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MASSEY PRODUCTS AND LOWER CENTRAL SERIES OF FREE GROUPS

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1. Introduction. The purpose of this paper is to continue the investigation into the relationships amongst Massey products, lower central series of free groups and the free differential calculus (see [**4**], [**9**], [**12**]). In particular we set forth the notion of a universal Massey product $\ll\alpha_1, \ldots, \alpha_k\gg$, where the α_i are one dimensional cohomology classes. This product is defined with zero indeterminacy, natural and multilinear in its variables.

In order to state the results we need some notation. Throughout F will denote the free group on fixed generators $x_1, \ldots, x_n$ and

$$F = F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots, \quad F_{k+1} = [F, F_k]$$

will denote the lower central series of F. If $I = (i_1, \ldots, i_k)$ is a sequence such that $1 \leqq i_1, \ldots, i_k \leqq n$ then ∂_I is the iterated Fox derivative $\partial_{i_1} \circ \ldots \circ \partial_{i_k}$ and $\epsilon_I = \epsilon \circ \partial_I$, where $\epsilon: ZF \rightarrow Z$ is the augmentation. By convention we set $\partial_I =$ identity if I is empty.

Let G be a one relator group presented by $\{x_1, \ldots, x_n | W\}$, where $W \in [F, F]$ is not a proper power. Then H^1G is free abelian on generators $u_1, \ldots, u_n$ which are dual to $x_1, \ldots, x_n$ and H^2G is infinite cyclic with generator denoted by $\{W\}$ (see [**3**]). If $\alpha_1, \ldots, \alpha_l \in H^1G$ are arbitrary cohomology classes, say

$$\alpha_i = \sum_{j=1}^{n} a_{ij}u_j,$$

then we will prove (see also [**9**])

THEOREM. *Suppose $W \in F_k$, $k \geqq 2$. If $2 \leqq l < k$ then the Massey product $\langle\alpha_1, \ldots, \alpha_l\rangle$ is identically equal to* 0. *On the other hand Massey products of length k are defined, have zero indeterminacy, and are evaluated on $\{W\}$ according to the formula*

$$(\langle\alpha_1, \ldots, \alpha_k\rangle, \{W\}) = \sum_{(j_1, \ldots, j_k)} a_{1j_1} \ldots a_{kj_k}\epsilon_{j_1 \ldots j_k}(W).$$

If we set

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$$a_J = a_{1j_1}a_{2j_2} \dots a_{kj_k} \quad \text{for } J = (j_1, \dots, j_k)$$

then this last formula amounts to the multilinearity of Massey products, i.e.,

$$\langle \alpha_1, \dots, \alpha_k \rangle = \sum_J a_J \langle u_J \rangle, \quad \text{where } u_J = \langle u_{j_1}, \dots, u_{j_k} \rangle.$$

Here the summation is over all sequences J of length k.

We also prove theorems about more general groups. Let G be presented by $\{x_1, \dots, x_n | W_1, \dots, W_p\}$, where the relators W_j belong to $[F, F]$. If X is the 2 dimensional CW complex associated to this presentation and $\alpha_1, \dots, \alpha_k \in H^1X$ then Massey products $\langle \alpha_1, \dots, \alpha_k \rangle$ can be defined in certain cases and evaluated in $H^2(X)$ by similar formulas (see Section 4).

In Section 2 we collect the preliminary material we need on the cohomology of groups. Then in Section 3 we treat Massey products via the approach in Dwyer's paper [**1**] and finally Section 4 is concerned with Massey products in 2 complexes.

2. Group cohomology. For any discrete group G let B_*G denote the normalized bar resolution (see [7]). Thus, for any $p > 0$, B_pG is the free left G module on generators $[g_1| \dots |g_p]$, where the $g_i \in G$ and $g_i \neq 1$. If $p = 0$ $B_0G = \Lambda = ZG$ is free on one generator [ ]. We can also define B_pG to be the left G module presented by

generators $[g_1| \dots |g_p]$, $g_i \in G$;

relators $[g_1| \dots |g_p]$, some $g_i = 1$.

In either case the boundary operators are the Λ module homomorphisms

$$d_p:B_pG \rightarrow B_{p-1}G$$

defined by

$$d_p[g_1| \dots |g_p] = g_1[g_2| \dots |g_p] + \sum_{i=1}^{p-1} (-1)^i [g_1| \dots |g_ig_{i+1}| \dots |g_p] + (-1)^p [g_1| \dots |g_{p-1}].$$

In particular

$$d_1[g] = (g - 1)[\ ] \quad \text{and} \quad d_2[g_1|g_2] = g_1[g_2] - [g_1g_2] + [g_1].$$

If $\epsilon:B_0G \rightarrow Z$ is the augmentation then

$$0 \leftarrow Z \overset{\epsilon}{\leftarrow} B_0G \overset{d_1}{\leftarrow} B_1G \leftarrow \dots$$

is a free Λ resolution of Z, and therefore can be used to define homology and cohomology of G. For a right Λ module A we have

$$H_*(G; A) = H_*(A \otimes_\Lambda B_*G)$$

and for a left Λ module A we set

$$H^*(G; A) = H^*(\mathrm{Hom}_\Lambda(B_*G, A)).$$

An element of the cochain group

$$B^p(G; A) = \mathrm{Hom}_\Lambda(B_p(G), A)$$

can be identified with a set function $u:G^p \to A$ which satisfies the normalization condition

$$u(g_1, \dots, g_p) = 0 \quad \text{if some } g_i = 1.$$

Using the normalized bar resolution permits a simple description of cup products of cochains. Suppose $u \in B^p(G; A)$ and $u' \in B^{p'}(G; A')$. Then

$$u \cup u' \in B^{p+p'}(G; A \otimes A')$$

is that cochain given by

$$u \cup u'(g_1, \dots, g_p, g'_1, \dots, g'_{p'}) = u(g_1, \dots, g_p) \otimes u'(g'_1, \dots, g'_{p'}).$$

This then induces the cup product pairing

$$H^p(G; A) \otimes H^{p'}(G; A') \to H^{p+p'}(G; A \otimes A').$$

As an example consider the case $A = A' = Z$ with the trivial module structure. Then

$$H^1G \cong B^1(G; Z) \cong \mathrm{Hom}_Z(G, Z)$$

and the cup product pairing of 1 dimensional classes is given by

$$u \cup u'[g|g'] = u(g)u'(g').$$

These formulas are particularly amenable when G is a 1-relator group $\{x_1, \dots, x_n | W\}$, where $W \in [F, F]$ is not a proper power. Let X be the 2 complex associated to this presentation of G and let $\widetilde{X}$ be its universal covering space. Thus X is a bouquet of n circles with a 2 cell sewn on by means of W. Since W is not a proper power it follows that $\widetilde{X}$ is contractible (see [**2**]) and therefore the cellular chain complex $C_*(\widetilde{X})$ can also be used to compute homology and cohomology of G. In fact augmenting $C_*(\widetilde{X})$ in the usual way results in the Lyndon resolution [**6**]

$$0 \leftarrow Z \overset{\epsilon}{\leftarrow} C_0(\widetilde{X}) \overset{d_1}{\leftarrow} C_1(\widetilde{X}) \overset{d_2}{\leftarrow} C_2(\widetilde{X}) \leftarrow 0,$$

where

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$$C_0(\tilde{X}) = \Lambda,\ C_1(\tilde{X}) = \Lambda^n,\ C_2(\tilde{X}) = \Lambda$$

$$d_1(\lambda_1, \ldots, \lambda_n) = \sum_i \lambda_i(\phi(x_i) - 1)$$

$$d_2(\lambda) = (\lambda\phi(\partial_1 W), \ldots, \lambda\phi(\partial_n W)).$$

Here ϕ is just the presenting homomorphism

$$F \xrightarrow{\phi} G.$$

We also let ϕ denote the corresponding ring homomorphism $\phi:ZF \to ZG$.

Since both $C_*(\tilde{X})$ and B_*G are resolutions of Z by free Λ modules we can find a chain transformation

$$T:C_*(\tilde{X}) \to B_*G$$

commuting with augmentation. Thus define

$$T_0:C_0(\tilde{X}) \to B_0G \quad \text{and} \quad T_1:C_1(\tilde{X}) \to B_1G$$

by

$$T_0(\lambda) = \lambda[\ ],\ T_1(\lambda_1, \ldots, \lambda_n) = \sum_i \lambda_i[\phi(x_i)].$$

In order to define $T_2:C_2(\tilde{X}) \to B_2G$ first consider the abelian group homomorphisms

$$s_0:B_0G \to B_1G,\ s_0:g[\ ] \to [g]$$

$$s_1:B_1G \to B_2G,\ s_1:g[g_1] \to [g|g_1].$$

Then define $T_2:C_2(\tilde{X}) \to B_2G$ by $T_2(\lambda) = \lambda s_1 T_1 d_2(1)$.

(2.1) THEOREM. *$T:C_*(\tilde{X}) \to B_*G$ is a chain transformation commuting with augmentation.*

Proof. We must show that the following diagram commutes

$$\begin{array}{ccccccccccccc}
0 & \leftarrow & Z & \xleftarrow{\epsilon} & C_0(\tilde{X}) & \xleftarrow{d_1} & C_1(\tilde{X}) & \xleftarrow{d_2} & C_2(\tilde{X}) & \leftarrow & 0 & \leftarrow & \cdots \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle T_0} & & \downarrow{\scriptstyle T_1} & & \downarrow{\scriptstyle T_2} & & \downarrow & & \\
0 & \leftarrow & Z & \xleftarrow{\epsilon} & B_0G & \xleftarrow{d_1} & B_1G & \xleftarrow{d_2} & B_2G & \xleftarrow{d_3} & B_3G & \leftarrow & \cdots
\end{array}$$

But this follows easily from the definitions and the identity

$$d_2 s_1 + s_0 d_1 = \mathrm{id}.$$

The homomorphism $T_2:C_2(\tilde{X}) \to B_2G$ can also be described in terms of the Fox derivatives of W. An immediate consequence of the definitions is

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$$T_2(\lambda) = \lambda \sum_{\substack{1 \leqq i \leqq n \\ x \in F}} a_{ix}[\phi(x)\,|\phi(x_i)\,], \quad \text{where } \partial_i(W) = \sum_{x \in F} a_{ix}x.$$

Now consider the chain complex

$$Z \otimes_\Lambda C_*(\tilde{X}) \cong C_*(X),$$

where Z is the trivial Λ module. Since $W \in [F, F]$ it follows that all boundary operators are zero and therefore $H_2G \cong H_2X \cong Z$ has a fundamental cycle, namely

$$1 \otimes_\Lambda 1 \in Z \otimes_\Lambda C_2(\tilde{X}).$$

Also $(\mathrm{id} \otimes T_2)(1 \otimes_\Lambda 1)$ is a fundamental cycle in $Z \otimes_\Lambda B_2G$. Now $Z \otimes_\Lambda B_2G$ is the free abelian group on generators $[g_1|g_2]$, where $g_1 \neq 1$, $g_2 \neq 1$, and

$$(\mathrm{id} \otimes T_2)(1 \otimes_\Lambda 1) = \sum_{\substack{1 \leqq i \leqq n \\ x \in F}} a_{ix}[\phi(x)\,|\phi(x_i)\,].$$

Definition.

$$\{W\} = \sum_{\substack{1 \leqq i \leqq n \\ x \in F}} a_{ix}[\phi(x)\,|\phi(x_i)\,].$$

(2.2) THEOREM. *$\{W\} \in Z \otimes_\Lambda B_2G$ is a fundamental cycle, i.e., it is a cycle with homology class a generator for $H_2G \cong Z$.*

The next theorem completely describes the cup product pairing

$$H^1G \otimes H^1G \to H^2G.$$

Similar results, with different proofs, can also be found in [**3**], [**5**], [**10**], [**11**]. The evaluation pairing $H^2G \otimes H_2G \to Z$ will be denoted by (,). Recall that $u_i \in H^1G$ is the class dual to x_i.

(2.3) THEOREM. $(u_i \cup u_j, \{W\}\,) = \epsilon_{ij}(W)$.

Proof. $u_i \cup u_j$ is represented by the cocyle in $B^2(G; Z)$ defined by

$$[g_1|g_2] \to u_i(g_1)u_j(g_2).$$

Evaluating on $\{W\}$ gives

$$\begin{aligned}(u_i \cup u_j, \{W\}\,) &= \sum_{\substack{1 \leqq k \leqq n \\ x \in F}} a_{kx}u_i(\phi(x)\,)u_j(\phi(x_k)\,) \\ &= \sum_{x \in F} a_{jx}u_i(\phi(x)\,) \\ &= \epsilon_i \sum_{x \in F} a_{jx}x\end{aligned}$$

since $u_i\phi = \epsilon_i$

$$= \epsilon_i\partial_j(W) = \epsilon_{ij}(W).$$

Now we generalize this result to finite 2 dimensional CW complexes. Thus let

$$G = \{x_1, \dots, x_n | W_1, \dots, W_p\}$$

be a finitely presented group and let X be the associated 2 complex. If $\phi: F \to G$ is the presenting homomorphism and $\tilde{X}$ is the universal covering space then the cellular chain complex $C_*(\tilde{X})$ is given by

$$0 \leftarrow C_0(\tilde{X}) \overset{d_1}{\leftarrow} C_1(\tilde{X}) \overset{d_2}{\leftarrow} C_2(\tilde{X}) \leftarrow 0,$$

where

$$C_0(\tilde{X}) = \Lambda,\ C_1(\tilde{X}) = \Lambda^n,\ C_2(\tilde{X}) = \Lambda^p,$$

$$d_1(\lambda_1, \dots, \lambda_n) = \sum_{i=1}^{n} \lambda_i(\phi(x_i) - 1)$$

$$d_2(\mu_1, \dots, \mu_p) = \Big(\dots, \underbrace{\sum_{j=1}^{p} \mu_j\phi(\partial_i W_j)}_{i^{\text{th}} \text{ coordinate.}}, \dots\Big)$$

In general $C_*(\tilde{X})$ is not exact since X may not be a $K(G, 1)$. However, we still have a chain transformation

$$T: C_*(\tilde{X}) \to B_*G$$

commuting with augmentation:

$$\begin{array}{ccccccccccc}
0 \leftarrow & Z & \overset{\epsilon}{\longleftarrow} & C_0(\tilde{X}) & \overset{d_1}{\longleftarrow} & C_1(\tilde{X}) & \overset{d_2}{\longleftarrow} & C_2(\tilde{X}) & \leftarrow & 0 & \leftarrow \cdots \\
 & \downarrow \text{id} & & \downarrow T_0 & & \downarrow T_1 & & \downarrow T_2 & & \downarrow & \\
0 \leftarrow & Z & \overset{\epsilon}{\longleftarrow} & B_0G & \overset{d_1}{\longleftarrow} & B_1G & \overset{d_2}{\longleftarrow} & B_2G & \leftarrow & B_3G & \leftarrow \cdots
\end{array}$$

T_0, T_1 are defined as before and

$$T_2: C_2(\tilde{X}) \to B_2G$$

is defined by

$$T_2(\mu_1, \dots, \mu_p) = \sum_{j=1}^{p} \mu_j s_1 T_1 d_2(e_j),$$

where

$$e_j = (0, \dots, 0, 1, 0, \dots, 0) \in \Lambda^p, \quad 1 \leqq j \leqq p.$$

$$\uparrow \text{ } j\text{th coordinate}$$

If we put

$$\partial_i(W_j) = \sum_{x \in F} a_{ijx}x \quad \text{for } 1 \leqq i \leqq n,\ 1 \leqq j \leqq p,$$

then

$$\begin{aligned} T_2(e_j) &= s_1 T_1 d_2(e_j) = s_1 T_1(\dots, \phi(\partial_i W_j), \dots) \\ &= s_1 \sum_{i=1}^{n} \phi(\partial_i W_j)[\phi(x_i)] = \sum_{\substack{1 \leqq i \leqq n \\ x \in F}} a_{ijx}[\phi(x)\,|\phi(x_i)]. \end{aligned}$$

Now assume that the relators $W_j \in [F, F]$, $1 \leqq j \leqq p$. Then H^1X is free abelian on generators $u_1, \dots, u_n$ and H_2X is free abelian on generators

$$1 \otimes_\Lambda e_k, \quad 1 \leqq k \leqq p.$$

The cup product structure in X is given by the following theorem.

(2.4) THEOREM. $(u_i \cup u_j, 1 \otimes_\Lambda e_k) = \epsilon_{ij}(W_k)$.

Proof. The chain transformation $T:C_*(\tilde{X}) \to B_*G$ induces an isomorphism

$$T^*:H^1G \to H^1X.$$

Thus we may consider u_i, u_j as elements of H^1G and compute their cup product as follows:

$$\begin{aligned} (u_i \cup u_j, 1 \otimes_\Lambda e_k) &= (u_i \cup u_j, 1 \otimes T_2(1 \otimes_\Lambda e_k)) \\ &= \Big(u_i \cup u_j, \sum_{\substack{1 \leqq s \leqq n \\ x \in F}} a_{skx}[\phi(x)\,|\phi(x_s)]\Big) \\ &= \sum_{\substack{1 \leqq s \leqq n \\ x \in F}} a_{skx} u_i(\phi(x)) u_j(\phi(x_s)) \\ &= \sum_{x \in F} a_{jkx} u_i(\phi(x)) \\ &= \epsilon_i \sum_{x \in F} a_{jkx} x = \epsilon_i \partial_j(W_k) = \epsilon_{ij}(W_k). \end{aligned}$$

It is also possible to further generalize these results by dropping the restrictions $W_j \in [F, F]$. Theorem (2.4) remains valid if we change the coefficients from Z to Z_d where

$$d = \text{L.C.M.}\{\epsilon_i(W_j) \mid 1 \leqq i \leqq n,\ 1 \leqq j \leqq p\}.$$

3. Massey products in groups. First we recall the definition of the Massey product of 1 dimensional classes $\alpha_1, \dots, \alpha_k \in H^1G$, see [**1**], [**4**] or [**8**]. Thus suppose given a $(k + 1) \times (k + 1)$ "matrix"

$$M = \left| \begin{array}{cccccc} 1 & m_{12} & m_{13} & \cdots & m_{1k} & \phi \\ & 1 & m_{23} & \cdots & m_{2k} & m_{2,k+1} \\ & & 1 & \ddots & & \\ & & & \ddots & & \\ & & & & \ddots & m_{k,k+1} \\ & & & & & 1 \end{array} \right|$$

where the m_{ij} are defined for only those (i, j) satisfying $1 \leqq i < j \leqq k + 1$ and $(1, j) \neq (1, k + 1)$. Moreover the following conditions must hold:

(1) $m_{ij} \in B^1(G; Z)$;

(2) $m_{i,i+1}$ is a cocycle representing α_i;

(3) if $\delta: B^1(G; Z) \to B^2(G; Z)$ is the coboundary then

$$\delta_{m_{ij}} = \sum_{s=i+1}^{j-1} m_{is} \cup m_{sj}.$$

M is called a defining system for the Massey product $\langle \alpha_1, \dots, \alpha_k \rangle$. It readily follows that the element

$$\sum_{s=2}^{k} m_{is} \cup m_{s,k+1} \in B^2(G; Z)$$

is a cocycle. The cohomology class of this cocycle is defined to be the value of M and is denoted by $\langle \alpha_1, \dots, \alpha_k \rangle_M$. In general the Massey product $\langle \alpha_1, \dots, \alpha_k \rangle$ is defined only if some defining system exists, in which case

$$\langle \alpha_1, \dots, \alpha_k \rangle = \{ \langle \alpha_1, \dots, \alpha_k \rangle_M \mid M \text{ a defining system} \} \subseteq H^2G.$$

Thus Massey products have indeterminacy.

Next we relate Massey products to triangular matrices as in [**1**]. Thus let T_{k+1} be the group of all $(k + 1) \times (k + 1)$ triangular matrices over Z having the form

$$\begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

and let Z_{k+1} be the central subgroup consisting of those matrices having the form

$$\begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 & \lambda \\ & 1 & 0 & \cdots & & 0 & 0 \\ & & \ddots & \ddots & & \vdots & \vdots \\ & & & \ddots & \ddots & 0 & \vdots \\ & 0 & & & \ddots & 1 & 0 \\ & & & & & \ddots & 1 \end{bmatrix} \quad \lambda \in Z \text{ arbitrary.}$$

Now let $\bar{T}_{k+1}$ denote the quotient group T_{k+1}/Z_{k+1}.

If G is any discrete group then a group homomorphism $\theta:G \to T_{k+1}$ consists of set maps

$$\theta_{ij}:G \to Z, \quad 1 \leqq i \leqq j \leqq k + 1,$$

satisfying

(4) $\theta_{ii}(g) = 1$ for all $g \in G$;

(5) $\theta_{ij}(g_1g_2) = \theta_{ij}(g_1) + \theta_{ij}(g_2) + \sum_{s=i+1}^{j-1} \theta_{is}(g_1)\theta_{sj}(g_2)$

for all $g_1, g_2 \in G$ if $i < j$.

Thus the near diagonal entries $\theta_{i,i+1}$, $1 \leqq i \leqq k$, are homomorphisms and therefore represent cohomology classes $\alpha_i \in H^1G$. Likewise a group homomorphism $\theta:G \to \bar{T}_{k+1}$ consists of set maps $\theta_{ij}:G \to Z$ defined for all (i, j) with $1 \leqq i \leqq j \leqq k + 1$, except for $(i, j) = (1, k + 1)$, and satisfying (4) and (5). Again the near diagonal entries are homomorphisms $G \to Z$, and hence represent cohomology classes in H^1G.

The main reason for introducing these groups is the following theorem (see [**1**]).

(3.1) THEOREM. *Suppose $\alpha_1, \ldots, \alpha_k \in H^1G$ are given. Then there exists a one to one correspondence $M \leftrightarrow \theta$ between defining systems M for the Massey product $\langle -\alpha_1, \ldots, -\alpha_k \rangle$ and group homomorphisms $\theta:G \to \bar{T}_{k+1}$ having near diagonal entries $\alpha_1, \ldots, \alpha_k$.*

To see how this correspondence works suppose $\theta:G \to \bar{T}_{k+1}$ is such a homomorphism. Then set

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$$m_{ij} = -\theta_{ij} \in B^1(G;\ Z),$$

and consider the calculation

$$\begin{aligned}\delta(m_{ij})[g_1|g_2] &= m_{ij}(g_1[g_2] - [g_1 g_2] + [g_1]) \\ &= -\theta_{ij}(g_2) + \theta_{ij}(g_1 g_2) - \theta_{ij}(g_1) \\ &= \sum_{s=i+1}^{j-1} \theta_{is}(g_1)\theta_{sj}(g_2) \\ &= \sum_{s=i+1}^{j-1} (-m_{is}) \cup (-m_{sj})[g_1|g_2].\end{aligned}$$

This implies that the m_{ij} satisfy the coboundary condition (3). The value of the defining system corresponding to $\theta:G \rightarrow \overline{T}_{k+1}$ is on the cocycle level given by

$$\langle -\alpha_1, \dots, -\alpha_k \rangle_M = \sum_{s=2}^{k} \theta_{1s} \cup \theta_{s,k+1}.$$

Now we consider an example of this theorem. Let $I = (i_1, \dots, i_k)$ be a fixed sequence, where $1 \leqq i_j \leqq n$ for $1 \leqq j \leqq k$. Then define $\xi:F \rightarrow T_{k+1}$ by

$$\xi(x) = \left| \begin{array}{ccccc} 1 & \epsilon_{i_1}(x) & \epsilon_{i_1 i_2}(x) & & \epsilon_I(x) \\ & 1 & \epsilon_{i_2}(x) & & \vdots \\ & & & \ddots & \vdots \\ & & & & \epsilon_{i_k}(x) \\ & & & & 1 \end{array} \right|$$

In other words the (s, t) entry, $1 \leqq s \leqq t \leqq k + 1$, is given by the formula

$$\xi_{st}(x) = \epsilon_{i_s \dots i_{t-1}}(x), \quad x \in F.$$

(3.2) LEMMA. *$\xi:F \rightarrow T_{k+1}$ is a homomorphism.*

Proof.

$$(\xi(x)\xi(y))_{st} = \sum_{j=s}^{t} \xi_{sj}(x)\xi_{jt}(y)$$

$$= \sum_{j=s}^{t} \epsilon_{i_s \ldots i_{j-1}}(x)\epsilon_{i_j \ldots i_{t-1}}(y)$$

$$= \sum_{I_1 \cdot I_2 = (i_s, \ldots, i_{t-1})} \epsilon_{I_1}(x)\epsilon_{I_2}(y)$$

where the summation is over all ordered pairs (I_1, I_2) satisfying

$$I_1 \cdot I_2 = (i_s, \ldots, i_{t-1}).$$

But according to **[4]** we have

$$\xi_{st}(xy) = \epsilon_{i_s \ldots i_{t-1}}(xy) = \sum_{I_1 \cdot I_2 = (i_s, \ldots, i_{t-1})} \epsilon_{I_1}(x)\epsilon_{I_2}(y).$$

Therefore

$$F \xrightarrow{\xi} T_{k+1} \rightarrow \overline{T}_{k+1}$$

yields a defining system for a Massey product. However the value of this product is automatically zero, and so not interesting. On the other hand suppose $W \in F$ is an element satisfying

(6) $\epsilon_{i_s \ldots i_{t-1}}(W) = 0$ for $1 \leqq s < t \leqq k + 1, (s, t) \neq (1, k + 1)$.

Then ξ induces a homomorphism $\theta:G \rightarrow \overline{T}_{k+1}$ such that

$$\begin{array}{ccc} F & \xrightarrow{\xi} & T_{k+1} \\ \downarrow{\scriptstyle \phi} & & \downarrow \\ G & \xrightarrow{\theta} & \overline{T}_{k+1} \end{array}$$

is commutative, where $G = \{x_1, \ldots, x_n | W\}$ and $\phi:F \rightarrow G$ is the presenting homomorphism. Thus θ yields a defining system M for the Massey product $\langle -u_{i_1}, \ldots, -u_{i_k} \rangle$ and its value is determined by the cocycle

$$\sum_{s=2}^{k} \theta_{1s} \cup \theta_{s,k+1} = \sum_{s=2}^{k} \epsilon_{i_1 \ldots i_{s-1}} \cup \epsilon_{i_s \ldots i_k}.$$

(3.3). THEOREM. *Suppose* $W \in [F, F]$ *is not a proper power,* $I = (i_1, \ldots, i_k)$ *is a fixed sequence and* (6) *above holds. Then*

$$\theta:G \rightarrow \overline{T}_{k+1}$$

yields a defining system M *for the Massey product* $\langle -u_{i_1}, \ldots, -u_{i_k} \rangle$. *Moreover*

$$(\langle -u_{i_1}, \ldots, -u_{i_k} \rangle_M, \{W\}) = \epsilon_I(W).$$

Proof. Only the last equation requires proof. Recall that there is a fundamental cycle $\{W\} \in H_2G$ since we are assuming that $W \in [F, F]$ and that W is not a proper power. In fact

$$\{W\} = \sum_{\substack{1 \leqq i \leqq n \\ x \in F}} a_{ix}[\phi(x) \,|\phi(x_i)\,], \quad \text{where } \partial_i(W) = \sum_{x \in F} a_{ix}x.$$

Thus we have

$$\begin{aligned} &(\langle -u_{i_1}, \ldots, -u_{i_k}\rangle_M, \{W\}\,) \\ &= \sum_{s=2}^{k} \epsilon_{i_1 \ldots i_{s-1}} \cup \epsilon_{i_s \ldots i_k} \sum_{\substack{1 \leqq i \leqq n \\ x \in F}} a_{ix}[\phi(x) \,|\phi(x_i)\,] \\ &= \sum_{s=2}^{k} \sum_{\substack{1 \leqq i \leqq n \\ x \in F}} a_{ix}\epsilon_{i_1 \ldots i_{s-1}}(x)\epsilon_{i_s \ldots i_k}(x_i) \\ &= \sum_{x \in F} a_{i_k x}\epsilon_{i_1 \ldots i_{k-1}}(x) = \epsilon_{i_1 \ldots i_{k-1}}\partial_{i_k}(W) = \epsilon_I(W). \end{aligned}$$

Definition. Assuming the hypothesis of (3.3) we set

$$\ll -u_{i_1}, \ldots, -u_{i_k} \gg \; = \langle -u_{i_1}, \ldots, -u_{i_k}\rangle_M.$$

We refer to $\ll -u_{i_1}, \ldots, -u_{i_k}\gg$ as a universal Massey product since it is defined with zero indeterminacy even though $\langle -u_{i_1}, \ldots, -u_{i_k}\rangle$ might have non-zero indeterminacy.

As an example of this consider the Massey product $\langle -u_1, -u_2, -u_3\rangle$ for the group

$$G = \{x_1, x_2, x_3 | W = [x_1, [x_2, x_3]\,][x_1, x_4]\,\}.$$

Then it is easy to check that the universal Massey product $\ll -u_1, -u_2, -u_3 \gg$ is defined. However, the Massey product $\langle -u_1, -u_2, -u_3\rangle$ will have non-zero indeterminacy since $u_1 \cup u_4 \neq 0$ (see **[4]**).

(3.4) COROLLARY. *Suppose $G = \{x_1, \ldots, x_n | W\}$, where $W \in F_k$ is not a proper power, $k \geqq 2$. Let $I = (i_1, \ldots, i_k)$ be any sequence. Then $\ll -u_{i_1}, \ldots, -u_{i_k}\gg$ is defined and*

$$(\ll -u_{i_1}, \ldots, -u_{i_k}\gg, \{W\}\,) = \epsilon_I(W).$$

In order to extend the definition of universal Massey products we again assume that $G = \{x_1, \ldots, x_n | W\}$ is such that $W \in [F, F]$ is not a proper power. Suppose $\alpha_1, \ldots, \alpha_k : G \to Z$ are homomorphisms, say

(7) $$\alpha_i = \sum_{j=1}^{n} a_{ij}u_j, \quad 1 \leqq i \leqq k.$$

Then define $\xi:F \to T_{k+1}$ by

(8) $$\xi_{st}(x) = \sum_{j_1,\dots,j_{t-s}} a_{sj_1}\dots a_{t-1,j_{t-s}}\epsilon_{j_1\dots j_{t-s}}(x),$$

$$1 \leqq s < t \leqq k+1.$$

Again ξ is a homomorphism.

If we make the assumption

(9) $$\epsilon_{j_1\dots j_{t-s}}(W) = 0$$

for all $(j_1, \dots, j_{t-s})$ satisfying

$$1 \leqq s < t \leqq k+1,\ (s, t) \neq (1, k+1),\ a_{sj_1} \dots a_{t-1,j_{t-s}} \neq 0$$

then it follows that ξ induces a homomorphism $\theta:G \to \bar{T}_{k+1}$ such that

$$\begin{array}{ccc} F & \xrightarrow{\ \xi\ } & T_{k+1} \\ \downarrow{\scriptstyle \phi} & & \downarrow \\ G & \xrightarrow{\ \theta\ } & \bar{T}_{k+1} \end{array}$$

is commutative.

In analogy with (3.3) we have

(3.5) THEOREM. *Suppose $G = \{x_1, \dots, x_n|W\}$, where $W \in [F, F]$ is not a proper power. Let $\alpha_1, \dots, \alpha_k:G \to Z$ be homomorphisms as in* (7) *and suppose* (9) *is valid. Then $\theta:G \to \bar{T}_{k+1}$ yields a defining system M for $\langle -\alpha_1, \dots, -\alpha_k\rangle$ and*

$$(\langle -\alpha_1, \dots, -\alpha_k\rangle_M, \{W\}) = \sum_J a_J\epsilon_J(W),$$

where the summation is over all sequences J of length k.

Definition. Assuming the hypothesis of (3.5) we define the universal Massey product $\ll -\alpha_1, \dots, -\alpha_k \gg$ by

$$\ll -\alpha_1, \dots, -\alpha_k \gg = \langle -\alpha_1, \dots, -\alpha_k\rangle_M.$$

Then the content of (3.5) is that the universal Massey product is uniquely defined and multilinear in its variables, i.e.,

$$\ll -\alpha_1, \dots, -\alpha_k \gg = \sum_J a_J \ll -u_J \gg.$$

(3.6) COROLLARY. *Suppose $G = \{x_1, \dots, x_n|W\}$, where $W \in F_k$ is not a proper power, $k \geqq 2$. Then $\ll -\alpha_1, \dots, -\alpha_k \gg$ is well defined and is evaluated on $\{W\}$ as in* (3.5).

Next we will show that universal Massey products are natural with respect to degree 1 maps. To explain this let F, F' be free groups on finitely many generators and suppose $W \in F_k$, $W' \in F'_k$ are not proper powers. Let G, G' be the corresponding 1 relator groups and suppose $f:F \to F'$ is a homomorphism satisfying $f(W) = W'$. Then f induces a homomorphism $g:G \to G'$ which is a degree 1 map, i.e.,

$$g_*:H_2G \to H_2G'$$

is an isomorphism. This last statement follows from the Hopf formula.

(3.7) THEOREM. *If* $\alpha'_1, \dots, \alpha'_k \in H^1G'$ *then*

$$g^*\ll\alpha'_1, \dots, \alpha'_k\gg = \ll g^*(\alpha'_1), \dots, g^*(\alpha'_k)\gg.$$

Proof. Included in this statement is the fact that both sides are well defined. The proof is an immediate consequence of (3.6) and the chain rule for Fox derivatives (see (5.6) of **[4]**).

In general Massey products have indeterminacy. However, the next theorem proves that all Massey products of length $\leqq k$ in

$$G = \{x_1, \dots, x_n | W\},$$

where $W \in F_k$ is not a proper power, are defined with zero indeterminacy.

(3.8) THEOREM. *If* $\alpha_1, \dots, \alpha_l \in H^1G$ *and* $l \leqq k$ *then*

$$\langle\alpha_1, \dots, \alpha_l\rangle = \ll\alpha_1, \dots, \alpha_l\gg.$$

Proof. First note that $\ll\alpha_1, \dots, \alpha_l\gg$ is defined and therefore so is $\langle\alpha_1, \dots, \alpha_k\rangle$. It is thus necessary to show that the indeterminacy is zero. But this follows from a result of May (see **[8]** or (2.2) of **[4]**) and induction on l.

4. Massey products in 2-complexes. In this final section we extend the results of the last section to Massey products of one dimensional classes in 2-complexes. Our notation is the same as in Section 2. That is

$$G = \{x_1, \dots, x_n | W_1, \dots, W_p\},$$

X is the 2-complex associated to this presentation and $\tilde{X}$ is its universal covering. We also assume throughout that the relators $W_j \in [F, F]$, $1 \leqq j \leqq p$.

Let $\alpha_1, \dots, \alpha_k:G \to Z$ be elements of H^1G as in (7) of Section 3 and let $\xi:F \to T_{k+1}$ be the homomorphism defined in (8) of Section 3. To insure that ξ induces a homomorphism $\theta:G \to \overline{T}_{k+1}$ we make the assumption $W_j \in F_k$, $1 \leqq j \leqq p$. Of course this is stronger than necessary. Now set $\ll -\alpha_1, \dots, -\alpha_k\gg$ equal to the cohomology class in H^2G which corresponds to θ. Then the chain transformation $T:C_*(\tilde{X}) \rightarrow B_*G$ induces an isomorphism $T^*:H^1G \rightarrow H^1X$ and therefore we may define universal Massey products in X by means of the formula

$$\begin{aligned}&\ll -\beta_1, \dots, -\beta_k \gg \\ &= T^* \ll -\alpha_1, \dots, -\alpha_k \gg \in H^2X, \quad \beta_i = T^*(\alpha_i).\end{aligned}$$

To evaluate this Massey product we note that

$$H^2X \cong \mathrm{Hom}_Z(H_2X, Z)$$

and H_2X is free abelian on the generators

$$1 \otimes_\Lambda e_j, \quad 1 \leqq j \leqq p$$

(see Section 2). By arguments similar to (3.3) we can prove the following theorem:

(4.1) THEOREM. *For* $1 \leqq j \leqq p$ *we have*

$$(\ll -\beta_1, \dots, -\beta_k \gg, 1 \otimes_\Lambda e_j) = \sum_J a_J \epsilon_J (W_j)$$

where the summation is over all J *having length* k.

Finally, in analogy with (3.8) we can prove that Massey products of length $\leqq k$ are defined with zero indeterminacy and given by the universal Massey product.

(4.2) THEOREM. *Let* $\beta_1, \dots, \beta_l \in H^1X$ *be arbitrary*, $l \leqq k$. *Then*

$$\langle -\beta_1, \dots, -\beta_l \rangle = \ll -\beta_1, \dots, -\beta_l \gg.$$

This implies that Massey products of length $<k$ vanish identically and products of length k are computed in terms of Fox derivatives of the relators. Finally we note that the universal Massey product defined in this paper is identical to the minimal Massey product defined in [**4**].

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