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A THEOREM ON APPROXIMATION OF IRRATIONAL NUMBERS BY SIMPLE CONTINUED FRACTIONS

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1. Introduction

Let ξ be an irrational number with simple continued fraction expansion $\xi=[a_0;a_1,a_2,\ldots]$, p_n/q_n be its nth convergent, $|\xi-p_n/q_n|=1/(M_nq_n^2)$. The following two theorems were proved by Müller [**9**] and rediscovered by Bagemihl and McLaughlin [**1**]:

Theorem 1. *For* $n>1$, $\max(M_{n-1},M_n,M_{n+1})>\sqrt{a_{n+1}^2+4}$.

Theorem 2. *For* $n>1$, *either* $M_n>a_{n+1}+1/a_{n+1}$ *or* $\min(M_{n-1},M_{n+1})>a_{n+1}+1/a_{n+1}$.

If $a_{n+1}\geqq 2$ in the above theorems, we have Fujiwara's theorems [**4**]. Since $a_{n+1}\geqq 1$, Theorem 1 implies Borel's theorem [**2**], and Vahlen's theorem [**22**] (that either M_n or M_{n+1} is greater than 2) follows from Theorem 2. For more information, cf. [**3**, **5**, 7, **10**, **12**, **16**, **17**].

In this paper, we use the method of papers [**20**, **21**] to prove a theorem, which includes Theorems 1, 2 and provides a new inequality $\max(M_{n-1},M_{n+1})>4a_{n+1}(a_{n+1}^2+1)/(2a_{n+1}^2+1)$. The proof is elementary. This theorem can be used to investigate asymmetric approximation.

2. Preliminaries

It is well known [**5**, **7**] that if $M_n=[a_{n+1};a_{n+2},\ldots]+[0;a_n,a_{n-1},\ldots,a_1]$, then $\xi-p_n/q_n=(-1)^n/(M_nq_n^2)$, $|\xi-p_n/q_n|=1/(M_nq_n^2)$.

Let $P=[a_{n+2};a_{n+3},\ldots]$; $Q=[a_n;a_{n-1},\ldots,a_1]$. Then we have the following relations:

$$M_{n-1}=\frac{1}{a_{n+1}+P^{-1}}+Q; \tag{1}$$

$$M_n=a_{n+1}+\frac{1}{P}+\frac{1}{Q}; \tag{2}$$

$$M_{n+1}=\frac{1}{a_{n+1}+Q^{-1}}+P. \tag{3}$$

We need the following two simple lemmas. They may be easily proved by evaluating the derivatives of the functions involved.

Lemma 1. *Let* $f(x)=1/(a+x^{-1})+1/(r-a-x^{-1})$ *with* $0<a<r$.

(i) *The function* $f(x)$ *is decreasing on the interval* $(1/(r-a),\ 2/(r-a))$. *Thus, if* $1/(r-a)<x<2/(r-a)$, $f(x)>4r/(r^2-a^2)$.

(ii) *If* $r\leqq 2a$ *then* $f(x)$ *is decreasing on the interval* $(1/(r-a),\ \infty)$, so that if $1/(r-a)<x$, $f(x)>r/(a(r-a))$.

Lemma 2. *If* $r>0$ *then the function* $g(x)=1/(r-x^{-1})+x$ *is increasing on the interval* $(2/r,\infty)$. *In particular, if* $0<a<r$ *and* $x>2/(r-a)$ *then* $g(x)>4r/(r^2-a^2)$.

3. Main results

Theorem 3.

(i) *Let* $r>a_{n+1}$ *be a constant. If* $M_n<r$, *then* $\max(M_{n-1},M_{n+1})>4r/(r^2-a_{n+1}^2)$.

(ii) *Let* r *be a constant such that* $a_{n+1}<r<2a_{n+1}$. *If* $M_n<r$, *then* $\min(M_{n-1},M_{n+1})>r/(a_{n+1}(r-a_{n+1}))$.

Proof. Since $M_n<r$, by (2) we have $1/P+1/Q=M_n-a_{n+1}<r-a_{n+1}$. Hence

$$P>\frac{1}{r-a_{n+1}-Q^{-1}}>0; \tag{4}$$

$$Q>\frac{1}{r-a_{n+1}-P^{-1}}>0. \tag{5}$$

It is easily seen that the following inequalities hold:

$$M_{n+1}=\frac{1}{a_{n+1}+Q^{-1}}+P>\frac{1}{a_{n+1}+Q^{-1}}+\frac{1}{r-a_{n+1}-Q^{-1}}; \tag{6}$$

$$M_{n-1}=\frac{1}{a_{n+1}+P^{-1}}+Q>\frac{1}{r-Q^{-1}}+Q. \tag{7}$$

From (4), (5), we know

$$P>\frac{1}{r-a_{n+1}}; \tag{8}$$

$$Q>\frac{1}{r-a_{n+1}}. \tag{9}$$

If $Q\leqq 2/(r-a_{n+1})$, by Lemma 1(i), (8) and (6) we have $M_{n+1}>4r/(r^2-a_{n+1}^2)$.

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If $Q>2/(r-a_{n+1})$, by Lemma 2 and (7) we have $M_{n-1}>4r/(r^2-a_{n+1}^2)$. Therefore $\max(M_{n-1},M_{n+1})>4r/(r^2-a_{n+1}^2)$.

To prove (ii), we notice that by (5)

$$M_{n-1}=\frac{1}{a_{n+1}+P^{-1}}+Q>\frac{1}{a_{n+1}+P^{-1}}+\frac{1}{r-a_{n+1}-P^{-1}}. \tag{10}$$

Since $a_{n+1}<r<2a_{n+1}$, by Lemma 1(ii), (9), (8), (6) and (10), we have $\min(M_{n-1},M_{n+1})>r/(a_{n+1}(r-a_{n+1}))$.

Remark 0. If $M_n=r$ and $Q\neq 2/(r-a_{n+1})$, we still have $\max(M_{n-1},M_{n+1})>4r/(r^2-a_{n+1}^2)$ by the proof of Theorem 3(i).

If $M_n=r$ and $Q=2/(r-a_{n+1})$, then by (2) we have $P=2/(r-a_{n+1})=Q$. It is impossible because Q is rational but P is irrational. Therefore Theorem 3(i) can be strengthened as follows:

Theorem 3(i′), *If* $M_n\leqq r$, *then* $\max(M_{n-1},M_{n+1})>4r(r^2-a_{n+1}^2)$.

Remark 1. Letting $r=a_{n+1}^2+4$ in Theorem 3(i) we have Theorem 1.

Letting $r=a_{n+1}+1/a_{n+1}$ in Theorem 3(ii) gives Theorem 2 and also the result that either $M_n>a_{n+1}+1/a_{n+1}$ or $\max(M_{n-1},M_{n+1})>4a_{n+1}(a_{n+1}^2+1)/(2a_{n+1}^2+1)$.

The following result is a generalization of Theorem 2.

Corollary 1. *For* $n>1$ *and* $k>0$, *either* $M_n>a_{n+1}+1/(ka_{n+1})$ *or* $\min(M_{n-1},M_{n+1})>ka_{n+1}+1/a_{n+1}$.

Proof. This is a special case of Theorem 3(ii) for $r=a_{n+1}+1/(ka_{n+1})$.

Remark 2. If k is sufficiently large, comparing Theorem 2 and Corollary 1, we have an interesting conclusion: if M_n loses a little, both M_{n-1} and M_{n+1} will gain a lot.

The following result, which is a corollary of Theorem 3, was obtained by Szüsz [**19**, Theorem 2.1].

Corollary 2. *Let* $\tau>0$ *be a constant and* $a_{n+1}>\bar{\tau}^1$. *Then either* $M_n>\sqrt{1+4\tau}/\tau$ *or* $\max(M_{n-1},M_{n+1})>\sqrt{1+4\tau}$.

Proof. If $a_{n+1}\geqq\sqrt{1+4\tau}/\tau$, then $M_n>\sqrt{1+4\tau}/\tau$ by (2). If $a_{n+1}<\sqrt{1+4\tau}/\tau$, let $r=\sqrt{1+4\tau}/\tau$, then by Theorem 3(i), we have either

$$M_n>\sqrt{1+4\tau}/\tau \quad \text{or} \quad \max(M_{n-1},M_{n+1})>\frac{4\sqrt{1+4\tau}/\tau}{(1+4\tau)/\tau^2-\tau^{-2}}=\sqrt{1+4\tau}.$$

We give another application of Theorem 3. Kurosu [**6**, p. 253] proved that if $a_{n+1}\geqq 2$, then either $M_n>8/3$ or $\max(M_{n-1},M_{n+1})>10/3=3.3333$. We improve 10/3 to be $24/7=3.4285$ without the restriction $a_{n+1}\geqq 2$.

Corollary 3. *Either* $M_n > 8/3$ *or* $\max(M_{n-1}, M_{n+1}) > 24/7$.

Proof. From formula (2), we know that if $a_{n+1} \geqq 3$, then $M_n > 3 > 8/3$. Therefore we need only consider the case $a_{n+1} \leqq 2$. Let $r = 8/3$. By Theorem 3(i), we have either

$$M_n > 8/3 \quad \text{or} \quad \max(M_{n-1}, M_{n+1}) > \frac{4(8/3)}{\left(\frac{8}{3}\right)^2 - 2^2} = 24/7.$$

4. Applications to asymmetric approximation

Segre [18] proved a theorem on asymmetric approximation, which was investigated in [11, 13, 14, 15, 19]. A version of LeVeque's statement [8] of Segre's theorem is the following result.

Theorem 4. *Let* τ *be a fixed positive number. Then in the five consecutive convergents* $p_i/q_i (i = n-2,\ n-1,\ n,\ n+1,\ n+2)$ *of an irrational number* ξ, *at least one of them satisfies the following inequality:*

$$-\frac{1}{\sqrt{1+4\tau}\, q_i^2} < \xi - \frac{p_i}{q_i} < \frac{\tau}{\sqrt{1+4\tau}\, q_i^2}. \tag{11}$$

LeVeque [8] pointed out that in the above theorem, five convergents cannot be replaced by three. A natural question arises: when is Theorem 4 true if five is replaced by three? Using Theorem 3, we obtain some results in this connection. We first prove the following theorem, which was obtained in [8] by using Farey's series.

Theorem 5. *If* n *is an odd positive integer, then in three consecutive convergents* p_i/q_i $(i = n-1, n, n+1)$ *of an irrational number* ξ, *at least one satisfies* (11).

Proof. Since $\xi - p_i/q_i = (-1)^i/(M_i q_i^2)$, it is easily seen that inequality (11) is equivalent to $M_i > \sqrt{1+4\tau}$ if i is odd, and $M_i > \sqrt{1+4\tau}/\tau$ if i is even.

If $a_{n+1} \geqq \sqrt{1+4\tau}$, then $M_n > \sqrt{1+4\tau}$ by formula (2); if $a_{n+1} < \sqrt{1+4\tau}$, let $r = \sqrt{1+4\tau}$, by Theorem 3(i) we have either

$$M_n > \sqrt{1+4\tau} \qquad \text{or} \qquad \max(M_{n-1}, M_{n+1}) > \frac{4\sqrt{1+4\tau}}{(1+4\tau) - 1^2} = \sqrt{1+4\tau}/\tau.$$

Theorem 5 is not correct for even n. But for some special value of τ we have an affirmative result.

Theorem 6. *If* $1 \leqq \tau < 2 + \sqrt{5}$, *then at least one of the three consecutive convergents* p_i/q_i $(i = n-1, n, n+1)$ *of an irrational number* ξ *satisfies inequality* (11).

Proof. We need only prove the theorem for even n.

Since $1 \leqq \tau < 2+\sqrt{5}$, we know that $1+4\tau-\tau^2>0$ and $4\tau/(1+4\tau-\tau^2) \geqq 1$. If $a_{n+1} \geqq \sqrt{1+4\tau}/\tau$, we have $M_n > \sqrt{1+4\tau}/\tau$ by formula (2); if $a_{n+1} < \sqrt{1+4\tau}/\tau$, let $r=\sqrt{1+4\tau}/\tau$, by Theorem 3(i), we have either

$$M_n > \sqrt{1+4\tau}/\tau \qquad \text{or} \qquad \max(M_{n-1}, M_{n+1}) > \frac{4\sqrt{1+4\tau}/\tau}{(1+4\tau)/\tau^2-1^2} = \frac{4\tau\sqrt{1+4\tau}}{1+4\tau-\tau^2} \geqq \sqrt{1+4\tau}.$$

Remark 3. Letting $\tau=1$ in Theorem 6, we obtain Borel's theorem again. But if $\tau=1$ in Theorem 4, we cannot obtain Borel's theorem.

Two real numbers ξ and ξ' are said to be equivalent if there are integers a, b, c, d such that $\xi'=(a\xi+b)/(c\xi+d)$ and $ad-bc=\pm 1$. If ξ is not equivalent to $(\sqrt{5}+1)/2=[1; 1, 1, \ldots]$, then for certain values of τ, inequality (11) can be sharpened.

Theorem 7. *If ξ is an irrational number not equivalent to $(\sqrt{5}+1)/2$, and $7/4<\tau<(5+2\sqrt{5})/4$, then there are infinitely many convergents p_n/q_n satisfying the following inequality:*

$$-\frac{1}{\sqrt{1+4\tau}\,q_n^2} < \xi - \frac{p_n}{q_n} < \frac{\tau-3/4}{\sqrt{1+4\tau}\,q_n^2} < \frac{\frac{6\sqrt{5}-10}{5}}{\sqrt{1+4\tau}\,q_n^2}. \tag{12}$$

Proof. Since ξ is not equivalent to $(\sqrt{5}+1)/2=[1; 1, 1, \ldots]$, there are infinitely many $a_{n+1} \geqq 2$.

Let n be odd. If $a_{n+1} \geqq \sqrt{1+4\tau}$, then $M_n > \sqrt{1+4\tau}$ by formula (2); if $a_{n+1} < \sqrt{1+4\tau}$, let $r=\sqrt{1+4\tau}$, by Theorem 3(i) either

$$M_n > \sqrt{1+4\tau} \qquad \text{or} \qquad \max(M_{n-1}, M_{n+1}) > \frac{4\sqrt{1+4\tau}}{(1+4\tau)-2^2} = \frac{\sqrt{1+4\tau}}{\tau-3/4}.$$

Let n be even. Since $7/4<\tau<(5+2\sqrt{5})/4$, we know that $-16\tau^2+40\tau-5>0$ and $4(4\tau-3)/(-16\tau^2+40\tau-5) \geqq 1$. If $a_{n+1} \geqq \sqrt{1+4\tau}/(\tau-3/4)$, then $M_n > \sqrt{1+4\tau}/(\tau-3/4)$ by formula (2); if $a_{n+1} < \sqrt{1+4\tau}/(\tau-3/4)$, let $r=\sqrt{1+4\tau}/(\tau-3/4)$, by Theorem 3(i), we have either $M_n > \sqrt{1+4\tau}/(\tau-3/4)$ or

$$\max(M_{n-1}, M_{n+1}) > \frac{4r}{r^2-2^2} = \frac{4(4\tau-3)\sqrt{1+4\tau}}{-16\tau^2+40\tau-5} \geqq \sqrt{1+4\tau}.$$

It is easily seen that

$$\tau - 3/4 = \left(1-\frac{3}{4\tau}\right)\tau < \left(1-\frac{3}{4(5+2\sqrt{5})/4}\right)\tau = \frac{6\sqrt{5}-10}{5}\tau.$$

Therefore inequality (12) is correct.

Another theorem of asymmetric approximation was given by Robinson [**14**]. We state it as Theorem 8.

Theorem 8. *Given any irrational number ξ and any positive number ε, there are infinitely many rational numbers p/q satisfying the following inequality:*

$$-\frac{1}{(\sqrt{5}-\varepsilon)q^2}<\xi-\frac{p}{q}<\frac{1}{(\sqrt{5}+1)q^2}. \tag{13}$$

For certain values of ε we can show that one of any three consecutive convergents of ξ satisfies inequality (13). In fact we have the following theorem.

Theorem 9. *If* $(20-7\sqrt{5})/5<\varepsilon<\sqrt{5}-1$, *then one of any three consecutive convergents of ξ satisfies inequality* (13).

Proof. Let n be odd. Since $\varepsilon<\sqrt{5}-1$, we have $4-2\sqrt{5}\varepsilon+\varepsilon^2>0$; since

$$\varepsilon>(20-7\sqrt{5})/5=0.8695>(3+\sqrt{5}\quad\sqrt{10+2\sqrt{5}})/(\sqrt{5}+1)=0.4425,$$

we have

$$\frac{4(\sqrt{5}-\varepsilon)}{4-2\sqrt{5}\varepsilon+\varepsilon^2}>\sqrt{5}+1.$$

If $a_{n+1}\geqq\sqrt{5}-\varepsilon$, then $M_n>\sqrt{5}-\varepsilon$ by formula (2); if $a_{n+1}<\sqrt{5}-\varepsilon$, let $r=\sqrt{5}-\varepsilon$, by Theorem 3(i), we have either $M_n>\sqrt{5}-\varepsilon$ or

$$\max(M_{n-1},M_{n+1})>\frac{4(\sqrt{5}-\varepsilon)}{(\sqrt{5}-\varepsilon)^2-1^2}=\frac{4(\sqrt{5}-\varepsilon)}{4-2\sqrt{5}\varepsilon+\varepsilon^2}>\sqrt{5}+1.$$

Let n be even. Since $\varepsilon>(20-7\sqrt{5})/5$, we have $4(\sqrt{5}+1)/(5+2\sqrt{5})>\sqrt{5}-\varepsilon$. If $a_{n+1}\geqq\sqrt{5}+1$, then $M_n>\sqrt{5}+1$ by formula (2); if $a_{n+1}<\sqrt{5}+1$, let $r=\sqrt{5}+1$, by Theorem 3(i), we have either

$$M_n>\sqrt{5}+1 \qquad \text{or} \qquad \max(M_{n-1},M_{n+1})>\frac{4(\sqrt{5}+1)}{(\sqrt{5}+1)^2-1^2}=\frac{4(\sqrt{5}+1)}{5+2\sqrt{5}}>\sqrt{5}-\varepsilon.$$

Therefore Theorem 9 is true.

Now two problems arise naturally.

Problem 1. Find all the values of τ such that for any irrational number ξ, at least one of any three consecutive convergents satisfies inequality (11).

Problem 2. Find all the values of ε such that for any irrational number ξ, at least one of any three consecutive convergents satisfies inequality (13).

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