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ON VMOA FOR RIEMANN SURFACES

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1. Introduction. Let $\Delta = \{z| \ |z| < 1\}$ be the unit disk and f an analytic function in Δ. The Dirichlet integral $D_\Delta(f)$ of f on Δ is defined by

$$D_\Delta(f) = \frac{1}{\pi} \int\int_\Delta |f'(z)|^2 dxdy,$$

and we denote by $AD(\Delta)$ the space of all functions f analytic on Δ for which $D_\Delta(f) < \infty$. We denote by $BMOA(\Delta)$ the space of analytic functions f in Δ for which

$$\sup_{a \in \Delta} \frac{2}{\pi} \int\int_\Delta |f'(z)|^2 \log\left|\frac{1 - \bar{a}z}{z - a}\right| dxdy < \infty$$

and by $VMOA(\Delta)$ the space of those analytic functions f in $BMOA(\Delta)$ satisfying the condition

$$\lim_{|a| \to 1} \int\int_\Delta |f'(z)|^2 \log\left|\frac{1 - \bar{a}z}{z - a}\right| dxdy = 0.$$

Other equivalent ways to define these spaces can be found in ([**2**], [**4**], [**12**] e.g.). The following inclusion chain

(1) $\quad AD(\Delta) \subset VMOA(\Delta) \subset \mathscr{B}_0$

is well-known where

$$\mathscr{B}_0 = \left\{f| \ f \text{ analytic in } \Delta \text{ and } \lim_{|z| \to 1} (1 - |z|^2)|f'(z)| = 0\right\}.$$

The first inclusion is proved by Yamashita in [**14**, Remark, p. 366] and the second one (e.g. [**11**, p. 200]).

Let R be an open Riemann surface which possesses a Green's function, i.e., $R \notin O_G$ and let F be an analytic function defined on R. The space $AD(R)$ is defined as above,

$$AD(R) = \left\{F| \ F \text{ analytic on } R \text{ and } D_R(F) = \frac{1}{\pi} \int\int_R |F'(z)|^2 dxdy < \infty\right\}.$$

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Following Metzger [**9**] we define BMOA for Riemann surfaces in the following way: We denote by BMOA(R) the space of functions F analytic on R for which

$$\mathrm{B}_R(F) = \sup_{a \in R} \frac{2}{\pi} \iint_R |F'(z)|^2 g(z, a) dxdy < \infty$$

where $g(z, a)$ denotes a Green's function of R with logarithmic singularity at a. Further, for analytic functions F on R, we define VMOA(R) as follows: Let ∂R be an ideal boundary of R and F an analytic function on R. Then $F \in$ VMOA(R) if and only if

$$\lim_{a \to \partial R} \iint_R |F'(z)|^2 g(z, a) dxdy = 0.$$

In Section 3 we will state the definition of VMOA(R) as "pulled back" to the universal covering surface Δ and point out the connection between VMOA(R) and BMOA(Δ) $\cap$ {automorphic functions}. In [**9**] Metzger proved the inclusion relation AD(R) $\subset$ BMOA(R) using the theory of covering surfaces. Kobayashi [**7**] showed that

$$\mathrm{B}_R(F) \leqq \mathrm{D}_R(F),$$

which implies Metzger's result, using calculation technique on Riemann surfaces.

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2. The spaces AD(R) **and** VMOA(R). In the BMO-seminar in Joensuu (1987) Metzger asked if it is true

(2) AD(R) $\subset$ VMOA(R)

(cf. also [**15**, (VII) p. 481]). In this paper we will show that the inclusion relation (2) does not necessarily hold. However, in some special cases it will be valid. In proving we will exploit the technique of Kobayashi. However, before proving our main results we will discuss some preliminary proposition and lemmas. Since we have defined the space VMOA(R) in the above-mentioned way, we must prove the relation between VMOA(R) and BMOA(R). This result we will need in proving Theorem 2.

PROPOSITION. *For any open Riemann surface R which possesses a Green's function*

VMOA(R) $\subset$ BMOA(R).

Proof. Let $F \in \mathrm{VMOA}(R)$ and $u_a(z) = |F(z) - F(a)|^2$. We denote by $h_a(z)$ the least harmonic majorant of $u_a(z)$ on R where, for convention, we set $h_a(z) = \infty$ if u_a admits no harmonic majorants. By the assumption,

$$\lim_{a \to \partial R} \int\int_R |F'(z)|^2 g(z, a)dxdy = 0.$$

Using Kobayashi's lemma [7, Lemma 1]

$$(3) \quad h_a(a) = \frac{2}{\pi} \int\int_R |F'(z)|^2 g(z, a)dxdy, \quad a \in R,$$

we find a compact set $R_0 \subset R$ such that

$$(4) \quad h_a(a) < 1$$

for all $a \in R \setminus R_0$.

Suppose, on the contrary, that $F \notin \mathrm{BMOA}(R)$. Then, by (3) and (4),

$$(5) \quad \sup_{a \in R_0} h_a(a) = \infty.$$

Now we have the following alternatives:

1) At least for one $a_0 \in R_0$ the condition

$$h_{a_0}(a_0) = \infty$$

holds. We choose a point $a_1 \in R \setminus R_0$. Let

$$L = \{\psi(t) \mid \ 0 \leqq t \leqq 1, \psi(0) = a_0, \psi(1) = a_1\}$$

be a path connecting the points a_0 and a_1. Let $\psi(t_0) \in L$, $0 \leqq t_0 < 1$, correspond to the largest value t_0 of t for which

$$h_{\psi(t_0)}(\psi(t_0)) = \infty.$$

We take a parametric disk $U_{\psi(t_0)} \ni \psi(t_0)$ such that

$$u_{\psi(t_0)}(z) = |F(z) - F(\psi(t_0))|^2 < 1 \quad \text{for all } z \in U_{\psi(t_0)}.$$

Further, we choose t_1, $0 \leqq t_0 < t_1 \leqq 1$, $\psi(t_1) \in U_{\psi(t_0)}$ for which the least harmonic majorant $h_{\psi(t_1)}$ of the subharmonic function

$$u_{\psi(t_1)}(z) = |F(z) - F(\psi(t_1))|^2$$

satisfies

$$h_{\psi(t_1)}(\psi(t_1)) < \infty.$$

Thus $h_{\psi(t_1)}(z) < \infty$ on R. Now

$$\begin{aligned} u_{\psi(t_0)}(z) = |F(z) - F(\psi(t_0))|^2 &\leqq 2^2(|F(z) - F(\psi(t_1))|^2 \\ &\quad + |F(\psi(t_1)) - F(\psi(t_0))|^2) \leqq 4(u_{\psi(t_1)}(z) + 1) \\ &\leqq 4(h_{\psi(t_1)}(z) + 1) < \infty \end{aligned}$$

on R. Hence $4(h_{\psi(t_1)}(z) + 1)$ is a harmonic majorant of the subharmonic function $u_{\psi(t_0)}(z)$. Therefore there exists the finite least harmonic majorant $h_{\psi(t_0)}(z)$ and it satisfies

$$h_{\psi(t_0)}(z) \leqq 4(h_{\psi(t_1)}(z) + 1) < \infty$$

on R. This implies that

$$h_{\psi(t_0)}(\psi(t_0)) < \infty$$

which is a contradiction.

2) We suppose that, for all $a \in R_0$,

$$h_a(a) < \infty.$$

By (5) there exists a sequence of points $(a_n) \subset R_0$, $a_n \rightarrow a_0 \in R_0$ (since R_0 is compact) such that

$$h_{a_n}(a_n) \rightarrow \infty$$

as $n \rightarrow \infty$. By the assumption,

$$h_{a_0}(a_0) < \infty.$$

Hence $h_{a_0}(z) < \infty$ on R. We take a parametric disk $U_{a_0} \ni a_0$ such that

$$|F(z) - F(z')|^2 < 1 \quad \text{for all } z, z' \in U_{a_0}.$$

Further, we may suppose that $(a_n) \subset U_{a_0}$. Then

$$u_{a_n}(z) = |F(z) - F(a_n)|^2 \leqq 2^2(|F(z) - F(a_0)|^2 + |F(a_0) - F(a_n)|^2) \leqq 4(u_{a_0}(z) + 1) \leqq 4(h_{a_0}(z) + 1).$$

Hence the least harmonic majorant satisfies

$$h_{a_n}(z) \leqq 4(h_{a_0}(z) + 1).$$

Thus

$$(6) \quad h_{a_n}(a_n) \leqq 4(h_{a_0}(a_n) + 1)$$

for each n. Since we may choose U_{a_0} to be a compact set,

$$h_{a_0}(z) \leqq K < \infty$$

for all $z \in U_{a_0}$. Therefore, by (6),

$$h_{a_n}(a_n) \leqq 4(K + 1) < \infty$$

which is a contradiction. Thus the antithesis is incorrect and the proposition is proved.

For our first theorem we need the following lemmas:

LEMMA 1. [7, Lemma 2]. *For any Riemann surface R,*

(7) $$\frac{1}{2}h_a(a) \leqq \frac{1}{\pi} \iint_R |F'(z)|^2(1 - e^{-2g(z,a)})dxdy,$$

where h_a *is the least harmonic majorant of the subharmonic function* $|F(z) - F(a)|^2$.

Remark. Kobayashi's lemma has been applied by the value $k = 1$.

LEMMA 2. *If*

$$\lim_{a \to \partial R} g(z, a) = 0 \quad \textit{for all } z \in R,$$

then for given compact set $R_0 \subset R$ *and any* $\epsilon > 0$ *there exists a compact set* $S_0 \subset R$ *such that* $a \in R \setminus S_0$ *implies*

$$g(z, a) < \epsilon \quad \textit{for all } z \in R_0.$$

Lemma 2 is well-known and its proof may be left for the reader.

The Riemann surface R is called *regular* if

$$\lim_{a \to \partial R} g(z, a) = 0 \quad \text{for each } z \in R.$$

The set of constant functions on R will be denoted by **C**.

THEOREM 1. *Let R be any open Riemann surface possessing a Green's function. Then we have the following possibilities*:

(a) *If R is regular, then*

$$\mathrm{AD}(R) \subset \mathrm{VMOA}(R).$$

(b) *If R is not regular, then*

$$\mathrm{VMOA}(R) = \mathbf{C}.$$

Proof. (a) Let $F \in \mathrm{AD}(R)$ and let $\epsilon > 0$. We may choose a compact set $R_\epsilon \subset R$ such that

(8) $$\iint_{R \setminus R_\epsilon} |F'(z)|^2 dxdy < \frac{\epsilon\pi}{2}.$$

By Lemma 2 there exists a compact set $S_0 \subset R$ such that $a \in R \setminus S_0$ implies

(9) $$1 - e^{-2g(z,a)} < \frac{\epsilon}{2\mathrm{D}_R(F)}$$

for all $z \in R_\epsilon$. Let $a \in R \setminus S_0$. Then, by (7), (8) and (9),

(10) $$\frac{1}{2}h_a(a) \leqq \frac{1}{\pi} \iint_{R_\epsilon} |F'(z)|^2(1 - e^{-2g(z,a)})dxdy$$

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$$+ \frac{1}{\pi} \iint_{R \setminus R_\epsilon} |F'(z)|^2 (1 - e^{-2g(z,a)}) dxdy$$

$$< \frac{\epsilon}{2D_R(F)} \frac{1}{\pi} \iint_{R_\epsilon} |F'(z)|^2 dxdy + \frac{1}{\pi} \iint_{R \setminus R_\epsilon} |F'(z)|^2 dxdy$$

$$< \frac{\epsilon}{2D_R(F)} D_R(F) + \frac{\epsilon}{2} = \epsilon.$$

Hence, by [7, Lemma 1] and (10), we have

$$\lim_{a \to \partial R} h_a(a) = \frac{2}{\pi} \lim_{a \to \partial R} \iint_R |F'(z)|^2 g(z, a) dxdy = 0.$$

Thus $F \in \mathrm{VMOA}(R)$ and (a) is proved.

(b) Let F be a non-constant analytic function on R. Since R is not regular, there exist a sequence of points $(a_n) \subset R$ converging to ∂R and a point $z_0 \in R$ such that

$$\lim_{n \to \infty} g(z_0, a_n) = c > 0.$$

We will consider the integral

$$\iint_R |F'(z)|^2 g(z, a_n) dxdy.$$

Let us take a parametric disk

$$U_r = \{z | \ |z - z_0| < r\}$$

on R and an annulus

$$V_{r_1 r_2} = \{z | \ r_1 < |z - z_0| < r_2\}$$

where $0 \leqq r_1 < r_2 \leqq r$. Since $g(z, a_n)$ is a harmonic function in U_r (for sufficiently large n), we have

$$g(z_0, a_n) = \frac{1}{2\pi} \int_0^{2\pi} g(z_0 + te^{i\varphi}, a_n) d\varphi, \quad 0 < t < r.$$

Thus, by integrating

$$\frac{1}{2} g(z_0, a_n)(r_2^2 - r_1^2) = \int_{r_1}^{r_2} g(z_0, a_n) t dt$$

$$= \frac{1}{2\pi} \int_{t=r_1}^{r_2} \left(\int_{\varphi=0}^{2\pi} g(z_0 + te^{i\varphi}, a_n) d\varphi \right) t dt$$

and

$$(11) \quad \iint_{V_{r_1 r_2}} g(z, a_n) dxdy = \pi g(z_0, a_n)(r_2^2 - r_1^2) \to \pi c(r_2^2 - r_1^2)$$

as $n \to \infty$. Now

$$(12) \quad \liminf_{n\to\infty} \iint_R |F'(z)|^2 g(z, a_n)dxdy \geqq \liminf_{n\to\infty} \iint_{U_r} |F'(z)|^2 g(z, a_n)dxdy.$$

Since F is a non-constant analytic function, we find r_1, r_2 such that $0 \leqq r_1 < r_2 \leqq r$ and $|F'(z)| \geqq \alpha > 0$ for all $z \in V_{r_1 r_2}$. Hence, by (11), (12) and the inclusion $V_{r_1 r_2} \subset U_r$

$$\begin{aligned} \liminf_{n\to\infty} \iint_R |F'(z)|^2 g(z, a_n)dxdy &\geqq \liminf_{n\to\infty} \iint_{V_{r_1 r_2}} |F'(z)|^2 g(z, a_n)dxdy \\ &\geqq \alpha^2 \lim_{n\to\infty} \iint_{V_{r_1 r_2}} g(z, a_n)dxdy \\ &= \pi\alpha^2 c(r_2^2 - r_1^2) > 0. \end{aligned}$$

By definition $F \notin \text{VMOA}(R)$ and thus $\text{VMOA}(R) = \mathbf{C}$.

The existence of (a). The simplest regular Riemann surface is the unit disk. For other Riemann surfaces satisfying the assumption of (a) (cf. [**3**, Folgesatz 13.3, p. 141]).

The existence of (b). Let R be a bounded, connected, open set in the complex plane with a Green's function. If R has irregular boundary points, then the assumption of (b) is satisfied (cf. [**6**, Satz 8.34, p. 200]). Further, on these Riemann surfaces R there exist non-constant functions $F \in \text{AD}(R)$ but as we showed the space $\text{VMOA}(R)$ consists only of constant functions. Thus $\text{AD}(R) \not\subset \text{VMOA}(R)$.

3. The spaces $\text{VMOA}(R)$ **and** $\mathscr{B}_0(R)$. In this section we want to consider if in the formula (1) the latter corresponding inclusion is valid on Riemann surfaces. Since there is a Green's function on R, the universal covering surface of R is Δ and if $\Pi:\Delta \to R$ denotes the universal covering map then the group of deck transformations is a Fuchsian group Γ. Let Ω be the Ford fundamental polygon for Γ so that $\Pi:\Omega \to R$ is a $(1 - 1)$ into map and $\Pi:\bar{\Omega} \to R$ is onto and the area measure of $\partial\Omega$ is zero.

The space

$$\mathscr{B}_0(R) = \left\{F| \quad F \text{ analytic on } R \text{ and } \lim_{z\to\partial R} \frac{|F'(z)|\,|dz|}{d\sigma(z)} = 0\right\},$$

where $d\sigma(z)$ is the Poincaré metric of R, corresponds to $\mathscr{B}_0$ on R. We want to prove the inclusion

$$\mathrm{VMOA}(R) \subset \mathscr{B}_0(R).$$

The proof of the inclusion is based on the fact that F can be "pulled back" to a function f acting on Δ by the following method: if F is analytic on R, define $f(z)$ by $f(z) = F(\Pi(z))$ for all z in Δ. It follows immediately that f is an analytic function on Δ which satisfies

(13) $f(\gamma(z)) = f(z)$ for all γ in Γ and z in Δ.

Functions satisfying (13) are called *automorphic functions with respect to* Γ. Define the spaces $\mathrm{AD}(\Delta/\Gamma)$, $\mathrm{BMOA}(\Delta/\Gamma)$, $\mathrm{VMOA}(\Delta/\Gamma)$ and $\mathscr{B}_0(\Delta/\Gamma)$ as the pull backs of the spaces $\mathrm{AD}(R)$, $\mathrm{BMOA}(R)$, $\mathrm{VMOA}(R)$ and $\mathscr{B}_0(R)$, respectively. We note that the space $\mathrm{BMOA}(\Delta/\Gamma)$ corresponding to $\mathrm{BMOA}(R)$ could be defined as

$$\mathrm{BMOA}(\Delta) \cap \{\text{automorphic functions with respect to } \Gamma\}.$$

However, in the case of $\mathrm{VMOA}(R)$ the corresponding space $\mathrm{VMOA}(\Delta/\Gamma)$ is in general different from

$$\mathrm{VMOA}(\Delta) \cap \{\text{automorphic functions with respect to } \Gamma\}.$$

In fact, the latter space consists only of constant functions. Instead the space $\mathrm{VMOA}(\Delta/\Gamma)$ includes those functions f of $\mathrm{BMOA}(\Delta) \cap \{$automorphic functions with respect to $\Gamma\}$ which satisfy in one fundamental polygon Ω (in all fundamental polygons) the condition

$$\lim_{\substack{|\omega|\to 1\\ \omega\in\Omega}} \iint_{\Omega} |f'(z)|^2 g(z, \omega; \Gamma)\,dxdy = 0$$

where $g(z, \omega; \Gamma)$ is a Green's function of the Riemann surface Δ/Γ. Now

$$\mathscr{B}_0(\Delta/\Gamma) = \left\{ f \,\middle|\, f \text{ automorphic with respect to } \Gamma \text{ and } \lim_{\substack{|z|\to 1\\ z\in\Omega}} (1 - |z|^2)|f'(z)| = 0 \right\}$$

(cf. [**1**, Definition 1] where we denoted $\mathscr{B}_0(\Delta/\Gamma)$ by $\mathscr{B}_0(\Gamma)$).

In [**1**, Example (a)] we proved $\mathrm{AD}(\Delta/\Gamma) \subset \mathscr{B}_0(\Delta/\Gamma)$. Further, in [**1**, Theorem 1] we gave some equivalent conditions implying an analytic automorphic function to belong to $\mathscr{B}_0(\Delta/\Gamma)$. In Section 2 we showed that $\mathrm{AD}(\Delta/\Gamma) \not\subset \mathrm{VMOA}(\Delta/\Gamma)$ in some cases. However, the following is valid:

THEOREM 2. $\mathrm{VMOA}(\Delta/\Gamma) \subset \mathscr{B}_0(\Delta/\Gamma)$.

Proof. It is well-known that (cf. Proposition)

$$\mathrm{VMOA}(\Delta/\Gamma) \subset \mathrm{BMOA}(\Delta/\Gamma) \subset \mathrm{BMOA}(\Delta) \subset \mathscr{B},$$

where the inclusion $\mathrm{BMOA}(\Delta/\Gamma) \subset \mathrm{BMOA}(\Delta)$ has been proved by Metzger [**9**, Proposition 2] and

$$\mathscr{B} = \left\{ g \mid \; g \text{ analytic in } \Delta \text{ and } \sup_{z \in \Delta} (1 - |z|^2)\,|g'(z)\,| < \infty \right\}$$

is the space of Bloch functions. Let $f \in \mathrm{VMOA}(\Delta/\Gamma)$. Suppose, on the contrary, that

$$f \in \mathscr{B} \backslash \mathscr{B}_0(\Delta/\Gamma).$$

Then there exists a sequence of points $(\omega_n) \subset \Omega$, $|\omega_n| \to 1$, such that

$$\lim_{n \to \infty} (1 - |\omega_n|^2)\,|f'(\omega_n)\,| = c > 0.$$

Set

$$f_n(\zeta) = f\left(\frac{\zeta + \omega_n}{1 + \bar{\omega}_n \zeta}\right) - f(\omega_n).$$

Since $f \in \mathscr{B}$, we may suppose that

$$\lim_{n \to \infty} f_n(\zeta) = f_0(\zeta)$$

uniformly in compact sets of Δ where $f_0(\zeta)$ is an analytic function in Δ. It follows that

$$|f_n'(0)\,| = (1 - |\omega_n|^2)\,|f'(\omega_n)\,| \to c \quad \text{as } n \to \infty.$$

Since

$$f_0'(0) = \lim_{n \to \infty} f_n'(0),$$

$f_0(\zeta)$ is a non-constant function.

If $g_\Delta(z, \omega) = \log(\,|1 - \bar{\omega} z|/|z - \omega|\,)$ is a Green's function of Δ, then by a Myrberg's theorem [**13**, p. 522]

$$g(z, \omega; \Gamma) = \sum_{\gamma \in \Gamma} g_\Delta(z, \gamma(\omega)\,)$$

is a Green's function of the Riemann surface Δ/Γ. By a calculation

$$\int\!\!\int_\Omega |f'(z)\,|^2 g(z, \omega; \Gamma) dxdy = \int\!\!\int_\Delta |f'(z)\,|^2 g_\Delta(z, \omega) dxdy.$$

Further, by the assumption

$$(14) \quad \lim_{n \to \infty} \int\!\!\int_\Delta |f'(z)\,|^2 g_\Delta(z, \omega_n) dxdy$$
$$= \lim_{n \to \infty} \int\!\!\int_\Omega |f'(z)\,|^2 g(z, \omega_n; \Gamma) dxdy = 0.$$

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Choose the pseudohyperbolic disks

$$U(\omega_n, R) = \left\{ z \,\middle|\, \left| \frac{z - \omega_n}{1 - \bar{\omega}_n z} \right| < R \right\} \quad (R < 1).$$

Then, by (14),

$$(15) \quad \lim_{n \to \infty} \iint_{U(\omega_n, R)} |f'(z)|^2 g_\Delta(z, \omega_n) dxdy = 0.$$

By the change of variable

$$z = T_n(\zeta) = \frac{\zeta + \omega_n}{1 + \bar{\omega}_n \zeta}, \quad z = x + iy, \zeta = \xi + i\eta,$$

$$\begin{aligned} 0 &= \lim_{n \to \infty} \iint_{U(\omega_n, R)} |f'(z)|^2 g_\Delta(z, \omega_n) dxdy \\ &= \lim_{n \to \infty} \iint_{U(0, R)} |f_n'(\zeta)|^2 g_\Delta(\zeta, 0) d\xi d\eta \\ &= \iint_{U(0, R)} |f_0'(\zeta)|^2 g_\Delta(\zeta, 0) d\xi d\eta \\ &\geqq \log \frac{1}{R} \iint_{U(0, R)} |f_0'(\zeta)|^2 d\xi d\eta > 0 \end{aligned}$$

since $f_0(\zeta)$ is non-constant. But this is a contradiction and thus $f \in \mathscr{B}_0(\Delta/\Gamma)$. Hence $\mathrm{VMOA}(\Delta/\Gamma) \subset \mathscr{B}_0(\Delta/\Gamma)$.

4. The space BMOH(R). Let U be a real-valued harmonic function on the Riemann surface R. We denote by HD(R) the space of all harmonic functions U on R with finite Dirichlet integral, that is,

$$\mathrm{HD}(R) = \left\{ U \,\middle|\, U \text{ harmonic on } R \text{ and } \iint_R |\mathrm{grad}\, U(z)|^2 dxdy < \infty \right\}.$$

Further we denote

$$\mathrm{BMOH}(R) = \left\{ U \,\middle|\, U \text{ harmonic on } R \text{ and } \sup_{a \in R} \iint_R |\mathrm{grad}\, U(z)|^2 g(z, a) dxdy < \infty \right\}.$$

(All the time we suppose that R admits Green's functions.)

Kusunoki and Taniguchi [**8**] showed that a harmonic function U in HD(R) belongs to BMOH(R) in some special cases. Gotoh [**5**] constructed an infinitely connected plane domain R and a harmonic function U such that $U \in \mathrm{HD}(R)$ but $U \notin \mathrm{BMOH}(R)$. The meaning of this section is to show that Pommerenke [**10**, Theorem 2] has implicitly constructed a harmonic function $U \in \mathrm{HD}(R)$ but not belonging to $\mathrm{BMOH}(R)$.

We proceed by pulling back functions to the universal covering surface Δ. Pommerenke constructed an analytic non-Bloch function f satisfying

$$\int\int_{\Omega} |f'(z)|^2 dxdy < \infty.$$

The function f satisfies the equality

$$f(\gamma(z)) = f(z) + r(\gamma)$$

where $\gamma \in \Gamma$ (Γ of convergence type) and $r(\gamma)$ is a real number. If we set

$$g(z) = -if(z) = u(z) + iv(z),$$

then u is a harmonic automorphic function with respect to Γ for which

$$\int\int_{\Omega} |\mathrm{grad}\ u(z)|^2 dxdy = \int\int_{\Omega} |f'(z)|^2 dxdy < \infty.$$

Since $\mathrm{BMOA}(\Delta) \subset \mathscr{B}$ and $g \notin \mathscr{B}$, we have

$$(16) \quad \sup_{\omega \in \Delta} \int\int_{\Delta} |g'(z)|^2 g_{\Delta}(z, \omega) dxdy = \infty.$$

The equality (16) is equivalent to (we may suppose that $\omega \in \bar{\Omega}$)

$$(17) \quad \sup_{\omega \in \bar{\Omega}} \int\int_{\Omega} |g'(z)|^2 g(z, \omega; \Gamma) dxdy = \infty$$

where

$$g(z, \omega; \Gamma) = \sum_{\gamma \in \Gamma} g_{\Delta}(z, \gamma(\omega)).$$

As transferred to the surface R the expression (17) is same as

$$\sup_{a \in R} \int\int_{R} |\mathrm{grad}\ U(z)|^2 g(z, a) dxdy = \infty$$

where $\Pi(\omega) = a$ and $u(z) = U(\Pi(z))$. By the definition $U \notin \mathrm{BMOH}(R)$.

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