ON CLOSURE AND FACTORIZATION PROPERTIES OF SUBEXPONENTIAL AND RELATED DISTRIBUTIONS

PAUL EMBRECHTS and CHARLES M. GOLDIE *

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Abstract

For a distribution function $F$ on $[0, \infty)$ we say $F \in \mathcal{S}$ if $\{1 - F(z^2)\}/(1 - F(z)) \to 2$ as $z \to \infty$, and $F \in \mathcal{S}_\gamma$ if for some fixed $\gamma > 0$, and for each real $y$, $\lim_{z \to \infty} \{1 - F(z+y)\}/(1 - F(z)) = e^{-\gamma}$. Sufficient conditions are given for the statement $F \in \mathcal{S} \Rightarrow F \in \mathcal{S}_\gamma$, and when both $F$ and $G$ are in $\mathcal{S}$ it is proved that $F \ast G \in \mathcal{S} \Rightarrow pF + (1 - p)G \in \mathcal{S}$ for some (all) $p \in (0, 1)$. The related classes $\mathcal{L}$, are proved closed under convolutions, which implies the closure of the class of positive random variables with regularly varying tails under multiplication (of random variables). An example is given that shows $\mathcal{S}$ to be a proper subclass of $\mathcal{L}_0$.

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1. Statement of results and discussion

We consider probability distribution functions $F$ on $[0, \infty)$ satisfying $F(0) = 0$, $F(\infty) = 1$, $F(x) < 1$ for $x < \infty$, and will write $\overline{F}$ for the tail function $1 - F$, similarly $F^{(2)}$ for $1 - F^{(2)}$, $F \ast \overline{G}$ for $1 - F \ast G$, and so on, where $F^{(2)}$ denotes the convolution $F \ast F$.

**Definition.** $F$ and $G$ are said to be *max-sum-equivalent*, written $F \sim_M G$ if

$$F \ast \overline{G}(x) \sim \overline{F}(x) + \overline{G}(x), \quad x \to \infty.$$ (1.1)

Note that $\overline{F}(x) + \overline{G}(x) \sim \overline{F}(x) + \overline{G}(x) - \overline{F}(x) \overline{G}(x)$, so that if $X$ and $Y$ are independent random variables with distribution functions $F$, $G$ respectively, we may rewrite (1.1) as

$$P(X + Y > x) \sim P(\max(X, Y) > x), \quad x \to \infty.$$ 

Hence the terminology.

*On leave from University of Sussex.*
DEFINITION. $F$ is said to belong to the subexponential class $\mathcal{S}$ if $F \sim M F$.

DEFINITION. For each $\gamma \geq 0$, the class $\mathcal{L}_\gamma$ is the set of $F$ for which for each fixed real $y$, \( \lim_{x \to \infty} \frac{F(x+y)}{F(x)} = e^{-\gamma y} \). Write $\mathcal{L}$ for $\mathcal{L}_0$.

The class $\mathcal{S}$ originated in branching processes (Chistyakov (1964)) has found applications in queues, random walks and transient renewal theory, and has recently entered the theory of infinite divisibility, where (Embrechts et al. (1979)) it characterizes those infinitely divisible distributions on $[0, \infty)$ whose tail functions are asymptotically equal to the tails of their Lévy measures. See the last-mentioned article for references to the other applications. Work on applications has led to interest in, and discovery of, many properties of $\mathcal{S}$, but one major conjecture remains unverified, that $\mathcal{S}$ is closed under convolutions: $F \in \mathcal{S}$, $G \in \mathcal{S} \Rightarrow F * G \in \mathcal{S}$. We have not solved this problem, but give some partial results about it, and relate it to the closure of $\mathcal{S}$ under finite mixing. Our results also give some sufficient conditions for the converse proposition, that $F * G \in \mathcal{S}$, $G \in \mathcal{S} \Rightarrow F \in \mathcal{S}$. In this connection it is appropriate to mention that there is no problem when $F = G$, because for any positive integer $n$, $F \in \mathcal{S} \Rightarrow F^{(n)} \in \mathcal{S}$ (see Embrechts et al. (1979), the left-to-right implication being Chistyakov’s). This carries over to when $G(x) \sim cF(x)$ for some $c > 0$, because $\mathcal{S}$ is known to be closed under tail equivalence.

**Theorem 1.** Let $F \in \mathcal{L}$, $G \in \mathcal{S}$, and $\sup_x G(x)/F(x) < \infty$. Then $F \in \mathcal{S} \Rightarrow F * G \in \mathcal{S}$.

A related result is in Pitman (1979): if $F$ and $G$ have densities whose ratio $G'/F'$ is bounded, then $F \in \mathcal{S} \Rightarrow F * G \in \mathcal{S}$.

If the boundedness of $G/F$ in Theorem 1 is strengthened to the requirement $\lim_{x \to \infty} G(x)/F(x) = 0$ then the condition $F \in \mathcal{L}$ can be omitted (Embrechts et al. (1979), Proposition 1). However this is not so in general, for in some circumstances the conclusion $F * G \in \mathcal{S}$ implies $F \in \mathcal{L}$, as follows.

**Proposition 1.** If $\sup_x G(x)/F(x) < \infty$, $F \in \mathcal{S}$, and $F \cdot G \in \mathcal{L}$, then $F \in \mathcal{S}$.

For the next result on convolution closure, $\mathcal{D}$ is to be the class of $F$ with dominated-variation tails: $\sup_x F(\frac{1}{2}x)/F(x) < \infty$.

**Proposition 2.** If $F \in \mathcal{D} \cap \mathcal{S}$ and $G \in \mathcal{D} \cap \mathcal{S}$ then $F * G \in \mathcal{D} \cap \mathcal{S}$.

We now give equivalent forms of convolution closure.

**Theorem 2.** Let $F \in \mathcal{S}$, $G \in \mathcal{S}$ and $H = F * G$. Then $\limsup_{x \to \infty} H^{(2)}(x)/H(x) \leq 4$, and the following are equivalent:

(i) $H \in \mathcal{S}$,
(ii) \( F \sim_M G \),
(iii) \( pF+(1-p)G \in \mathcal{S} \) for some (all) \( p \) satisfying \( 0<p<1 \).

An immediate consequence is that the following three conjectures are equivalent (all true or all false):

(a) \( \mathcal{S} \) is closed under convolutions;
(b) \( \mathcal{S} \) is closed under finite mixtures;
(c) \( F \in \mathcal{S}, \ G \in \mathcal{S} \Rightarrow F \sim_M G \).

As regards finite mixtures, Pitman (1979) proves that under the density condition quoted above, \( F \in \mathcal{S} \) implies \( pF+(1-p)G \in \mathcal{S} \).

We turn to the classes \( \mathcal{L}_\gamma \). It is well known that \( \mathcal{S} \subset \mathcal{L} \), and \( \mathcal{L} \) plays an important role in the study of \( \mathcal{S} \). The classes \( \mathcal{L}_\gamma \) for \( \gamma > 0 \) bear a similar relation to the classes \( \mathcal{S}_\gamma \) of Chover et al. (1973). We prove convolution closure. For this theorem only we drop the restriction that the distribution functions have support in \( [0, \infty) \), that is we no longer insist that \( F(0)=0, \ G(0)=0 \). The definition of \( \mathcal{L}_\gamma \) is unaltered, and \( \gamma \) can be any nonnegative number. Let \( \mathcal{R}_\alpha \) denote the class of functions regularly varying at \( \infty \) with exponent \( \alpha \).

**Theorem 3.** (a) If \( F \in \mathcal{L}_\gamma \) and \( \bar{G} = o(\bar{F}) \), in particular if \( G \in \mathcal{L}_{\gamma'} \) for \( \gamma' > \gamma \), then \( F \ast G \in \mathcal{L}_{\gamma'} \).

(b) If \( F \in \mathcal{L}_\gamma \) and \( G \in \mathcal{L}_\gamma \) then \( F \ast G \in \mathcal{L}_\gamma \).

**Corollary.** Let \( X^* \) and \( Y^* \) be independent positive random variables with distribution functions \( F^*, \ G^* \) respectively, and let \( H^* \) be the distribution function of the product \( X^*Y^* \). Let \( \bar{F}^* \in \mathcal{R}_{-\gamma} \). Then \( \bar{H}^* \in \mathcal{R}_{-\gamma} \) if either (a) \( G^* = o(\bar{F}^*) \), or (b) \( G^* \in \mathcal{R}_{-\gamma} \).

It is of interest to compare Theorem 3 with the corresponding well-knowns result in which \( \mathcal{L}_\gamma, \mathcal{L}_{\gamma'} \) are replaced by \( \mathcal{R}_{-\gamma}, \mathcal{R}_{-\gamma'} \). In that case one concludes not just that \( F \ast G \in \mathcal{R}_{-\gamma} \) but also that \( F \ast G \sim \bar{F} + \bar{G} \). However in the circumstances of Theorem 3 one does not have that sort of extra information, because if for instance \( G = F \) and \( \gamma = 0 \), then the conclusion \( F \ast G \sim \bar{F} + \bar{G} \) would mean that \( F \in \mathcal{S} \), and we know (see below) that \( \mathcal{S} \) is a proper subclass of \( \mathcal{L}_0 \).

The classes \( \mathcal{R}_{-\gamma}, \mathcal{S}, \mathcal{D} \) are closed under convolution roots (see Embrechts et al. (1979) for the first two; the case of \( \mathcal{D} \) is elementary). We conjecture that for each \( \gamma \geq 0 \), \( \mathcal{L}_\gamma \) is closed under convolution roots: \( F^{(b)} \in \mathcal{L}_\gamma \Rightarrow F \in \mathcal{L}_\gamma \).

The corollary may be related to domains of attraction. First, \( \mathcal{R}_{-\gamma} \) for \( \gamma > 0 \) is the class of tail functions of the domain of attraction for maxima of Gnedenko's canonical law \( \Phi_\gamma \). Second, \( \mathcal{R}_{-\gamma} \) for \( 0 < \gamma < 1 \) is the class of tail functions of nonnegative random variables in the domain of attraction for sums of a completely asymmetric stable law of exponent \( \gamma \). Thus the corollary asserts the closure of
these domains of attraction under \textit{multiplication} by certain random variables, or (the same thing), under the taking of certain scale mixtures.

One may without difficulty relax the positivity of \(X^*\) and \(Y^*\) required by the corollary, and instead allow \(Y^*\) to take any real values and \(X^*\) to be nonnegative. The conclusion then applies to the upper tail of \(X^*Y^*\). If the same condition as on \(G^*\) is imposed on \(G^*(-x)\) as \(x \to \infty\), then one also obtains a lower-tail conclusion. However, there is not necessarily any balance between the tails, so the extended corollary is not a domain-of-attraction theorem. Under more stringent conditions one can balance the tails; see Breiman (1965), Proposition 3.

Another view of the corollary is obtained by realizing that \(\overline{H^*}\) is the Mellin–Stieltjes convolution of \(G^*\) and \(F^*\):

\[
\overline{H^*}(x) = \int_0^\infty \overline{G^*}(x/t) F^*(dt).
\]

The corollary is thus an Abelian theorem for Mellin–Stieltjes convolutions, of a rather different sort, it appears, from those mainly studied (see Bingham and Teugels (1979)).

Our final result, forming Section 3, is an example of a distribution function in \(\mathscr{L}\setminus\mathscr{S}\). We found this example last year, but it then seemed over-complicated in comparison with \(F_\beta(x) = \exp\{-x(\log x)^{-\beta}\}\), \(x \geqslant 1\), which was thought to be an element of \(\mathscr{L}\setminus\mathscr{S}\) when \(0 < \beta \leqslant 1\). However Pitman (1979) shows that in fact \(F_\beta \in \mathscr{S}\) for each \(\beta > 0\). Pitman also gives a new example in \(\mathscr{L}\setminus\mathscr{S}\), and his example and ours are therefore the only known elements of that class. Neither is particularly simple.

2. Proofs

Denote \(\max(x,y)\) by \(x \vee y\) and \(\min(x,y)\) by \(x \wedge y\). The operators \(\vee, \wedge\) are to bind more tightly than \(+, \,-\). Intervals of integration will include \{exclude\} finite right \{left\} endpoints, unless otherwise indicated. We always write \(H\) for \(F^*G\) and \(K\) for \(F \cdot G\). Independent random variables \(X, X_1, X_2, Y, Y_1, Y_2\) will be used to aid explanations, the \(X\)'s each having distribution function \(F\), the \(Y\)'s each having distribution function \(G\). Note that

\[
K(x) = P(X \vee Y > x) = \overline{F}(x) + \overline{G}(x) - \overline{F}(x) \overline{G}(x) \sim \overline{F}(x) + \overline{G}(x), \quad x \to \infty.
\]

\textbf{Lemma 1.} If \(K \in \mathscr{L}\) and \(H \in \mathscr{S}\) then \(F \sim M G\).

\textbf{Proof.} Define \(\bar{X} = X_1 \vee X_2\), \(\bar{Y} = Y_1 \vee Y_2\), and denote their distribution functions by \(F_2, G_2\) respectively, so that

\[
F_2 = (2 - \overline{F}) \overline{F}, \quad G_2 = (2 - \overline{G}) \overline{G}.
\]
Fix $w > 0$ and $x \geq 2w$, then

\[(2.2) \quad H(x) = P(X \leq w, X + Y > x) + P(Y \leq w, X + Y > x) + P(X > w, Y > w, X + Y > x) \]

\[\leq P(Y > x - w) + P(X > x - w) + P(X > w, Y > w, X + Y > x)\]

Next,

\[(2.3) \quad H^{(2)}(x) = P(X + Y, X + Y > x) \]

\[\geq P(\hat{X} + \hat{Y} > x)\]

\[= P(\hat{X} \leq w, \hat{X} + \hat{Y} > x) + P(\hat{Y} \leq w, \hat{X} + \hat{Y} > x) + P(\hat{X} > w, \hat{Y} > w, \hat{X} + \hat{Y} > x)\]

\[= F_2(w)G_2(x) + G_2(w)F_2(x) + \int_{w}^{\infty} G_2((x - u) \vee w) F_2(du),\]

and then, using (2.1),

\[(2.4) \quad \int_{w}^{\infty} G_2((x - u) \vee w) F_2(du) \geq \{2 - G(w)\} \int_{w}^{\infty} G((x - u) \vee w) F_2(du)\]

\[= \{2 - G(w)\} P(\hat{X} > w, Y > w, \hat{X} + Y > x)\]

\[= \{2 - G(w)\} \int_{w}^{\infty} F_2((x - u) \vee w) G(du)\]

\[\geq \{2 - G(w)\} \{2 - F(w)\} \int_{w}^{\infty} F((x - u) \vee w) G(du)\]

\[= \{2 - F(w)\} \{2 - G(w)\} P(X > w, Y > w, X + Y > x)\]

Combining (2.2), (2.3) and (2.4),

\[H^{(2)}(x) \geq F_2(w)G_2(x) + G_2(w)F_2(x)\]

\[+ \{2 - F(w)\} \{2 - G(w)\} \{H(x) - F(x - w) - G(x - w)\}\]

\[\geq \{F_2(w)(2 - G(x))\} \land \{G_2(w)(2 - F(x))\} \{F(x) + G(x)\}\]

\[+ \{2 - F(w)\} \{2 - G(w)\} H(x) - 4\{F(x - w) + G(x - w)\},\]

using (2.1). Rearranging,

\[\{2 - F(w)\} \{2 - G(w)\} - H^{(2)}(x)/H(x) \leq 4\{F(x - w) + G(x - w)\}\]

\[\leq \{F_2(w)(2 - G(x))\} \land \{G_2(w)(2 - F(x))\} \{F(x) + G(x)\}.\]

Divide through by $K(x)$ and let $x \to \infty$. We have

\[F(x - w) + G(x - w) \sim K(x - w) \sim K(x) \quad \text{since } K \in \mathcal{L}.\]
Also $\bar{H}^{(2)}(x)/\bar{H}(x) \to 2$ by our assumption that $H \in \mathcal{S}$. Hence
\[\left\{ 2 - \bar{F}(w) \right\} \left\{ 2 - \bar{G}(w) \right\} - 2 \limsup_{x \to \infty} \bar{H}(x)/\bar{K}(x) \leq 4 - 2\{F_2(w) \wedge G_2(w)\} .\]

Now let $w \to \infty$, and then $\limsup_{x \to \infty} \bar{H}(x)/\bar{K}(x) \leq 1$. However $\bar{H} \geq \bar{K}$ and so the lemma is proved.

**Lemma 2.** If $G \in \mathcal{L}$, $F \in \mathcal{S}$ and $c = \sup_x \bar{G}(x)/\bar{F}(x) < \infty$, then $F \sim_M G$.

**Proof.** For any fixed $w > 0$,
\[\int_w^\infty \left\{ \bar{F}(x-t)/\bar{F}(x) \right\} F(dt) = \bar{F}^{(2)}(x) - 1 - \int_0^w \left\{ \bar{F}(x-t)/\bar{F}(x) \right\} F(dt) \to -2 - 1 - F(w), \quad x \to \infty ,\]

the terms on the right converging by subexponentiality and dominated convergence, respectively. Then
\[
F \ast \bar{G}(x) = \bar{F}(x) + \int_0^w \bar{G}(x-t) F(dt) + \int_w^\infty \bar{G}(x-t) F(dt) \\
\leq \bar{F}(x) + \bar{G}(x-w) + \bar{F}(x) \cdot c \int_w^\infty \left\{ \bar{F}(x-t)/\bar{F}(x) \right\} F(dt) \\
\leq \left\{ \bar{F}(x) + \bar{G}(x) \right\} \cdot \left[ 1 + c \int_w^\infty \left\{ \bar{F}(x-t)/\bar{F}(x) \right\} F(dt) \right] \vee \{ \bar{G}(x-w)/\bar{G}(x) \} .
\]

Consequently
\[
\limsup_{x \to \infty} \frac{F \ast \bar{G}(x)}{\{ \bar{F}(x) + \bar{G}(x) \}} \leq 1 + c \bar{F}(w) .
\]

Let $w \to \infty$, then $\limsup_{x \to \infty} \frac{F \ast \bar{G}(x)}{\{ \bar{F}(x) + \bar{G}(x) \}} \leq 1$ and the lemma follows.

**Proof of Theorem 1.** (a) Suppose
\[F \in \mathcal{L}, \quad G \in \mathcal{L}, \quad H \in \mathcal{S} \quad \text{and} \quad c = \sup_x \bar{G}(x)/\bar{F}(x) < \infty .\]

By Lemma 1, $\bar{H}(x) \sim \bar{F}(x) + \bar{G}(x)$. Since $H \in \mathcal{S}$ we have $H^{(2)} \in \mathcal{S}$; also $F^{(2)} \in \mathcal{L}$ and $G^{(2)} \in \mathcal{L}$ by Theorem 3, and so the distribution function of
\[X_1 + X_2 \vee (Y_1 + Y_2) ,\]

whose tail is asymptotically equal to $\bar{F}^{(2)} + \bar{G}^{(2)}$, is also in $\mathcal{L}$. So Lemma 1 applies to $F^{(2)}$, $G^{(2)}$ and $H^{(2)} = F^{(2)} * G^{(2)}$, giving $\bar{H}^{(2)}(x) \sim \bar{F}^{(2)}(x) + \bar{G}^{(2)}(x)$. Since $\bar{H} \sim 2 \bar{H}$ it follows that
\[\{ \bar{F}^{(2)}(x) + \bar{G}^{(2)}(x) \}/\{ \bar{F}(x) + \bar{G}(x) \} \to 2, \quad x \to \infty .\]
So for all large $x$ the left side is at most $2+\varepsilon$, and hence
\[ F^{(2)}(x)/F(x) \leq 2+\varepsilon + \{2+\varepsilon - G^{(2)}(x)/G(x)\} G(x)/F(x). \]
Since $\liminf_x G^{(2)}(x)/G(x) \geq 2$ (Chistyakov (1964)) we have
\[ \limsup_x F^{(2)}(x)/F(x) \leq 2+\varepsilon + (2+\varepsilon - 2)c. \]
Hence this lim sup is at most 2, whence $F \in \mathcal{S}$.

(b) Suppose $F \in \mathcal{S}$, $G \in \mathcal{S}$ and $\sup_x G(x)/F(x) < \infty$. Now $F^{(2)} \in \mathcal{S}$, $G^{(2)} \in \mathcal{S}$, and $F^{(2)} \sim 2F$, $G^{(2)} \sim 2G$, whence $\sup_x G^{(2)}(x)/F^{(2)}(x) < \infty$. By Lemma 2, $F^{(2)} \sim M G^{(2)}$. Then
\[ \limsup_{x \to \infty} \frac{H^{(2)}(x)}{H(x)} = \limsup_{x \to \infty} \frac{\{F^{(2)}(x) + G^{(2)}(x)\}}{\{F(x) + G(x)\}} \leq \limsup_{x \to \infty} \frac{\{F^{(2)}(x) + G^{(2)}(x)\}}{\{F(x) + G(x)\}} = 2 \]
since $F \in \mathcal{S}$, $G \in \mathcal{S}$. Thus $H \in \mathcal{S}$, concluding the proof.

Note. In (a), that is, the ‘$\leq$’ statement of Theorem 1, we used only $G \in \mathcal{S}$ rather than the stronger assumption $G \in \mathcal{S}$.

Proof of Proposition 1. By Lemma 1, $F \sim M G$. Suppose $F \notin \mathcal{S}$, then for some $t_0 > 0$ there exists $x_n \to \infty$ such that $\frac{F(x_n - t_0)}{F(x_n)} \to l > 1$ as $n \to \infty$. Passing to a subsequence, we may take it that also $\frac{G(x_n)}{F(x_n)} \to c < \infty$. Define
\[ \mu(t) = \liminf_n \frac{F(x_n - t)}{F(x_n)} \geq 1, \]
then $\mu(t) \geq l$ for all $t \geq t_0$, so $\int \mu(t) G(dt) > 1$. We have
\[ H(x_n) = G(x_n) + F(x_n) \int_{x_0}^{x_n} \{\frac{F(x_n - t)}{F(x_n)}\} G(dt) \]
whence
\[ \liminf_n \frac{H(x_n)}{\{F(x_n) + G(x_n)\}} = c/(1+c) + (1+c)^{-1} \liminf_n \int_{x_0}^{x_n} \{\frac{F(x_n - t)}{F(x_n)}\} G(dt) \]
and by Fatou’s lemma the right side is at least
\[ c/(1+c) + (1+c)^{-1} \int \mu(t) G(dt) > 1. \]
This contradicts $F \sim M G$. So it must be the case that $F \in \mathcal{S}$, proving the proposition.
PROOF OF PROPOSITION 2. First,

\[ \frac{P(X + Y > x)}{P(X + Y > 2x)} \leq \frac{P(X > \frac{1}{2}x)}{P(X > 2x)} + \frac{P(Y > \frac{1}{2}x)}{P(Y > 2x)} \]

which is bounded above as \( x \to \infty \), and so \( H \in \mathcal{D} \). Therefore to show \( H \in \mathcal{E} \) we have only, by Goldie (1978), Theorem 1, to show \( H \in \mathcal{E} \). Now

\[ (2.5) \quad H(x) = P(X + Y > x, X < \frac{1}{2}x)P(X + Y > x, Y < \frac{1}{2}x) + P(X > \frac{1}{2}x)P(Y > \frac{1}{2}x). \]

Taking the last two terms on the right,

\[ (2.6) \quad \left\{ \frac{P(X + Y > x, Y < \frac{1}{2}x) + P(X > \frac{1}{2}x)P(Y > \frac{1}{2}x)}{F(x)} \right\}\]

and the integrand is bounded by a constant and converges pointwise to 1, so that the integral of (2.6) converges to 1. The other term on the right of (2.6) is \( O(1) \). Thus the right side of (2.6) tends to 1. Similarly,

\[ P(X + Y > x, X > \frac{1}{2}x)/G(x) \to 1. \]

Returning to (2.5), we conclude \( \overline{H}(x) \sim \overline{F}(x) + \overline{G}(x) \). Since \( F \in \mathcal{L} \) and \( G \in \mathcal{L} \), the conclusion \( H \in \mathcal{L} \) follows easily, which is enough to complete the proof as indicated earlier.

PROOF OF THEOREM 2. Assume \( F \in \mathcal{E} \) and \( G \in \mathcal{E} \). We prove

\[ (2.7) \quad \frac{H^{(2)}(x)}{H(x)} = 4 - 2\left\{ \frac{F(x) + G(x)}{H(x)} + o(1) \right\}, \quad x \to \infty. \]

Thus, let \( \overline{f}(y) = \sup \{ F^{(2)}(x)/F(x) ; x \geq y \} \), \( \overline{f}(y) = \inf \{ F^{(2)}(x)/F(x) ; x \geq y \} \), and define \( \overline{g}, \overline{g} \) similarly in terms of \( G \). For brevity, write \( X^{(2)} \) for \( X_1 + X_2 \), and \( Y^{(2)} \) for \( Y_1 + Y_2 \). Fix \( w > 0 \), then for \( x \geq 2w \),

\[ (2.8) \quad \overline{H}^{(2)}(x) = P(X^{(2)} \leq w, X^{(2)} + Y^{(2)} > x) + P(Y^{(2)} \leq w, X^{(2)} + Y^{(2)} > x) + P(X^{(2)} > w, Y^{(2)} > w, X^{(2)} + Y^{(2)} > x). \]

The sum of the first two terms on the right is at least

\[ \overline{G}^{(2)}(x) F^{(2)}(w) + \overline{F}^{(2)}(x) G^{(2)}(w), \]

which is bounded below by

\[ g(x) \overline{G}(x) F^{(2)}(w) + \overline{f}(x) \overline{F}(x) G^{(2)}(w). \]

Similarly, an upper bound for the same quantity is

\[ \overline{g}(x-w) \overline{G}(x-w) F^{(2)}(w) + \overline{f}(x-w) \overline{F}(x-w) G^{(2)}(w). \]
The third term on the right of (2.8) is equal to
\[
\int_{w}^{\infty} G^{(2)}((x-u) \lor w) F^{(2)}(du) \leq \bar{g}(w) \int_{w}^{\infty} G((x-u) \lor w) F^{(2)}(du)
\]
\[
= \bar{g}(w) P(X^{(2)} > w, Y > w, X^{(2)} + Y > x)
\]
\[
= \bar{g}(w) \int_{w}^{\infty} F^{(2)}((x-u) \lor w) G(du)
\]
\[
\leq \bar{g}(w) f(w) \int_{w}^{\infty} F((x-u) \lor w) G(du)
\]
\[
= f(w) \bar{g}(w) \left\{ H(x) - \int_{0}^{w} G(x-u) F(du) \right\}
\]
\[
- \int_{0}^{w} F(x-u) G(du)
\]
\[
\leq f(w) \bar{g}(w) \{ H(x) - G(x) F(w) - F(x) G(w) \}.
\]
Similarly, a lower bound for the third term on the right of (2.8) is
\[
f(w) g(w) \{ H(x) - \bar{G}(x-w) F(w) - \bar{F}(x-w) G(w) \}.
\]
Replacing the right side of (2.8) by its upper bounds we calculate
\[
H^{(2)}(x) + 2F(x) + 2\bar{G}(x) \leq f(w) \bar{g}(w) H(x)
\]
\[
+ [2 - f(w) \bar{g}(w) \{ F(w) \land G(w) \}] \{ F(x) + \bar{G}(x) \}
\]
\[
+ [[\bar{g}(x-w) F^{(2)}(w) \lor \{ f(x-w) G^{(2)}(w) \}] \{ F(x-w) + \bar{G}(x-w) \}.
\]
Divide by \( H(x) \) and take \( \limsup_{x \to \infty} \). We know \( F \in \mathcal{L}, G \in \mathcal{L} \), so that
\[
F(x-w) + \bar{G}(x-w) \sim F(x) + \bar{G}(x).
\]
Further, \( \limsup_{x \to \infty} (F + G)/H \leq 1 \). Hence
\[
\limsup_{x \to \infty} \{ H^{(2)}(x) + 2F(x) + 2\bar{G}(x) \}/H(x)
\]
\[
\leq f(w) \bar{g}(w) + |2 - f(w) \bar{g}(w) \{ F(w) \land G(w) \} + 2\{ F^{(2)}(w) \lor G^{(2)}(w) \}| \rightarrow 4 + 0, \quad w \to \infty.
\]
Similarly, using the lower bounds, the \( \liminf \) is at least 4. So (2.7) is proved. This immediately gives the first conclusion of the theorem, and also \( H \in \mathcal{S} \Leftrightarrow H \sim F + G \), that is, (i) \( \Leftrightarrow \) (ii). To complete the theorem we show that if \( F \sim M G \) then \( pF + qG \in \mathcal{S} \) for all \( p \), \( 0 < p < 1 \), \( q = 1 - p \), and conversely if \( pF + qG \in \mathcal{S} \) for some \( p \in (0, 1) \) then \( F \sim M G \).
First assume \( F \sim_M G \), then
\[
\frac{(pF+qG)^{(2)}}{(pF+qG)} = \frac{\{p^2 F^{(2)} + 2pqF*G + q^2 G^{(2)}\}}{(pF+qG)}
\sim \frac{\{p^2 F^{(2)} + 2pq(F+G) + q^2 G^{(2)}\}}{(pF+qG)}
\sim \frac{\{2p^2 F + 2pq(F+G) + 2q^2 G\}}{(pF+qG)}
= 2.
\]

Thus \( pF+qG \in \mathcal{S} \).

For the converse, suppose it is not true that \( F \sim_M G \), so that there exists \( \varepsilon > 0 \) such that on an unbounded set of \( x \), \( F*G \geq (1+\varepsilon)(F+G) \). Then
\[
\limsup_{x \to \infty} \frac{(pF+qG)^{(2)}}{(pF+qG)} = \limsup_{x \to \infty} \frac{\{p^2 F^{(2)} + 2pqF*G + q^2 G^{(2)}\}}{(pF+qG)}
\geq \limsup_{x \to \infty} \frac{\{2p^2 F + 2pq(F+G) + 2q^2 G\}}{(pF+qG)}
= 2 + 2pq \varepsilon \limsup(F+G)/(pF+qG) > 2.
\]
and so \( pF+qG \notin \mathcal{S} \). Theorem 2 is proved.

**Proof of Theorem 3.** (a) Assume \( F \in \mathcal{L} \) and \( \lim_{x \to \infty} \frac{G(x)}{F(x)} = 0 \). Fix \( t > 0 \) and \( v > t \). We have
\[
\bar{H}(x) = P(X+Y>x) = P(X+Y>x, Y \leq x-v) + P(X+Y>x, Y>x-v)
\]
and
\[
0 \leq P(X+Y>x, Y>x-v)/P(X+Y>x, Y \leq x-v)
\leq P(Y>x-v)/P(Y>x, 0 < Y \leq x-v)
= \bar{G}(x-v)/[\bar{F}(x)\{G(x-v)-G(0)\}]
= \{G(x-v)/\bar{F}(x-v)\} \{\bar{F}(x-v)/\bar{F}(x)\} \{G(x-v)-G(0)\}^{-1}
\to 0. e^{\nu \varepsilon}. 1/\bar{G}(0) = 0.
\]
Thus
\[
(2.9) \quad \bar{H}(x) \sim \int_{-\infty}^{x-v} \bar{F}(x-u) G(du), \quad x \to \infty.
\]

Altering \( v \) to \( v-t \) and \( x \) to \( x-t \),
\[
(2.10) \quad \bar{H}(x-t) \sim \int_{-\infty}^{x-v} \{\bar{F}(x-u-t)/\bar{F}(x-u)\} \bar{F}(x-u) G(du).
\]
Defining

\[ M_F(y) = \sup \{ \frac{F(x-t)}{F(x)}, x \geq y \}, \quad m_F(y) = \inf \{ \frac{F(x-t)}{F(x)}, x \geq y \}, \]

the right side of (2.10) is between

\[ m_F(v) \int_{-\infty}^{v-x} \frac{F(x-u)}{G(du)} \quad \text{and} \quad M_F(v) \int_{-\infty}^{v-x} \frac{F(x-v)}{G(du)}. \]

Hence, using (2.9)

\[ m_F(v) \leq \liminf_{x \to \infty} \frac{H(x-t)}{H(x)} \leq \limsup_{x \to \infty} \frac{H(x-t)}{H(x)} \leq M_F(v). \]

Let \( v \to \infty \) and (a) follows.

(b) Assume \( F \in \mathcal{L}_\gamma \) and \( G \in \mathcal{L}_\gamma \). Define \( m_G \) and \( M_G \) for \( G \), exactly like \( m_F \) and \( M_F \) for \( F \). This time

(2.11) \[ H(x) = \int_{-\infty}^{x-v} \frac{F(x-u)}{G(du)} + \int_{-\infty}^{x-v} \frac{G(x-u)}{F(du)} \frac{F(x-v)}{G(x-v)}, \]

and altering \( x \) to \( x-t \) and \( v \) to \( v-t \),

(2.12) \[ H(x-t) = \int_{-\infty}^{x-v} \{ \frac{F(x-u-t)}{F(x-u)} \} \frac{F(x-u)}{G(du)} + \int_{-\infty}^{v-t} \{ \frac{G(x-u-t)}{G(x-u)} \} \frac{G(x-u)}{F(du)} \]

\[ + \{ \frac{F(v-t)}{F(v)} \} \frac{F(x-v)}{G(x-v)} \]

\[ \leq M_F(v) \int_{-\infty}^{x-v} \frac{F(x-u)}{G(du)} + M_G(x-v+t) \int_{-\infty}^{x-v} \frac{G(x-u)}{F(du)} \]

\[ + M_F(v) \frac{F(v)}{G(x-v)} \]

\[ \leq \{ M_F(v) + M_G(x-v+t) \} \frac{H(x)}{v-t}, \quad v \to \infty. \]

hence

\[ \limsup_{x \to \infty} \frac{H(x-t)}{H(x)} \leq M_F(v) \to e^\gamma, \quad v \to \infty. \]

Now

\[ \liminf_{x \to \infty} \frac{H(x)}{\int_{v-t}^{v} \frac{G(x-u)}{F(du)}} \]

\[ \geq \liminf_{x \to \infty} \int_{0}^{v} \frac{G(x-u)}{F(du)} \frac{F(v-t)}{\{ G(x-v+t) F(v-t) \}}, \]
which by Fatou's lemma is at least
\[
\{F(v-t)\}^{-1} \int_0^v \liminf_{x \to -\infty} \{G(x-u)/G(x-v+t)\} F(du) \leq \int_0^v e^{ru} F(du)/\{e^{r(v-t)} F(v-t)\}.
\]
As \(v \to \infty\) the last term tends to \(\infty\), because for \(v \geq nt\),
\[
\int_0^v e^{ru} F(du)/\{e^{r(v-t)} F(v-t)\} \geq \sum_{k=1}^n e^{r(v-kt)} \{F(v-kt) - F(v-(k-1)t)\}/\{e^{r(v-t)} F(v-t)\} \\
\to \sum_{k=1}^n e^{-(k-1)rt} (e^{r(k-1)x} - e^{r(k-2)x}) \\
= n(1 - e^{-r}).
\]
Thus by taking \(v\) sufficiently large we may ensure that
\[
\limsup_{x \to -\infty} \int_{v-t}^v G(x-u) F(du)/H(x) \leq \frac{1}{2} \varepsilon.
\]
For this \(v\) it will then be true for all large \(x\) that, from (2.11),
\[
\int_{-\infty}^{x-t} F(x-u) G(du) + \int_{-\infty}^{v-t} G(x-u) F(du) + F(v) G(x-v) \geq H(x)(1 - \varepsilon).
\]
Then from (2.12),
\[
H(x-t) \geq m_F(v) \int_{-\infty}^{x-v} F(x-u) G(du) + m_G(x-v+t) \int_{-\infty}^{v-t} G(x-u) F(du) \\
\quad + m_F(v) F(v) G(x-v) \\
\geq \{m_F(v) \wedge m_G(x-v+t)\} H(x)(1 - \varepsilon).
\]
Hence
\[
\liminf_{x \to -\infty} H(x-t)/H(x) \geq m_F(v)(1 - \varepsilon) \to e^r(1 - \varepsilon), \quad v \to \infty.
\]
Since \(\varepsilon > 0\) was arbitrary the above lim inf is at least \(e^r\), and \((b)\) is proved.

**Proof of Corollary.** Set \(X^* = e^X, \quad Y^* = e^Y\), so that \(F^*(x) = F(\log x), \quad G^*(x) = G(\log x)\), and the corollary becomes just a re-statement of the theorem.

3. A distribution function in \(\mathcal{L}\setminus\mathcal{I}\)

Let \(a_n\) be a sequence of positive numbers satisfying \(a_n \to \infty, \quad a_n < \frac{1}{2}(n+1)!\). We
specify the sequence fully later. Define $F$ by its tail function:

$$F(x) = 1, \quad -\infty < x \leq 2,$$

$$= 1/(n+1)!, \quad (n+1)! + na_n \leq x \leq (n+2)!, \quad n = 1, 2, \ldots,$$

$$F((n+1)! + u) = (1 + n - u/a_n)/(n+1)!, \quad 0 \leq u \leq na_n, \quad n = 1, 2, \ldots.$$

The idea is that $F$ decreases to zero by a sequence of slowly-flattening linear slopes, with increasingly long flats in between. We first show $F \in \mathcal{S}$. Fix $t$ and let $n$ be large enough for $na_n > t$. Then within the interval

$$(n+1)! \leq x \leq (n+2)!,$$

$F(x-t)/F(x)$ will be greatest in the sub-interval

$$(n+1)! + t \leq x \leq (n+1)! + na_n.$$

For such $x = (n+1)! + u$,

$$F(x-t)/F(x) = (1 + n - (u-t)/a_n)/(1 + n - u/a_n)$$

$$= 1 + (t/a_n)/(n+1 - u/a_n) \leq 1 + t/a_n \to 1.$$

Thus $F \in \mathcal{S}$.

Now we show $F \notin \mathcal{S}$. Let $X, X'$ be independent, each with distribution function $F$. Then for $0 < u < na_n$, using the fact that $2\{n! + (n-1)a_n\} < (n+1)!$,

$$F^{(2)}((n+1)! + u) = P(Y > (n+1)! + u) + P(Y' > (n+1)! + u) + P((n+1)! \leq Y$$

$$< (n+1)! + u < Y + Y') + P((n+1)! \leq Y' \leq (n+1)! + u$$

$$< Y + Y' - P(Y > (n+1)!)) P(Y' > (n+1)!))$$

$$= 2F((n+1)! + u) + \{2a_n^{-1}/(n+1)\} \int_0^u F(u-x) \, dx$$

$$- \{F((n+1)!)\}^2.$$ 

So

$$F^{(2)}((n+1)! + u)/F((n+1)! + u) = 2 + 2 \int_0^u F(x) \, dx/(a_n + na_n - u) + o(1).$$

Therefore

$$(3.1) \quad F^{(2)}(b_n)/F(b_n) = 2 + (2/a_n) \int_0^{na_n} F(x) \, dx + o(1),$$

where $b_n = (n+1)! + na_n$. To prevent $F \in \mathcal{S}$ we must ensure the right side of (3.1) does not converge to 2, and so intuitively $a_n$ must tend to $\infty$ very slowly. Set
\[ k_n = \max \{ k : (k+1)! \leq n a_n \}, \text{ then} \]

\[
F^{(2)}(b_n)/F(b_n) \geq 2 + (2/a_n) \sum_{k=1}^{k_n} \frac{(k+1)!}{k!} \int_k^{(k+1)!} F(x) \, dx
\]

\[
\geq 2 + (2/a_n) \sum_{k=1}^{k_n} \left\{(k+1)! - k! \right\}/k!
\]

\[
= 2 + (2/a_n) \sum_{k=1}^{k_n} k > 2 + k^2/a_n.
\]

So \( F \notin \mathcal{S} \) if \( k^2/a_n \to 0 \). This can be achieved by setting \( a_n = \max \{ k : (k+1)! \leq n \} \). For then \( a_n \to \infty \), and on comparing the definitions of \( k_n \) and \( a_n \) we see that \( k_n > a_n \), which suffices.

References


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Departement Wiskunde
Katholieke Universiteit Leuven
Celestijnenlaan 200B, B-3030 Heverlee
Belgium

Department of Mathematics
Westfield College
University of London
Kidderpore Avenue
London NW3 7ST
United Kingdom