This department welcomes short notes and problems believed to be new. Contributors should include solutions where known, or background material in case the problem is unsolved. Send all communications concerning this department to I. G. Connell, Department of Mathematics, McGill University, Montreal, P. Q.

ON ABEL'S BINOMIAL IDENTITY

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The identity of Abel [1] [2] we deal with here can be stated in the following form: If n is a positive integer,

$$(a+b)^n = a \sum_{i=0}^{n} \binom{n}{i} (a+i)^{i-1} (b-i)^{n-i} .$$

(In order that all terms be defined we require $a \neq 0$, $b \neq n$.)

This identity and deductions from it have been very useful in many problems, for instance in mathematical statistics [3]. Usually this identity is established by means of the Lagrange-Bürman theorem [4]. Here we will derive it very simply.

Clearly, $\frac{1}{a}(a+b)^n$ can always be written as

$$\frac{1}{a}(a+b)^n = \sum_{i=0}^{n} c_{n-i}(b-i)^{n-i} ,$$

where the c's are independent of b and $c_n = \frac{1}{a}$. To determine the c's conveniently, we employ a little device. Changing b to $b+n+1$ and then multiplying by b we have

$$\frac{b}{a}(a+b+n+1)^n = \sum_{i=0}^{n} bc_{n-i}(b+n+1-i)^{n-i}$$

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$$= \sum_{i=0}^{n} bc_i (b + i + 1)^i .$$

Differentiating (with respect to b) both sides r times (where $1 \leq r \leq n + 1$), we obtain

$$\frac{1}{a} \frac{n!}{(n-r+1)!} ((n+1)(b+r) + ar)(a+b+n+1)^{n-r}$$

$$= r!\, c_{r-1} + \sum_{i=r}^{n} \frac{i!\,(b+i+1)^{i-r} c_i (i+1)(b+r)}{(i-r+1)!} .$$

Putting $b = -r$ in this relation we deduce that

$$\frac{n!}{(n-r+1)!\,(r-1)!} (a+n-r+1)^{n-r} = c_{r-1}, \quad 1 \leq r \leq n+1$$

i.e., $\binom{n}{i} (a+i)^{i-1} = c_{n-i}, \quad 0 \leq i \leq n$.

This completes the proof of the identity.

We remark that any polynomial in a, b can be expressed in the form $\Sigma c_i (a+i)^i$ by the above method. One first changes a to a+1, multiplies by a, differentiates r times, and then puts $a = -r$ to get c_{r-1}.

We also note that another form of Abel's identity is

$$(\alpha + \gamma + n\beta)^n = \sum_{k=0}^{n} \binom{n}{k} (\alpha + \beta k)^k \gamma(\gamma + \beta(n-k))^{n-k-1} .$$

To obtain this put $a = \gamma/\beta$, $\gamma\beta \neq 0$, $b = n + \alpha/\beta$ in the identity given at the beginning. Then, after a little simplification, we get

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$$(\alpha + \gamma + n\beta)^n = \gamma \sum_{i=0}^{n} \binom{n}{i} (\gamma + \beta i)^{i-1} (\alpha + (n-i)\beta)^{n-i}$$

$$= \gamma \sum_{k=0}^{n} \binom{n}{k} (\alpha + \beta k)^k (\gamma + \beta(n-k))^{n-k-1} .$$

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