FINITE TYPE PSEUDO-UMBILICAL SUBMANIFOLDS IN A HYPERSPHERE

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The notion of finite type submanifolds was introduced by B.Y. Chen. In this article we study 2- and 3-type pseudo-umbilical submanifolds in a hypersphere. Two theorems in this respect are obtained.

1. INTRODUCTION

A submanifold \( M \) of a Euclidean \( m \)-space \( E^m \) is said to be of finite type if each component of its position vector \( X \) can be written as a finite sum of eigenfunctions of the Laplacian \( \Delta \) of \( M \), that is,

\[
X = X_0 + \sum_{i=1}^{k} X_i, 
\]

where \( X_0 \) is a constant vector and \( \Delta X_i = \lambda_i X_i \), \( t = 1, 2, \ldots, k \). If in particular all eigenvalues \( \{\lambda_1, \ldots, \lambda_k\} \) are mutually different, then \( M \) is said to be of \( k \)-type. A \( k \)-type submanifold is said to be null if one of the \( \lambda_i \), \( t = 1, 2, \ldots, k \), is null. It is easy to see that if \( M \) is compact, then \( X_0 \) in (1.1) is exactly the centre of mass in \( E^m \). A submanifold \( M \) of a hypersphere \( S^{m-1} \) of \( E^m \) is called mass-symmetric in \( S^{m-1} \) if the centre of mass of \( M \) in \( E^m \) is the centre of the hypersphere \( S^{m-1} \) in \( E^m \) (see [1] for details).

In terms of finite type submanifolds, a well-known result of Takahashi [6] says that a submanifold \( M \) in \( E^m \) is of 1-type if and only if it is either a minimal submanifold of \( E^m \) or a minimal submanifold of a hypersphere of \( E^m \). In the first case \( M \) is of null 1-type and in the last case \( M \) is mass-symmetric in \( S^{m-1} \). In [3] it was proved that a compact 2-type hypersurface of a hypersphere is mass-symmetric if and only if it has constant mean curvature. In [5] it was proved that every 2-type pseudo-umbilical submanifold of \( E^m \) with constant mean curvature is either spherical or null 2-type. And in [4] some examples were given for spherical 2-type pseudo-umbilical surfaces, which is mass-symmetric.

In this article we prove the following.

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THEOREM 1. Let $M$ be a pseudo-umbilical submanifold of a hypersphere $S^{m-1}$ in $E^m$. If $M$ is of 2-type, then $M$ is a non-null 2-type submanifold of $S^{m-1}$ with constant mean curvature. In particular, if $M$ is compact, then $M$ is mass-symmetric.

THEOREM 2. Let $M$ be a pseudo-umbilical submanifold of a hypersphere $S^{m-1}$ in $E^m$ with constant mean curvature. If $M$ is of 3-type, then $\|DH\|$ is constant if and only if $\|DH\|^2$ is harmonic. In particular, if $M$ is closed, then $M$ is mass-symmetric.

2. SOME BASIC FORMULAS

Let $M$ be an $n$-dimensional (connected) submanifold of the unit hypersphere $S^{m-1}(1)$ of $E^m$ centred at the origin. Then the position vector $X$ of $M$ in $E^m$ is normal to $M$ as well as to $S^{m-1}(1)$. Let $A$, $h$, $D$ and $H$ denote the Weingarten map, the second fundamental form, the normal connection and the mean curvature vector of $M$ in $E^m$, respectively, and by $A'$, $h'$, $D'$ and $H'$ the corresponding invariants of $M$ in $S^{m-1}(1)$. A submanifold $M$ is said to be pseudo-umbilical if its Weingarten map with respect to the mean curvature vector is proportional to the identity map. It is clear that $M$ is pseudo-umbilical in $E^m$ if and only if so it is in $S^{m-1}$. We put $\alpha^2 = \langle H, H \rangle$ and $\beta^2 = \langle H', H' \rangle$, where $\langle , \rangle$ denotes the inner product of $E^m$. We have

$$H = H' - X, \quad H' = \beta \xi, \quad \alpha^2 = \beta^2 + 1,$$

for some unit normal vector $\xi$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal tangent frame of $M$. For any vector $\eta$ normal to the submanifold $M$ we put

$$\bar{A}(\eta) = \sum_{i=1}^{n} h(e_i, A_\eta e_i),$$

$$\text{tr} \left( \overline{\nabla} A_\eta \right) = \sum_{i=1}^{n} \left\{ (\nabla_{e_i} A_\eta) e_i + \text{tr} A_{D_\eta} \right\},$$

where $\text{tr} A_{D_\eta} = \sum_{i=1}^{n} A_{De_i} \eta e_i$. Then we have the following useful formulas obtained in [1, 2].

$$\Delta H = \Delta^D H + \bar{A}(H) + \text{tr} \left( \overline{\nabla} A_H \right),$$

$$\text{tr} \left( \overline{\nabla} A_H \right) = \frac{n}{2} \text{grad} \alpha^2 + 2 \text{tr} A_{DH},$$

where $\Delta^D$ denotes the Laplacian operator associated with the normal connection $D$.

By using the same method of the proofs of (2.4) and (2.5) given in [1, 2] we may obtain the following.
LEMMA 2.1. Let $M$ be an $n$-dimensional submanifold of $E^m$. Then for any vector field $\eta$ normal to $M$ we have

\begin{align}
(2.6) & \quad \Delta \eta = \Delta^D \eta + \tilde{A}(\eta) + \text{tr} (\nabla A_{\eta}), \\
(2.7) & \quad \text{tr} (\nabla A_{\eta}) = n \sum_{i=1}^{n} (\eta, D_{e_i} H) e_i + 2 \text{tr} A_{D_{\eta}}.
\end{align}

From (2.4), (2.5) and Lemma 2.1, we may obtain the following.

LEMMA 2.2. ([5]). If $M$ is a pseudo-umbilical submanifold of $E^m$, then we have

\begin{equation}
(2.8) \quad \Delta H = \Delta^D H + n \alpha^2 H + \frac{4-n}{2} \text{grad} \alpha^2.
\end{equation}

3. PROOF OF THEOREM 1

Let $M$ be a 2-type submanifold of $S^{m-1}(1)$ of $E^m$ centred at the origin. Then (1.1) becomes

\begin{equation}
(3.1) \quad X = X_0 + X_{t_1} + X_{t_2}, \quad \Delta X_{t_i} = \lambda_i X_{t_i}, \quad t = 1, 2.
\end{equation}

Since $\Delta X = -nH$, we have

\begin{equation}
(3.2) \quad \Delta H = bH + c(X - X_0),
\end{equation}

where $b = \lambda_{i_1} + \lambda_{i_2}, \quad c = \lambda_{i_1} \lambda_{i_2}/n$.

If $M$ is pseudo-umbilical in $E^m$, then from Lemma 2.2 and (3.2) we have

\begin{equation}
(3.3) \quad \Delta^D H + n \alpha^2 H + \frac{4-n}{2} \text{grad} \alpha^2 = bH + c(X - X_0),
\end{equation}

which implies

\begin{align}
(3.4) & \quad \frac{4-n}{2} \text{grad} \alpha^2 = -(X_0)^T, \\
(3.5) & \quad \langle \Delta^D H, X \rangle - (n \alpha^2 - b + c) + c(X_0, X) = 0,
\end{align}

where $(X_0)^T$ denotes the tangent component of $X_0$.

Since $\langle \Delta^D H, X \rangle = \Delta(H, X) = 0$, (3.5) becomes

\begin{equation}
(3.6) \quad n \alpha^2 - b + c = c(X_0, X).
\end{equation}

Taking the gradient of both sides of (3.6), we have

\begin{equation}
(3.7) \quad n \text{grad} \alpha^2 = (X_0)^T.
\end{equation}
Combining (3.4) and (3.7), we obtain

$$\frac{4 + n}{2} \nabla \alpha^2 = 0,$$

which implies that $\alpha$ is constant on $M$ and, moreover, by (3.7)

$$c(X_0) = 0.$$

In (3.9), if $c = 0$, that is, if $M$ is of null 2-type, then by (3.6) and (3.3), we have $\Delta D H = 0$. Consequently, we have

$$0 = \langle \Delta D H, H \rangle = \frac{1}{2} \Delta \alpha^2 + \|DH\|^2 = \|DH\|^2.$$

Since we have shown above that $\alpha$ is constant on $M$, (3.10) and (2.1) imply that $D \xi = 0$ and $\beta = \text{constant}$. Put $\eta = \xi + \beta X$ and take the covariant derivative of $\eta$ in $E^m$ with respect to any vector $Y$ tangent to $M$, we have

$$\tilde{\nabla}_Y \eta = \tilde{\nabla}_Y \xi + \beta \tilde{\nabla}_Y X = -A_\xi Y - \beta A_X Y = 0.$$

Thus the normal vector field $\eta$ on $M$ is constant. It implies that $M$ is in the intersection of $S^{m-1}$ and a hyperplane of $E^m$, that is, in a $S^{m-2}$ of $E^m$. Now we may consider $M$ as a pseudo-umbilical submanifold of $S^{m-2}$ which is still of null 2-type. Then by induction $M$ would be an open portion of $S^n$ in $E^{n+1}$, but which is not of 2-type. This is a contradiction.

Now we may assume that $M$ is of non-null 2-type, then from (3.9) we have $(X_0)^T = 0$. Consequently, we have

$$\nabla (X - X_0, X_0) = (X_0)^T = 0,$$

which implies that either $X_0 = 0$, or $X_0 \neq 0$ and $M$ is in the intersection of $S^{m-1}$ and a hyperplane of $E^m$. But the latter case could not occur since by an argument similar to the one mentioned above it would lead to a contradiction. Consequently, if $M$ is compact, since in this case $X_0$ is the centre of mass of $M$ in $E^m$, then $M$ is mass-symmetric in $S^{m-1}$.

4. Proof of Theorem 2

Let $M$ be a 3-type submanifold of $S^{m-1}(1)$ of $E^m$ centred at origin. Then, as in Section 3, we have

$$X = X_0 + X_{i_1} + X_{i_2} + X_{i_3}, \quad \Delta X_{i_t} = \lambda_{i_t} X_{i_t}, \quad t = 1, 2, 3.$$

$$\Delta^2 H = a \Delta H + b H + c(X - X_0),$$
where \( a = \sum \lambda_i \), \( b = - \sum \lambda_i \lambda_i \), and \( c = \lambda_i \lambda_i \lambda_i / n \).

If \( M \) is a pseudo-umbilical submanifold in \( E^m \) with constant mean curvature, then by Lemmas 2.2 and 2.1, we have

\[
(4.3) \quad \Delta H = \Delta D H + n a^2 H,
\]

\[
(4.4) \quad \Delta^2 H = \Delta (\Delta D H) + n a^2 \Delta H
\]

\[
= \Delta D \Delta D H + \tilde{\alpha}(\Delta D H) + \text{tr} (\nabla A_{\Delta D H}) + n a^2 \Delta D H + n^2 \alpha^4 H.
\]

From (4.2)—(4.4), we have

\[
\langle \Delta D \Delta D H, X \rangle + \langle \tilde{\alpha}(\Delta D H), X \rangle + n a^2 \langle \Delta D H, X \rangle + n^2 \alpha^4 \langle H, X \rangle
\]

\[
= a \langle \Delta D H, X \rangle + (n a^2 a + b) \langle H, X \rangle + c - c(X_0, X).
\]

In (4.5), we have

\[
(4.6) \quad \langle \Delta D H, X \rangle = \Delta \langle H, X \rangle = 0, \quad \langle \Delta D \Delta D H, X \rangle = \Delta \langle \Delta D H, X \rangle = 0,
\]

\[
(4.7) \quad \langle \tilde{\alpha}(\Delta D H), X \rangle = \langle \Delta D H, \tilde{\alpha}(X) \rangle = \langle \Delta D H, \sum_i h(e_i, A X e_i) \rangle
\]

\[
= -n \langle \Delta D H, H \rangle = -n \|D H\|^2.
\]

Thus (4.5) implies

\[
(4.8) \quad \|D H\|^2 = -(n^2 \alpha^4 - n a^2 a - b + c)/n + c(X, X_0)/n.
\]

Taking the Laplacian of both sides of (4.8), we obtain

\[
(4.9) \quad \Delta \|D H\|^2 = c \Delta (X, X_0)/n = -c(H, X_0).
\]

Consequently, \( \|D H\|^2 \) is harmonic if and only if \( c(H, X_0) = 0 \), that is, (i) \( c = 0 \), or

(ii) \( c \neq 0 \) and \( \langle H, X_0 \rangle = 0 \).

In Case (i), \( M \) is of null 3-type and by (4.8) \( \|D H\|^2 = -(n^2 \alpha^4 - n a^2 a - b)/n \) which is a constant.

In Case (ii), from (4.2) we have

\[
(4.10) \quad \langle \Delta^2 H, X_0 \rangle = a \langle \Delta H, X_0 \rangle + b \langle H, X_0 \rangle + c(X - X_0, X_0).
\]

Since we have

\[
(4.11) \quad \langle \Delta H, X_0 \rangle = \Delta \langle H, X_0 \rangle, \quad \langle \Delta^2 H, X_0 \rangle = \Delta \langle \Delta H, X_0 \rangle = \Delta^2 \langle H, X_0 \rangle,
\]

(4.10) implies \( \langle X - X_0, X_0 \rangle = 0 \). With an argument similar to the one mentioned in the proof of Theorem 1, we may conclude that \( X_0 \) must be the origin of \( E^m \). Then (4.8) gives \( \|D H\|^2 = -(n^2 \alpha^4 - n a^2 a - b + c)/n \) which is a constant. In particular, if \( M \) is closed (that is, \( M \) is compact and without boundary), then because \( \lambda_i, \lambda_i \) and \( \lambda_i \) are positive, \( M \) cannot be null. Moreover, in this case, because \( X_0 \) is the centre of mass of \( M \) in \( E^m \), \( M \) is mass-symmetric in \( S^{m-1} \).
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