $A \times_\theta B$ in terms of such sets for A and B. In particular, we aim at a criterion for when spectral synthesis holds for $A \times_\theta B$. The crucial step towards solving these problems is the following theorem.

Theorem 4.1 *Let F be a closed subset of $\Delta(A)$ and G a closed subset of $\Delta(B)$.*

(i) *If F is compact, then $F \times \{\theta\}$ is a set of synthesis for $A \times_\theta B$ if and only if F is a set of synthesis for A.*

(ii) *Suppose that F is noncompact. If $\overline{F \times \{\theta\}}$ is a set of synthesis for $A \times_\theta B$, then F is a set of synthesis for A, and the converse holds when $\{(0,\theta)\}$ is a Ditkin set.*

(iii) *If $\{0\} \times G$ is a set of synthesis for $A \times_\theta B$, then G is a set of synthesis for B, and the converse holds whenever A is Tauberian.*

Proof (i) Suppose first that F is a set of synthesis for A and let $(a,b) \in k(F \times \{\theta\})$. Since A is regular, and hence normal [12, Corollary 4.2.9], and F is compact, there exists $y \in A$ such that $\widehat{y}(\phi) = \widehat{b}(\theta)$ for all ϕ in a relatively compact neighbourhood U of F in $\Delta(A)$. Now, given $\epsilon > 0$, since F is a spectral set and $a + y \in k(F)$, there exists

https://doi.org/10.4153/CJM-2014-028-4 Published online by Cambridge University Press

$x \in A$ such that $\|x-(a+y)\| \le \epsilon$ and $\widehat{x}$ vanishes on an open subset V with $F \subseteq V \subseteq U$. Then $\|(x-a,b)-(a,b)\| = \|x-(a+y)\| \le \epsilon$ and, by the choice of y, for every $\phi \in V$,

$$(x-y,b)^\wedge(\phi,\theta) = -\widehat{y}(\phi) + \widehat{b}(\theta) = 0.$$

Since $V \times \{\theta\}$ is open in $\Delta(A \times_\theta B)$ and $\epsilon > 0$ was arbitrary, this shows that $F \times \{\theta\}$ is a set of synthesis for $A \times_\theta B$.

Conversely, assume that $F \times \{\theta\}$ is a spectral set for $A \times_\theta B$ and let $a \in k(F)$ and $\epsilon > 0$ be given. Choose $y \in A$ such that $\widehat{y} = 1$ on a neighbourhood U of F is $\Delta(A)$. Since $(a,0) \in k(F \times \{\theta\})$, there exists $(u,v) \in A \times_\theta B$ such that $\widehat{u}(\phi) + \widehat{v}(\theta) = \widehat{(u,v)}(\phi,\theta) = 0$ for all ϕ in a neighbourhood V of F and

$$\|a-u\| + \|v\| = \|(a,0)-(u,v)\| \le \frac{\epsilon}{1+\|y\|}.$$

Now, let $x = u + \widehat{v}(\theta)y \in A$. Then $\widehat{x}(\phi) = \widehat{u}(\phi) + \widehat{v}(\theta)\widehat{y}(\phi) = 0$ for all $\phi \in V \cap U$ and

$$\|x-a\| \le \|u-a\| + |\widehat{v}(\theta)| \cdot \|y\| \le (1+\|y\|)(\|u-a\| + \|v\|) \le \epsilon.$$

Consequently, F is a set of synthesis for A.

(ii) Let $\overline{F \times \{\theta\}}$ be a set of synthesis, and let $a \in k(F)$ and $\epsilon > 0$. Then $(a,0) \in k(\overline{F \times \{\theta\}})$ and hence there exists $(u,v) \in j(\overline{F \times \{\theta\}})$ such that

$$\epsilon \ge \|(a,0)-(u,v)\| = \|a-u\| + \|v\|.$$

Since $0 = \widehat{(u,v)}(\phi,\theta) = \widehat{u}(\phi) + \widehat{v}(\theta)$ for all $\phi \in F$ and $\widehat{u} \in C_0(\Delta(A))$, we must have $\widehat{v}(\theta) = 0$ because F is noncompact. Thus $\widehat{u} = 0$ in a neighbourhood of F, as required.

Conversely, suppose that F is a spectral set and $\{(0,\theta)\}$ is a Ditkin set, and let $(a,b) \in k(\overline{F \times \{\theta\}})$. Then, since $(a,b) \in k((0,\theta))$, given $\epsilon > 0$, there exists $(u,v) \in A \times_\theta B$ such that $\widehat{(u,v)} = 0$ on a neighbourhood of $(0,\theta)$ in $\Delta(A \times_\theta B)$ and $\|(a,b)(u,v)-(a,b)\| \le \epsilon$. Since F is noncompact, we see as in the preceding paragraph that $b \in k(\theta)$ and $a \in k(F)$. As F is a set of synthesis, there exists $w \in j(F)$ with $\|a-w\| \le \epsilon/\|(u,v)\|$. Then

$$\begin{aligned}\|(a,b)-(w,b)(u,v)\| &\le \|(a,b)-(a,b)(u,v)\| \\ &\quad + \|(a,b)(u,v)-(w,b)(u,v)\| \\ &\le \epsilon + \|(u,v)\| \cdot \|a-w\| \le 2\epsilon.\end{aligned}$$

Finally, $\widehat{(w,b)}$ vanishes on a neighbourhood of $F \times \{\theta\}$ and $\widehat{(u,v)}$ vanishes on a neighbourhood of $(0,\theta)$. The description of the topology on $\Delta(A \times_\theta B)$ (Proposition 1.2) now shows that $(w,b)(u,v) \in j(\overline{F \times \{\theta\}})$.

(iii) Let $I = A \times \{0\}$ and recall that B is isometrically isomorphic to the quotient algebra $(A \times_\theta B)/I$, where $I = A \times \{0\}$. By the injection theorem for spectral sets of general regular and semisimple commutative Banach algebras [12, Theorem 5.2.7], if $\{0\} \times G$ is of synthesis for $A \times_\theta B$, then G is a spectral set for B, and the converse holds provided that the hull $h(I)$ is a spectral set for $A \times_\theta B$. To see that this latter condition is satisfied, let $(a,b) \in k(h(I))$ and $\epsilon > 0$ be given. Then $b = 0$ because $\widehat{b}(\psi) = \widehat{(a,b)}(0,\psi) = 0$ for all $\psi \in \Delta(B)$ and B is semisimple. Since A is Tauberian, there exists $u \in A$ such that $\|u-a\| \le \epsilon$ and $\widehat{u}$ has compact support, say C, in $\Delta(A)$.

Then $\|(u,0)-(a,b)\| \le \epsilon$ and $\widehat{(u,0)}$ vanishes outside the compact subset $C \times \{\theta\}$ of $\Delta(A \times_\theta B)$, which is disjoint from $\{0\} \times G$. Thus $(u,0) \in j(\{0\} \times G)$. ■

Remark 4.2 (1) The preceding theorem considerably extends Theorem 3.5 of [19], which is one of the main results of [19], in several respects. In [19, Theorem 3.5], B was assumed to be the Fourier algebra of a locally compact group, A was required to be unital and only singletons in $\Delta(A \times_\theta B)$ were considered.

(2) The proof of part (i) of Theorem 4.1 also shows that $F = \varnothing$ is a set of synthesis for A if and only if the singleton $\{(0,\theta)\}$ is a set of synthesis for $A \times_\theta B$. This does not only cover [19, Proposition 3.3], but also proves its converse.

The first one of the following three corollaries is an immediate consequence of Theorem 4.1.

Corollary 4.3 *If spectral synthesis holds for $A \times_\theta B$, then it also holds for A and B.*

It is obvious, and apparent from Theorem 4.1, that the nature of the topology on $\Delta(A \times_\theta B)$ indicates that the point $(0,\theta)$ plays a special role if $\Delta(A)$ is noncompact.

Corollary 4.4 *Let E be a closed subset of $\Delta(A \times_\theta B)$ and let*

$$F = \{\phi \in \Delta(A) : (\phi,\theta) \in E\} \quad \text{and} \quad G = \{\psi \in \Delta(B) : (0,\psi) \in E\}.$$

Suppose that F and G are sets of synthesis for A and B, respectively. Then E is a set of synthesis for $A \times_\theta B$ provided that one of the following two conditions is fulfilled.

(i) *The singleton $\{(0,\theta)\}$ is a Ditkin set.*

(ii) *A is unital and B satisfies Ditkin's condition at infinity.*

Proof Assuming first that (i) holds, since $\{(0,\theta)\}$ is a Ditkin set, it follows from Theorem 4.1(i) and (ii) that $E_1 = \overline{F \times \{\theta\}}$ is a set of synthesis for $A \times_\theta B$. Moreover, as A is Tauberian (compare Lemma 4.7 below), $E_2 = \{0\} \times G$ is of synthesis by Theorem 4.1(iii). Now, $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \{(0,\theta)\}$, which is a Ditkin set by hypothesis. This implies that E is a set of synthesis [12, Theorem 5.2.5].

Now assume that (ii) holds and note that $\varnothing$ is a Ditkin set for $A \times_\theta B$ since $\Delta(A)$ is compact and B satisfies Ditkin's condition at infinity. The proof then proceeds analogously to that in case (i), using [12, Theorem 5.2.5] again and the facts that F is compact and E is the disjoint union of the two spectral sets $F \times \{\theta\}$ and $\{0\} \times G$. ■

From Corollary 4.4 we conclude at once with the following.

Corollary 4.5 *Suppose that spectral synthesis holds for A and B and that one of conditions* (i) *and* (ii) *of Corollary 4.4 is fulfilled. Then spectral synthesis holds for $A \times_\theta B$.*

As exemplified by the above results, it is important to know when $A \times_\theta B$ satisfies Ditkin's condition at infinity, and when the singleton $\{(0,\theta)\}$ is a Ditkin set. The following three lemmas deal with these questions.

https://doi.org/10.4153/CJM-2014-028-4 Published online by Cambridge University Press

Lemma 4.6 *Suppose that A has an approximate identity, B satisfies Ditkin's condition at infinity and the ideal $k(\theta)$ has an approximate identity. Then $A \times_\theta B$ satisfies Ditkin's condition at infinity.*

Proof Let $(a, b) \in A \times_\theta B$ and $\epsilon > 0$ be given, and assume first that $\theta(b) \neq 0$. Since $\varnothing$ is a Ditkin set for B, there exists $v \in B$ such that $\widehat{v}$ has compact support and

$$\|vb - b\| \leq \min\{\epsilon, \epsilon|\theta(b)|\}.$$

It then follows that

$$|\theta(v) - 1| = |\theta(vb) - \theta(b)| \cdot |\theta(b)|^{-1} \leq \|vb - b\| \cdot |\theta(b)|^{-1} \leq \epsilon$$

and therefore

$$\begin{aligned} \|(a,b)(0,v) - (a,b)\| &= \|\theta(v)a - a, vb - b)\| \\ &\leq |\theta(v) - 1| \cdot \|a\| + \|vb - b\| \\ &\leq \epsilon(\|a\| + 1). \end{aligned}$$

Clearly, $\widehat{(0,v)}$ has compact support.

Now, assume that $b \in k(\theta)$. Then, again using that B satisfies Ditkin's condition at infinity, there exists $w_1 \in B$ such that $\|w_1 b - b\| \leq \epsilon$ and $\widehat{w}_1$ has compact support in $\Delta(B)$. Moreover, since $k(\theta)$ has an approximate identity, there exists $w_2 \in k(\theta)$ with $\|w_2(w_1 b) - w_1 b\| \leq \epsilon$. Then $v = w_1 w_2 \in k(\theta)$ satisfies

$$\|vb - b\| \leq \|w_2(w_1 b) - w_1 b\| + \|w_1 b - b\| \leq 2\epsilon,$$

and $\widehat{v}$ has compact support in $\Delta(B)$. Next, since A has an approximate identity, there exists $u \in A$ such that $\|ua - a\| \leq \epsilon$. Then

$$\|(a,b)(u,v) - (a,b)\| = \|(au - a, bv - b)\| = \|au - a\| + \|vb - b\| \leq 3\epsilon.$$

Finally, $\widehat{(u,v)}$ has compact support in $\Delta(A \times_\theta B)$ since $\widehat{(u,v)}(0,\psi) = \widehat{v}(\psi)$, $\widehat{v}$ has compact support in $\Delta(A)$ and $\Delta(A) \times \{0\} \cup \{(0,\theta)\}$ is compact.

This completes the proof. ∎

Most likely, the hypotheses on θ and B in Lemma 4.6 do not imply that $\{\theta\}$ is a Ditkin set for B. However, as we shall see next, the converse conclusion is true.

Lemma 4.7 *The singleton $\{(0,\theta)\}$ is a Ditkin set for $A \times_\theta B$ if and only if $\varnothing$ is a Ditkin set for A and $\{\theta\}$ is a Ditkin set for B.*

Proof Suppose first that $\{(0,\theta)\}$ is a Ditkin set for $A \times_\theta B$. It then follows from [12, Theorem 5.2.7(i)] that $\{\theta\}$ is a Ditkin set for B. To see that also $\varnothing$ is a Ditkin set for A, let $a \in A$ and $\epsilon > 0$ be given. Since $(a, 0) \in k((0,\theta))$ and $\{(0,\theta)\}$ is a Ditkin set, there exist $(x, y) \in A \times_\theta B$ and an open subset V of $\Delta(A \times_\theta B)$ such that $(0,\theta) \in V$, V has compact complement and

$$\epsilon \geq \|(a,0) - (a,0)(x,y)\| = \|a - ax + \theta(y)\| = \|a - ax\|.$$

Here we have used that $\widehat{y}(\theta) = \widehat{(x,y)}(0,\theta) = 0$. For any $\phi \in V$, we have $\widehat{(x,y)}(\phi,\theta) = \widehat{x}(\phi) + \widehat{y}(\theta) = \widehat{x}(\phi)$. So $\widehat{x}$ vanishes on the open subset $U = V \cap (\Delta(A) \times \{\theta\})$ of

$\Delta(A) \times \{\theta\}$. Furthermore, U has compact complement since

$$(\Delta(A) \times \{\theta\}) \setminus U = [\Delta(A) \times \{\theta\} \cup \{(0,\theta)\}] \setminus U$$

and $(\Delta(A) \times \{\theta\}) \cup \{(0,\theta)\}$ is closed in $\Delta(A \times_\theta B)$. Since a and ϵ are arbitrary, this shows that $\varnothing$ is a Ditkin set for A.

Conversely, assume that $\{\theta\}$ is a Ditkin set for B and $\varnothing$ is a Ditkin set for A, and let $(a,b) \in k((0,\theta))$ and $\epsilon > 0$ be given. Then $\widehat{b}(\theta) = 0$ and hence there exists $v \in B$ such that $\|bv - b\| \le \epsilon$ and $\widehat{v}$ has compact support K disjoint from $\{\theta\}$. Also there exists $u \in A$ such that $\|au - a\| \le \epsilon$ and $\widehat{u}$ has compact support, say C, in $\Delta(A)$. Then, since $b, v \in k(\theta)$,

$$\|(a,b)(u,v) - (a,b)\| = \|au - a\| + \|bv - b\| \le 2\epsilon.$$

Moreover, $\widehat{(u,v)}(\phi,\theta) = \widehat{u}(\phi) = 0$ on $(\Delta(A) \setminus C) \times \{\theta\}$ and $\widehat{(u,v)}(0,\psi) = \widehat{v}(\psi) = 0$ for all $\psi \in \Delta(B) \setminus K$. Thus $\widehat{(u,v)}$ has compact support not containing $\{(0,\theta)\}$. Consequently, $\{(0,\theta)\}$ is a Ditkin set for $A \times_\theta B$. ∎

The following lemma might be folklore. However, being unaware of a reference, we include the short proof.

Lemma 4.8 *Let C be a regular commutative Banach algebra such that, for some $\omega \in \Delta(C)$, $\{\omega\}$ is a Ditkin set. Then $\varnothing$ is a Ditkin set.*

Proof By regularity of C, there exists $e \in C$ such that $\widehat{e}(\omega) = 1$ and $\widehat{e}$ has compact support in $\Delta(C)$. Now, let $x \in C$ and $\epsilon > 0$ be given. Then $y = x - ex \in k(\omega)$, and hence we find $v \in k(\omega)$ such that $\|yv - y\| \le \epsilon$ and $\widehat{v}$ has compact support disjoint from $\{\omega\}$. Then the element $u = v + e - ev$ of C satisfies

$$\|xu - x\| = \|(x - ex)v - (x - ex)\| \le \epsilon$$

and $\widehat{u}$ has compact support in $\Delta(C)$. ∎

5 Weak Spectral Sets for $A \times_\theta B$

In connection with the union problem, that is the question of whether the union of two sets of synthesis is again a set of synthesis, Warner [30] introduced and studied the following more general concept. A closed subset E of $\Delta(A)$ is called a *weak spectral set* or *set of weak synthesis* if there exists $n \in \mathbb{N}$ such that $a^n \in \overline{j(E)}$ for each $a \in k(E)$. As shown in [30, Theorem 1.2] and [1, footnote 7, p. 885], this is equivalent to $k(E)/\overline{j(E)}$ being nilpotent. Adopting the notation of [30], we let $\xi(E)$ denote the smallest such number n and call it the *characteristic* of E. So E is a spectral set if and only if $\xi(E) = 1$. We say that *weak spectral synthesis holds for A* if every closed subset of $\Delta(A)$ is a weak spectral set. The interest in and relevance of weak spectral sets and weak spectral synthesis arises from the fact that there exist several Banach algebras for which weak spectral synthesis holds, whereas spectral synthesis fails. For examples, see [11, Section 1]. Also, the sphere $S^{n-1} \subseteq \mathbb{R}^n$, $n \ge 3$, L. Schwartz's classical example of a nonspectral set for $L^1(\mathbb{R}^n)$ (see [23, 7.3.1 and 7.3.2]), is a weak spectral set with characteristic $\xi(S^{n-1}) = \lfloor \frac{n+1}{2} \rfloor$ [29, Theorem 3]. Another important feature of the class of weak spectral sets is that it is closed under the formation of finite unions. Actually, for any two weak spectral sets E and F, $\xi(E \cup F) \leq \xi(E) + \xi(F)$ [30, Theorem 2.2] (see [21, Corollary 3.11] for a different approach).

In this section we investigate weak spectral sets in the spectrum of products $A \times_\theta B$, where we continue to suppose throughout that A and B are semisimple and regular commutative Banach algebras.

As with sets of synthesis, our aim is to describe sets of weak synthesis for $A \times_\theta B$ through sets of weak synthesis for A and B and to establish relations between the respective characteristics. The main results are Theorems 5.5 and 5.7 below. However, the estimates for the characteristics of weak spectral sets in $\Delta(A \times_\theta B)$ are not sharp enough to subsume the results of Section 4 on sets of synthesis. We start with some kind of analogue of Lemma 4.7 for weak spectral sets.

Lemma 5.1 *The singleton $\{(0,\theta)\}$ is a set of weak synthesis for $A \times_\theta B$ if and only if $\{\theta\}$ is a set of weak for synthesis for B and $\varnothing_A$, the empty set of $\Delta(A)$, if of weak synthesis for A. In this case,*

$$\max\big\{\,\xi(\varnothing_A), \xi(\{\theta\})\big\} \leq \xi\big(\,\{(0,\theta)\}\big) \leq \xi(\varnothing_A)\xi(\{\theta\}).$$

Proof To prove the lemma, we apply the injection theorem [11, Theorem 2.2] to $A \times_\theta B$ and the ideal $I = A \times \{0\}$.

Suppose first that $\{(0,\theta)\}$ is a weak spectral set for $A \times_\theta B$ and let $n = \xi\big(\,\{(0,\theta)\}\big)$. Then, since $B = A \times_\theta B/I$, $\{\theta\}$ is a weak spectral set for B and $\xi(\{\theta\}) \leq n$ [11, Theorem 2.2(i)]. To see that also $\xi(\varnothing_A) \leq n$, let $a \in A$ and $\epsilon > 0$ be given. Then $(a,0) \in k((0,\theta))$ and hence there exists $(x,y) \in j((0,\theta))$ such that

$$\epsilon \geq \|(a,0)^n - (x,y)\| = \|a^n - x\| + \|y\|.$$

Moreover, $\widehat{x}$ has compact support in $\Delta(A)$ since, for $\phi \in \Delta(A)$,

$$\widehat{x}(\phi) = \widehat{x}(\phi) + \widehat{(x,y)}(0,\theta) = \widehat{x}(\phi) + \widehat{y}(\theta) = \widehat{(x,y)}(\phi,\theta),$$

and the set $\{\phi \in \Delta(A) : (\phi,\theta) \in \operatorname{supp}\widehat{(x,y)}\}$ is compact.

For the 'if part' of the lemma, assume that $n = \xi(\{\theta\}) < \infty$ and $m = \xi(\varnothing_A) < \infty$. Notice that then $\xi(h(I)) \leq m$. Indeed, since $k(h(I)) = k(\{0\} \times \Delta(B)) = I$, we get

$$\begin{aligned} k(h(I))^m &= A^m \times \{0\} \subseteq \overline{j(\varnothing_A)} \times \{0\} \\ &= \overline{j(\varnothing_I)} = \overline{j(\{0\} \times \Delta(B))} \\ &= \overline{j(h(I))}, \end{aligned}$$

where the second last equation follows from the fact that the set $\Delta(A) \times \{\theta\}$ is open in $\Delta(A \times_\theta B)$. Now [11, Theorem 2.2(ii)] implies that $\{(0,\theta)\}$ is a weak spectral set for $A \times_\theta B$ and $\xi\big(\,\{(0,\theta)\}\big) \leq nm$. ∎

Lemma 5.1 generalizes Proposition 3.3 of [19].

Lemma 5.2 *Let C be a regular and semisimple commutative Banach algebra and E a closed subset of $\Delta(C)$. Let $\omega \in \Delta(C)$ and $I = k(\omega)$, and suppose that $\{\omega\}$ is a weak spectral set for C.*

https://doi.org/10.4153/CJM-2014-028-4 Published online by Cambridge University Press

(i) *If E is a weak spectral set for C, then $E \cap \Delta(I)$ is a weak spectral set for I and $\xi(E \cap \Delta(I)) \leq \xi(E) + \xi(\{\omega\})$. If, in addition, $\varnothing$ is a Ditkin set for C, then conversely $\xi(E \cap \Delta(I)) \geq \xi(E)$.*

(ii) *If $E \cap \Delta(I)$ is a weak spectral set for I and $\varnothing$ is a Ditkin set for C, then E is weak spectral set for C with $\xi(E) \leq \xi(E \cap \Delta(I))$.*

Proof Observe that for any closed subset E of $\Delta(C)$,

(5.1) $$k(E \cup \{\omega\}) = k(E \cap \Delta(I)) \quad \text{and} \quad j(E \cup \{\omega\}) = j(E \cap \Delta(I)).$$

The first equation in (5.1) and $j(E \cup \{\omega\}) \subseteq j(E \cap \Delta(I))$ being clear, consider any $x \in j(E \cap \Delta(I))$. Then the support K of $\widehat{x}|_{\Delta(I)}$ is compact in $\Delta(C)$ and satisfies $K \cap (E \cap \Delta(I)) = \varnothing$. Then $\Delta(C) \setminus K$ is open in $\Delta(C)$ and contains $E \cup \{\omega\}$, and $\widehat{x}$ vanishes on $\Delta(A) \setminus K$. Consequently, $x \in j(E \cup \{\omega\})$.

Now (5.1) implies that $E \cap \Delta(I)$ is a weak spectral set for I if and only if $E \cup \{\omega\}$ is a weak spectral set for C, and in this case $\xi(E \cup \{\omega\}) = \xi(E \cap \Delta(I))$.

Turning to the proof of (i), suppose that E is a weak spectral set for C. Then, since $\{\omega\}$ is a weak spectral set, so is the union $E \cup \{\omega\}$ and $\xi(E \cup \{\omega\}) \leq \xi(E) + \xi(\{\omega\})$. The first part of the proof implies that $E \cap \Delta(I)$ is a weak spectral set for I and

$$\xi(E \cap \Delta(I)) = \xi(E \cup \{\omega\}) \leq \xi(E) + \xi(\{\omega\}).$$

Moreover, if in addition $\varnothing$ is a Ditkin set for C, then $\xi(E) \leq \xi(E \cup \{\omega\})$ by [11, Corollary 2.3], since E is open and closed in $E \cup \{\omega\}$. This completes the proof of (i).

(ii) If $E \cap \Delta(I)$ is a weak spectral set for I, then $E \cup \{\omega\}$ is one for C, and $\xi(E \cup \{\omega\}) = \xi(E \cap \Delta(I))$ by the first part of the proof. Finally, if $\omega \notin E$ then, since $\varnothing$ is a Ditkin set, by [13, Theorem 2.4] E is of weak synthesis and $\xi(E) \leq \xi(E \cup \{\omega\})$. ■

Combining (i) and (ii) of the preceding lemma, we obtain the following.

Corollary 5.3 *Let C be a regular and semisimple commutative Banach algebra and let $\omega \in \Delta(C)$ and $I = k(\omega)$. Suppose that $\{\omega\}$ is a set of synthesis and that $\varnothing$ is a Ditkin set. Then, for any closed subset E of $\Delta(C)$, E is a weak spectral set for C if and only if $E \cap \Delta(I)$ is a weak spectral set for I, and in this case*

$$\xi(E) = \xi(E \cap \Delta(I)) \leq \xi(E) + 1.$$

The direct product $A_1 \times A_2$ of two Banach algebras, A_1 and A_2 is defined to be the Cartesian product of A_1 and A_2 with componentwise operations and the norm $\|(a_1, a_2)\| = \|a_1\| + \|a_2\|$. Then

$$\Delta(A_1 \times A_2) = (\Delta(A_1) \times \{0\}) \cup (\{0\} \times \Delta(A_2)),$$

which is homeomorphic to the topological sum of $\Delta(A_1)$ and $\Delta(A_2)$.

Lemma 5.4 *Let A_1 and A_2 be regular and semisimple commutative Banach algebras, and for $i = 1, 2$, let E_i be a closed subset of $\Delta(A_i)$. Suppose that $A_1 \times A_2$ satisfies Ditkin's condition at infinity. Then $E_1 \cup E_2$ is a weak spectral set for $A_1 \times A_2$ if and only if E_i is a weak spectral set for A_i, $i = 1, 2$, and then $\xi(E_1 \cup E_2) = \max\{\xi(E_1), \xi(E_2)\}$.*

https://doi.org/10.4153/CJM-2014-028-4 Published online by Cambridge University Press

Proof Since $\Delta(A_1 \times A_2)$ is the disjoint union of the two open subsets $\Delta(A_1)$ and $\Delta(A_2)$ and $\varnothing$ is a Ditkin set for $A_1 \times A_2$, it follows from [13, Theorem 2.4] and [13, Theorem 2.8] that

$$\xi(E_1), \xi(E_2) \leq \xi(E_1 \cup E_2) \leq \max\{\xi(E_1), \xi(E_2)\},$$

as required. ■

Lemmas 5.2 and 5.4 and Corollary 5.3 now quickly lead to our first main result on weak spectral sets for $A \times_\theta B$.

Theorem 5.5 *Let A and B be regular and semisimple commutative Banach algebras and $\theta \in \Delta(B)$. Suppose that A satisfies Ditkin's condition at infinity and that $\{\theta\}$ is a Ditkin set for B. Let E be a closed subset of $\Delta(A \times_\theta B)$ and set, as in Corollary 4.4,*

$$F = \{\phi \in \Delta(A) : (\phi, \theta) \in E\} \quad \textit{and} \quad G = \{\psi \in \Delta(B) : (0, \psi) \in E\}.$$

If E is a weak spectral set for $A \times_\theta B$, then

(i) *F is of weak synthesis for A and $\xi(F) \leq \xi(E) + 1$;*
(ii) *G is a weak spectral set for B and $\xi(G) \leq \xi(E) + 1$.*

Proof We apply Lemma 5.2(i) to the algebra $C = A \times_\theta B$ and its ideal $I = A \times k(\theta) = k((0, \theta))$. Since, by Lemma 4.7, $\{(0, \theta)\}$ is a Ditkin set for $A \times_\theta B$, it follows from Corollary 5.3 that $E \cap \Delta(I)$ is a weak spectral set for I and $\xi(E \cap \Delta(I)) \leq \xi(E) + 1$.

Note next that I satisfies Ditkin's condition at infinity, since both A and $k(\theta)$ have this property, and that

$$E \cap \Delta(I) = [F \times \{\theta\}] \cup [\{0\} \times (G \setminus \{\theta\})].$$

Then Lemma 5.4 yields that F and $G \setminus \{\theta\}$ are weak spectral sets for A and $k(\theta)$, respectively, and

$$\max\big\{\, \xi(F), \xi(G \setminus \{\theta\})\big\} \leq \xi(E \cap \Delta(I)) \leq \xi(E) + 1.$$

This in particular proves (i), and applying Lemma 5.2(ii) to $C = B$ and the ideal $k(\theta)$, it follows that G is a weak spectral set and $\xi(G) \leq \xi(G \setminus \{\theta\})$ provided that B satisfies Ditkin's condition at infinity. However, in the present situation, this is true by Lemma 4.8 since $\{\theta\}$ is a Ditkin set. ■

For any commutative Banach algebra C, let $\xi_C \in \mathbb{N} \cup \{\infty\}$ be defined by

$$\xi_C = \sup\{\xi(E) : E \subseteq \Delta(C) \text{ closed}\}.$$

With this notation, we conclude the following from Theorem 5.5.

Corollary 5.6 *Assume A satisfies Ditkin's condition at infinity and $\{\theta\}$ is a Ditkin set for B. Then, if weak spectral synthesis holds for $A \times_\theta B$, it also holds for A and B, and $\max\{\xi_A, \xi_B\} \leq 1 + \xi_{A \times_\theta B}$.*

We continue this section with converses to Theorem 5.5 and Corollary 5.6.

Theorem 5.7 *Let E be a closed subset of $\Delta(A \times_\theta B)$, and let $F \subseteq \Delta(A)$ and $G \subseteq \Delta(B)$ be defined as in Theorem 5.5. Suppose that F and G are weak spectral sets for A and B, respectively.*

https://doi.org/10.4153/CJM-2014-028-4 Published online by Cambridge University Press

(i) *If* $(0,\theta) \in E$, *then the set* E *is of weak synthesis for* $A \times_\theta B$ *and*

$$\xi(E) = \max\{\xi(F), \xi(G)\}.$$

(ii) *Suppose that* $(0,\theta) \notin E$ *and that* $\{\theta\}$ *is a weak spectral set for* B *and that* $A \times_\theta B$ *satisfies Ditkin's condition at infinity. Then* E *is of weak synthesis for* $A \times_\theta B$ *and*

$$\xi(E) \le \max\big\{\, \xi(F), \xi(G), \xi(\{\theta\})\big\}.$$

Proof (i) Note first that for any $(a,b) \in A \times_\theta B$ and $n \in \mathbb{N}$, we have $(a,b)^n = \big(a^n + \theta(b)p(a), b^n\big)$, where $p(a) = 0$ if $n = 1$ and $p(a)$ is a polynomial in a for any $n \ge 1$. Now let $n = \max\{\xi(F), \xi(G)\}$ and assume that $(a,b) \in k(E)$. Then $\theta(b) = \widehat{(a,b)}(0,\theta) = 0$ as $(0,\theta) \in E$, and therefore

$$(a,b)^n = (a^n, b^n) \in \overline{j(F)} \times \overline{j(G)}.$$

Thus, given $\epsilon > 0$, there exist $x \in j(F)$ and $y \in j(G)$ such that $\|a^n - x\| \le \epsilon$ and $\|b^n - y\| \le \epsilon$, and hence $\|(a,b)^n - (x,y)\| \le 2\epsilon$.

It remains to show that $(x,y) \in j(E)$. To that end, let U be an open subset of $\Delta(A)$ such that $F \subseteq U$, $\Delta(A) \setminus U$ is compact and $\widehat{x}$ vanishes on U. Similarly, let V be an open subset of $\Delta(B)$ such that $G \subseteq V$, $\Delta(B) \setminus V$ is compact and $\widehat{y}$ vanishes on V. Set

$$W = [U \times \{\theta\}] \cup [\{0\} \times V] \subseteq \Delta(A \times_\theta B).$$

Then, using that $\theta \in V$, it is easily verified that $\widehat{(x,y)}$ vanishes on W. From the description of the topology on $\Delta(A \times_\theta B)$ (Proposition 1.2, in particular (iv)) it follows that W is open in $\Delta(A \times_\theta B)$. Finally, $\Delta(A \times_\theta B) \setminus W$ is compact since both $\Delta(A) \setminus U$ and $\Delta(B) \setminus V$ are compact. Since $\epsilon > 0$ was arbitrary, it follows that $(a,b)^n \in \overline{j(E)}$ for all $(a,b) \in k(E)$. On the other hand, n is minimal with this property, since $k(F) \times \{0\} \subseteq k(E)$ and $\{0\} \times k(G) \subseteq k(E)$. This completes the proof of (i).

(ii) Consider $\widetilde{E} = E \cup \{(0,\theta)\}$ and $\widetilde{G} = G \cup \{\theta\}$. Then $\xi(\widetilde{E}) = \max\{\xi(F), \xi(\widetilde{G})\}$ by (i), and since $\varnothing$ is a Ditkin set for B, $\xi(\widetilde{G}) \le \max\big\{\xi(G), \xi(\{\theta\})\big\}$ [13, Theorem 2.8]. On the other hand, since $A \times_\theta B$ satisfies Ditkin's condition at infinity, $\xi(E) \le \xi(\widetilde{E})$ [13, Corollary 2.5]. It follows that

$$\xi(E) \le \max\{\xi(F), \xi(\widetilde{G}\} \le \max\big\{\, \xi(F), \xi(G), \xi(\{\theta\})\big\},$$

as claimed. ∎

Corollary 5.8 *Suppose that* $A \times_\theta B$ *satisfies Ditkin's condition at infinity. Then weak spectral synthesis holds for* $A \times_\theta B$ *if weak spectral synthesis holds for both* A *and* B. *In this case,*

$$\xi_{A \times_\theta B} \le \max\{\xi_A, \xi_B\}.$$

Proof The statements are immediate consequences of parts (i) and (ii) of Theorem 5.7. ∎

Acknowledgement The author is very much obliged to the referee for detecting a serious error in Example 3.6 of the previous version of the paper.

References

[1] W. Bade and P. C. Curtis Jr., *The Wedderburn decomposition of commutative Banach algebras.* Amer. J. Math. **82**(1960), 851–866. http://dx.doi.org/10.2307/2372944

[2] L. Baggett and K. Taylor, *A sufficient condition for the complete reducibility of the regular representation.* J. Funct. Anal. **34**(1979), 250–265. http://dx.doi.org/10.1016/0022-1236(79)90033-8

[3] S. Bochner, *A theorem on Fourier–Stieltjes integrals.* Bull. Amer. Math. Soc. **40**(1934), 271–276. http://dx.doi.org/10.1090/S0002-9904-1934-05843-9

[4] M. Cowling, *The Fourier–Stieltjes algebra of a semisimple group.* Colloq. Math. **41**(1979), 89–94.

[5] W. F. Eberlein, *Characterizations of Fourier–Stieltjes transforms.* Duke Math. J. **22**(1955), 465–468. http://dx.doi.org/10.1215/S0012-7094-55-02251-1

[6] P. Eymard, *L'algèbre de Fourier d'une groupe localement compact.* Bull. Soc. Math. France **92**(1964), 181–236.

[7] A. Figà-Talamanca, *Positive definite functions which vanish at infinity.* Pacific J. Math. **69**(1977), 355–363. http://dx.doi.org/10.2140/pjm.1977.69.355

[8] C. A. Jones and C. D. Lahr, *Weak and norm approximate identities are different.* Pacific J. Math. **72**(1977), 99–104. http://dx.doi.org/10.2140/pjm.1977.72.99

[9] J. Inoue and S.-E. Takahasi, *On characterizations of the image of the Gelfand transform of commutative Banach algebras.* Math. Nachr. **280**(2007), 105–126. http://dx.doi.org/10.1002/mana.200410468

[10] Z. Kamali and M. Lashkarizadeh Bami, *The multiplier algebra and* BSE *property of the direct sum of Banach algebras.* Bull. Austral. Math. Soc. **88**(2013), 250–258. http://dx.doi.org/10.1017/S0004972712001001

[11] E. Kaniuth, *Weak spectral synthesis in commutative Banach algebras.* J. Funct. Anal. **254**(2008), 987–1002. http://dx.doi.org/10.1016/j.jfa.2007.10.002

[12] ———, *A course in commutative Banach algebras.* Graduate Texts in Math. **246**, Springer, New York, 2009.

[13] ———, *Weak spectral synthesis in commutative Banach algebras. II.* J. Funct. Anal. **259**(2010), 524–544. http://dx.doi.org/10.1016/j.jfa.2010.04.011

[14] E. Kaniuth and A. Ülger, *The Bochner–Schoenberg–Eberlein property for commutative Banach algebras, especially Fourier and Fourier–Stieltjes algebras.* Trans. Amer. Math. Soc. **362**(2010), 4331–4356. http://dx.doi.org/10.1090/S0002-9947-10-05060-9

[15] E. Kaniuth, A. T. Lau and A. Ülger, *Homomorphisms of commutative Banach algebras and extensions to multiplier algebras with applications to Fourier algebras*. Studia Math. **183**(2007), 35–62. http://dx.doi.org/10.4064/sm183-1-3

[16] E. Kaniuth, A. T. Lau and A. Ülger, *The Rajchman algebra $B_0(G)$ of a locally compact group*. Submitted.

[17] R. Larsen, *An introduction to the theory of multipliers*. Springer, New York, 1971.

[18] A. T. Lau, *Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups.* Fund. Math. **118**(1983), 161–175.

[19] M. S. Monfared, *On certain products of Banach algebras with applications to harmonic analysis.* Studia Math. **178**(2007), 277–294. http://dx.doi.org/10.4064/sm178-3-4

[20] M. S. Monfared, *Character amenability of Banach algebras.* Math. Proc. Camb. Phil. Soc. **144**(2008), 697–706. http://dx.doi.org/10.1017/S0305004108001126

[21] M. Muraleedharan and K. Parthasarathy, *Difference spectrum and spectral synthesis.* Tohoku Math. J. **51**(1999), 65–73. http://dx.doi.org/10.2748/tmj/1178224853

[22] J.-P. Pier, *Amenable locally compact groups.* Wiley Interscience, New York, 1984.

[23] W. Rudin, *Fourier analysis on groups.* Interscience, New York, 1962.

[24] I. J. Schoenberg, *A remark on the preceding note by Bochner.* Bull. Amer. Math. Soc. **40**(1934), 277–278. http://dx.doi.org/10.1090/S0002-9904-1934-05845-2

[25] S.-E. Takahasi and O. Hatori, *Commutative Banach algebras which satisfy a Bochner–Schoenberg–Eberlein-type theorem.* Proc. Amer. Math. Soc. **110**(1990), 149–158.

[26] ———, *Commutative Banach algebras and* BSE*-inequalities.* Math. Japonica **37**(1992), 47–52.

[27] S.-E. Takahasi, Y. Takahashi, O. Hatori and K. Tanahashi, *Commutative Banach algebras and* BSE*-norm.* Math. Japonica **46**(1997), 273–277.

[28] A. Ülger, *Multipliers with closed range on commutative Banach algebras.* Studia Math. **153**(2002), 59–80. http://dx.doi.org/10.4064/sm153-1-5

[29] N. Th. Varopoulos, *Spectral synthesis on spheres.* Math. Proc. Camb. Phil. Soc. **62**(1966), 379–387. http://dx.doi.org/10.1017/S0305004100039967

[30] C. R. Warner, *Weak spectral synthesis.* Proc. Amer. Math. Soc. **99**(1987), 244–248. http://dx.doi.org/10.1090/S0002-9939-1987-0870779-7

Institut für Mathematik, Universität Paderborn, D-33095 Paderborn, Germany
e-mail: kaniuth@math.uni-paderborn.de