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AN ABSTRACT VERSION OF A RESULT OF FONG AND SUCHESTON

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Nagel [3] has given a purely functional-analytic proof of Akcoglu and Sucheston's operator version [1] of the Blum-Hanson theorem. The purpose of this note is to show that the same techniques may be applied to obtain a proof, in the context of (AL)-spaces, of a more general result due to Fong and Sucheston [2]. By Kakutani's representation theorem, any (AL)-space can of course be represented as an L^1-space. Thus the present result is simply a reformulation of that of Fong and Sucheston.

A matrix (a_{ni}) is *uniformly regular* (UR) if the following conditions hold:

(i) $M = \sup_n \sum_i |a_{ni}| < \infty$

(ii) $\lim_n \sup_i |a_{ni}| = 0$

(iii) $\lim_n \sum_i a_{ni} = 1$.

Let T be a contraction on a Banach space E, and consider the statements

(A) (T^n) converges in the weak operator topology on E

(B) For each (UR)-matrix, $A_n = \sum_{i=1}^n a_{ni} T^i$ converges in the strong operator topology on E.

It was proved in [2] that (B)$\Rightarrow$(A) for any Banach space E. The converse implication was proved in [2] when E is $L^1(\mu)$ or $L^2(\mu)$. We now show how this implication may be obtained directly when E is an (AL)-space. (For terminology see [4]).

Let S be an order contraction ("strong contraction" in [3]) on a Banach lattice F with order-continuous norm, and quasi-interior point u in F_+. Then the principal ideal $F_u = \bigcup_{n \in N} n[-u, u]$ is dense in F and the norms p_u, p_μ defined in [3, ex. 7] give rise to the diagram

$$L^\infty(X, \mu) \cong C(X) \cong F_u \to F \to L^1(X, \mu), \qquad F_u \to L^2(X, \mu) \to L^1(X, \mu)$$

where the strong and weak topologies induced by F and $L^2(X, \mu)$ coincide on $\overline{co}(S)$. Since $1/M \sum_{i=1}^n a_{ni} S^i$ lies in the closed convex circled hull of (S^n), we can conclude from the L^2-result ([2; Th. 1.1]) that (A)$\Rightarrow$(B) for $T = S$, $E = F$.

Now let E be an (AL)-space and let T be any contraction on E. The following lemma is taken from [4; p. 347].

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LEMMA. *If (T^n) is order-bounded in $L(E)$ and the modulus $\tau=|T|$ has no fixed vectors other than O, then $\|T^nx\|\to 0$ for all $x\in E$.*

THEOREM (Fong and Sucheston). *If E is an (AL)-space and T is any contraction on E then (A) and (B) are equivalent.*

Proof. (A)$\Rightarrow$(B): As in [4; Th. V 8.7] we assume that E has quasi-interior elements and construct the largest ideal J on which T induces an order contraction. Then J is a band and $E=J\oplus J^\perp$. Let $Q:E\to J^\perp$ be the band projection. By the Nagel construction above, for any $x\in J$ and (UR)-matrix (a_{ni}), we know that $A_nx=\sum_{i=1}^n a_{ni}T^ix$ converges strongly.

On the (AL)-space $J^\perp$ consider the contraction $T_0=QTQ$. Then $z=|T_0|z$ implies $z\in J$ and $z=|T|z$, since $|T_0|=Q|T|Q$, hence $z=0$ by construction of J. By the lemma $\|T_0^nx\|\to 0$ for all $x\in J$.

Now let $\varepsilon>0$ and $x\in E$ be given. Find $m_0\in N$ such that $\|QT^{m_0}x\|<\varepsilon/M$. Let $Y=(I-Q)T^{m_0}x\in J$ and $b_{ni}=a_{n,i+m_0}$. The (UR)-matrix (b_{ni}) and y are now used to deduce that $g_n=\sum_i b_{ni}T^iy$ norm converges in J, as T is an order contraction on J. On the other hand, as in [2],

$$g_n=\sum_i b_{ni}T^{i+m_0}x-\sum_i b_{ni}T^i(QT^{m_0}x)=\sum_i a_{n,i+m_0}T^{i+m_0}x-h_n,$$

where $\|h_n\|=\|\sum b_{ni}T^iQT^{m_0}x\|\le(\sum_i|b_{ni}|\varepsilon/M\le\varepsilon$. Hence

$$\|g_n-A_nx\|\le\left\|\sum_i a_{n,i+m_0}T^{i+m_0}x-\sum_i a_{ni}T^ix\right\|+\varepsilon\le\left(\sum_i^{m_0}|a_{ni}|\right)\|x\|+\varepsilon.$$

But $\lim_n(\sum_{i=1}^{m_0}|a_{ni}|=0$. So, finally, $\lim_{n\to\infty}\sup\|g_n-A_nx\|<\varepsilon$ and hence (A_nx) converges strongly in E.

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