Canad. Math. Bull. Vol. **31** (3), 1988

A MULTILINEAR YOUNG'S INEQUALITY

BY
DANIEL M. OBERLIN

ABSTRACT. We prove an $(n + 1)$-linear inequality which generalizes the classical bilinear inequality of Young concerning the L^p norm of the convolution of two functions.

Let G be, say, a locally compact abelian group. For a function f on G and $t \in G$, define the translate f_t of f by $f_t(x) = f(x - t)$, $x \in G$. An n-linear operator M taking n-tuples of functions on G to functions on G is called a multilinear convolution (see [2], [3]) if

$$M((f_1)_t, \ldots, (f_n)_t) = (M(f_1, \ldots, f_n))_t,\ t \in G.$$

We are interested in certain multilinear convolutions M defined as follows. Let λ be a locally finite Borel measure on G^n. If $f_1, \ldots, f_n$ are continuous functions of compact support on G, put

$$M(f_1, \ldots, f_n)(x) = \int_{G^n} f_1(x - x_1) \ldots f_n(x - x_n) d\lambda(x_1, \ldots, x_n),\ x \in G.$$

Our question about such M is the question of L^p-boundedness: taking L^p norms with respect to Haar measure, when do we have an inequality

$$\|M(f_1, \ldots, f_n)\|_r \leqq C\|f_1\|_{p_1} \ldots \|f_n\|_{p_n}?$$

Along these lines, there is a general theorem with $r = 1$ in [5]. The case with $G = \mathbf{R}$ and λ equal to Lebesgue measure on the unit sphere in $\mathbf{R}^n$ is the subject of [4]. The case considered in this note occurs when λ is absolutely continuous with respect to Haar measure on G^n and has density K in $L^q(G^n)$. Thus

$$M(f_1, \ldots, f_n)(x) = \int_{G^n} f_1(x - x_1) \ldots f_n(x - x_n) \times K(x_1, \ldots, x_n) dx_1 \ldots dx_n,\ x \in G,$$

and the inequality

$$(1) \qquad \|M(f_1, \ldots, f_n)\|_r \leqq \|f_1\|_{p_1} \ldots \|f_n\|_{p_n} \|K\|_q$$

Received by the editors June 14, 1987, and, in revised form, September 15, 1987.
Partially supported by a grant from the National Science Foundation.
AMS Subject Classification (1985): 43A15.

is a multilinear analogue of Young's inequality.

THEOREM. *Inequality* (1) *holds when the conditions below are met*:

(a) $$0 \leqq \frac{1}{r}, \frac{1}{q}, \frac{1}{p_1}, \ldots, \frac{1}{p_n} \leqq 1;$$

(b) $$\frac{1}{r} \leqq \frac{1}{q};$$

(c) $$n + \frac{1}{r} = \frac{1}{p_1} + \ldots + \frac{1}{p_n} + \frac{n}{q};$$

(d) $$1 - \frac{1}{q} \leqq \frac{1}{p_j} \text{ for } j = 1, \ldots, n.$$

COMMENTS. (1) When $n = 1$, this is Young's inequality. Then (b) and (d) are redundant.

(2) If the p_j are equal, then (d) is redundant.

(3) If G is compact or discrete, then the inclusions between L^p spaces on G show that (1) will hold in cases not covered by the theorem. But (b)-(d) are all necessary for a general result. For example, if $G = \mathbf{R}$, let

$$D = \{ (x, x, \ldots, x) + (t_1, \ldots, t_n) \in \mathbf{R}^n : 0 \leqq x \leqq L, 0 \leqq t_i \leqq \delta \},$$
$$f_1 = \ldots = f_n = \chi_{[0,\delta]}, K = \chi_D.$$

If (1) holds it follows that

$$\delta^n L^{1/r} \leqq (\text{const.}) \delta^{(1/p_1)+\ldots+(1/p_n)} (L\delta^{n-1})^{1/q} \text{ for } 0 < \delta \leqq L < \infty.$$

Thus (b) and (c) must hold. To see that (d) holds, let

$$D = \{ (x_1, \ldots, x_n) : |x_j| \leqq L, |x_i| \leqq 1 \text{ if } i \neq j \},$$
$$f_j = \chi_{[0,L]}, f_i = \chi_{[0,1]} \text{ if } i \neq j, \text{ and } K = \chi_D.$$

Then (1) implies

$$L \leqq (\text{const.}) L^{1/q} L^{1/p_j} \text{ for } L \geqq 1.$$

Thus (d) holds.

PROOF OF THEOREM. Consider the closed convex subset K of $\mathbf{R}^{n+2}$ defined by

$$K = \{ k = (\alpha_1, \ldots, \alpha_n, \beta, \gamma) : n + \gamma = \alpha_1 + \ldots + \alpha_n + n\beta,$$
$$0 \leqq \gamma \leqq \beta \leqq 1, 1 - \beta \leqq \alpha_i \leqq 1 \text{ for } 1 \leqq i \leqq n \}.$$

By the multilinear Riesz-Thorin theorem (see [1]), it is enough to show that (1) holds whenever

https://doi.org/10.4153/CMB-1988-054-0 Published online by Cambridge University Press

$$\left(\frac{1}{p_1}, \ldots, \frac{1}{p_n}, \frac{1}{q}, \frac{1}{r}\right)$$

is an extreme point of K. Thus we begin by looking at the geometry of K.

The first step is to show that if $0 < \beta < 1$, then k is not an extreme point of K. So fix $k = (\alpha_1, \ldots, \alpha_n, \beta, \gamma)$ with $0 < \beta < 1$. Write $I_n = \{1, \ldots, n\}$ and define subsets S and T of I_n by

$$i \in S \text{ if and only if } \alpha_i = 1, \ i \in T \text{ if and only if } \alpha_i = 1 - \beta.$$

Consider first the case $S \cup T = I_n$. For $t \in \mathbf{R}$, define

$$\beta(t) = \beta + t,$$

$$\alpha_i(t) = \alpha_i \text{ if } i \in S, \ \alpha_i(t) = \alpha_i - t \text{ if } i \in T.$$

Define

$$\gamma(t) = \sum_1^n \alpha_i(t) + n\beta(t) - n = \sum_1^n \alpha_i + n\beta - n - \sum_{i \in T} t + nt = \gamma + |S|t.$$

If there is $\delta > 0$ such that

$$(\alpha_1(t), \ldots, \alpha_n(t), \beta(t), \gamma(t)) \in K \text{ for } |t| < \delta,$$

then it is clear that k is not extreme. By definition,

$$n + \gamma(t) = \alpha_1(t) + \ldots + \alpha_n(t) + n\beta(t) \text{ for all } t.$$

Also

$$1 - \beta(t) \leqq \alpha_i(t) \leqq 1 \text{ if } |t| < \delta_i \text{ for some } \delta_i > 0 \text{ and } 1 \leqq i \leqq n.$$

Thus it is only necessary to find $\delta_0 > 0$ such that for $|t| < \delta_0$

$$0 \leqq \gamma(t) \leqq \beta(t) \leqq 1, \text{ or}$$

$$0 \leqq \gamma + |S|t \leqq \beta + t \leqq 1.$$

Such a δ_0 exists if either

$$\gamma = 0 \text{ and } |S| = 0, \text{ or } \gamma = \beta \text{ and } |S| = 1, \text{ or } 0 < \gamma < \beta.$$

Now recall that $0 \leqq \gamma \leqq \beta$ since $k \in K$. If $\gamma = 0$, then $|S| = 0$ follows from

$$n + \gamma = \alpha_1 + \ldots + \alpha_n + n\beta \text{ and } S \cup T = I_n.$$

But if $0 < \gamma = \beta$, then $\alpha_1 + \ldots + \alpha_n = (n - 1)(1 - \beta) + 1$, so $|S| = 1$. Thus k is not extreme in the case $S \cup T = I_n$. If, on the other hand, $S \cup T \neq I_n$, fix l with $1 - \beta < \alpha_l < 1$. Define

$$\alpha_i(t) = \alpha_i \text{ if } i \in S, \ \alpha_i(t) = \alpha_i - t \text{ if } i \notin S \text{ and } i \neq l,$$

https://doi.org/10.4153/CMB-1988-054-0 Published online by Cambridge University Press

and $\alpha_l(t) = \alpha_l - mt$ where $m = |S| + 1$ if $\gamma = 0$, $m = |S|$ if $\gamma > 0$.

Define

$$\beta(t) = \beta + t, \gamma(t) = \sum_1^n \alpha_i(t) + n\beta(t) - n = \gamma + (|S| + 1 - m)t.$$

Then, as before

$$(\alpha_1(t), \ldots, \alpha_n(t), \beta(t), \gamma(t)) \in K$$

for small $|t|$. Thus k is not extreme whenever $0 < \beta < 1$.

Suppose now that k is extreme in K and we will check that (1) holds if

$$\left(\frac{1}{p_1}, \ldots, \frac{1}{p_n}, \frac{1}{q}, \frac{1}{r}\right) = k.$$

If $\beta = 0$, then $\gamma = 0$ and each α_i is 1. Thus $p_1 = p_2 = \ldots = p_n = 1$, $q = r = \infty$, and (1) is clear. So suppose $\beta = 1$. The set

$$\{k \in K:\beta = 1\}$$

is affinely isomorphic (by projection onto the first n coordinates) to

$$\left\{(\alpha_1, \ldots, \alpha_n):0 \leqq \alpha_i \leqq 1, \sum_1^n \alpha_i \leqq 1\right\}.$$

The extreme points of this set occur when one of the α_i is 1 and the rest are zero. So suppose $p_l = 1$, $p_i = \infty$ if $i \neq l$, and $q = r = 1$. Then

$$|M(f_1, \ldots, f_n)(x)| \leqq \prod_{i\neq l} \|f_i\|_\infty \int_G |f_l(x - x_l)|$$

$$\times \left[\int_{G^{n-1}} |K(x_1, \ldots, x_n)| \prod_{i\neq l} dx_i\right] dx_l.$$

Since

$$\int_G \left[\int_{G^{n-1}} |K(x_1, \ldots, x_n)| \prod_{i\neq l} dx_i\right] dx_l = \|K\|_1,$$

inequality (1) is true in this case and the proof of the theorem is complete.

References

1. J. Bergh and J. Löfström, *Interpolation Spaces*, Springer-Verlag, Berlin, 1976.

2. R. R. Coifman and Y. Meyer, *Fourier analysis of multilinear convolutions, Caldéron's theorem, and analysis on Lipschitz curves*, Euclidean harmonic analysis (College Park, Md., 1979), pp. 104-122, Lecture Notes in Math. **779**, Springer-Verlag, Berlin, 1980.

3. M. Murray, *Multilinear convolutions and transference*, Michigan Math. J. **31** (1984), pp. 321-330.

4. D. Oberlin, *Multilinear convolutions defined by measures on spheres*, Trans. Amer. Math. Soc., (to appear).

5. ———, *The size of sums of sets*, Studia Math. **83** (1986), pp. 139-146.

FLORIDA STATE UNIVERSITY
TALLAHASSEE, FL 32306