ABELIAN *P*-SUBGROUPS OF THE GENERAL LINEAR GROUP

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A. J. Weir [1] has found the maximal normal abelian subgroups of the Sylow p-subgroups of the general linear group over a finite field of characteristic p, and a theorem of J. L. Alperin [2] shows that the Sylow p-subgroups of the general linear group over finite fields of characteristic different from p have a unique largest normal abelian subgroup and that no other abelian subgroup has order as great. We shall prove the following:

THEOREM. *Let F be a finite field of order $q = p^k$ (p an odd prime). The maximal order of an abelian p-subgroup of $GL(n, F)$ is q^m where $m = [\frac{1}{4}n^2]$ = the integer part of $\frac{1}{4}n^2$, and this maximum is attained.*

This result shows that in the case where the characteristic is p, there are abelian p-subgroups which are normal in the Sylow p-groups and have order greater than any other abelian p-subgroups; however our proof shows that (in the case where n is odd) there may be several such subgroups.

PROOF. We firstly observe that a matrix in $GL(n, F)$ is a p-element if and only if it has n eigenvalues equal to 1, since 1 is the only p^{th} power root of unity in F.

So, let A be an abelian subgroup of $GL(n, F)$; we say A is of type (n, r) if the common (right) eigenspace of the matrices in A (for the unique eigenvalue 1) is of dimension $n-r$. Since there is at least one non-zero vector which is fixed by all the matrices in A, $n-r > 0$. This implies that A is conjugate in $GL(n, F)$ to a group A_0 so that the matrices in A_0 have the form

$$(1) \qquad x = \left\{\begin{array}{c|c} 1 & a \\ \hline 0 & y \end{array}\right\} \begin{array}{l} n-r \\ r \end{array} \qquad 0 \leqq r < n$$

Let the maximal order of an abelian p-subgroup of type (n, r) be $q^{0(n,r)}$.

Let A_1 be the group of $r \times r$ matrices $\{y^T : y \text{ occurs in (1) for some element of } A_0\}$ where y^T is the transpose of y. A is an abelian p-subgroup of $GL(n, F)$. Suppose A_1 is of type (r, s).

The subgroup N of A_0 consisting of all elements with matrices of the form

$$u = \left\{\begin{array}{c|c} 1 & b \\ \hline 0 & 1 \end{array}\right\} \tag{2}$$

is normal in A. Then since, in the notation of (1), $x \to y$ is a homomorphism of A_0, with kernel N, we have that

$$|A| = |A_0| = |N|\,|A_1|.$$

Since A_0 is abelian, $xu = ux$ ($u \in N$, $x \in A_0$) and we conclude that $y = b$. Thus $y^T b^T = b^T$ and each row of b is a right eigenvector of each y^T. Since A_1 is of type (r, s) the space spanned by the i^{th} rows of the b's in (2) is of dimension at most $r-s$. Since the b's have $n-r$ rows there are at most $(r-s)(n-r)$ linearly independent b's. Since N is isomorphic to the additive group $\{b : b$ occurs in (2) for some element of $N\}$

$$|N| \leqq q^{(r-s)(n-r)}.$$

Hence

$$|A| \leqq q^{(n-r)(r-s)} \cdot q^{0(r,s)}.$$

Thus

$$0(n, r) \leqq \max_{0 \leqq s < r} \{0(r, s) + (n-r)(r-s)\}.$$

We now prove by induction that

$$\begin{aligned} 0(n, r) &\leqq r(n-r) && \text{for } r \leqq \tfrac{1}{2}n \\ &\leqq \tfrac{1}{4}n^2 && \text{for } r > \tfrac{1}{2}n. \end{aligned}$$

It is necessary to consider four cases, depending on where the maximum of $\{0(r, s) + (r-s)(n-r)\}$ occurs.

Case (i): $s > \frac{1}{2}r$, $r > \frac{1}{2}n$:

$$\begin{aligned} 0(n, r) &\leqq \max_{0 \leqq s < r} \{0(r, s) + (n-r)(r-s)\} \\ &\leqq \tfrac{1}{4}r^2 + (n-r)(\tfrac{1}{2}r) \\ &< \tfrac{1}{4}n^2 \end{aligned}$$

Case (ii): $s \leqq \frac{1}{2}r$, $r \leqq \frac{1}{2}n$:

$$\begin{aligned} 0(n, r) &\leqq \max_{0 \leqq s < r} \{0(r, s) + (n-r)(r-s)\} \\ &\leqq \max_{s} \{s(r-s) + (n-r)(r-s)\} \\ &\leqq 0 + r(n-r) \quad \text{since } s \geqq 0 \\ &\leqq r(n-r) \end{aligned}$$

Case (iii): $s \leqq \frac{1}{2}r,\ r \geqq \frac{1}{2}n$:

$$\begin{aligned} 0(n, r) &\leqq \max_{0 \leqq s < r} \{0(r, s)+(n-r)(r-s)\} \\ &\leqq \max_{s} \{s(r-s)+(n-r)(r-s)\} \\ &\leqq \tfrac{1}{4}(2r-n)^2+r(n-r) \\ &\leqq \tfrac{1}{4}n^2 \end{aligned}$$

Case (iv): $s > \frac{1}{2}r,\ r \leqq \frac{1}{2}n$.

$$\begin{aligned} 0(n, r) &\leqq \max_{0 \leqq s < r} \{0(r, s)+(n-r)(r-s)\} \\ &\leqq \max_{s} \{\tfrac{1}{4}r^2+(n-r)(\tfrac{1}{2}r)\} \\ &\leqq r(n-r) \end{aligned}$$

Finally, the group of all matrices in $GL(n, F)$ of the form

$$\left\{\begin{array}{c|c} 1 & a \\ \hline 0 & 1 \end{array}\right\} \begin{array}{l} r \\ n-r \end{array} \quad \text{with } r = \left[\frac{n}{2}\right] \tag{3}$$

is a normal elementary abelian p-group of order $q^{[n^2/4]}$ and thus the maximum is attained. This completed the proof of the theorem.

On examining the case of equality, $0(n, r) = [\frac{1}{4}n^2]$ in (iii) we see that the group of matrices of the form (3) above is the only abelian p-group which attains this maximum.

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References

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