FINITE 3-GROUPS ACTING ON BORDERED SURFACES

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1. Introduction. Let $G$ be a finite group. The real genus $\rho(G)$ [8] is the minimum algebraic genus of any compact bordered Klein surface on which $G$ acts. There are now several results about the real genus parameter. The groups with real genus $\rho \leq 5$ have been classified [8,9,12], and genus formulas have been obtained for several classes of groups [8,9,10,11,12]. Most notably, McCullough calculated the real genus of each finite abelian group [13]. In addition, there is a good general lower bound for the real genus of a finite group [11].

Here we consider finite 3-groups acting on bordered Klein surfaces. We begin by specializing the approach in [11] to obtain a lower bound for the real genus of a 3-group. Then we determine the real genus of several infinite families of 3-groups. The lower bound is attained for most of these families. We also develop some general ideas about 3-groups acting on bordered surfaces. Finally, we determine the real genus of all groups with order 81.

We use the standard representation of a group $G$ as a quotient of a non-euclidean crystallographic group $\Gamma$ by a bordered surface group $K$; then $G$ acts on the Klein surface $U/K$, where $U$ is the open upper half-plane.

2. Preliminaries. We shall assume that all surfaces are compact. A bordered surface $X$ can carry a dianalytic structure [1, p. 46] and be considered a Klein surface or a non-singular real algebraic curve. Thus the surface $X$ has an algebraic genus $g$. The algebraic genus appears naturally in bounds for the order of the automorphism group of a Klein surface, and the real genus of a group is defined in terms of the algebraic genus.

There is a useful upper bound for the real genus of a finite group in terms of the orders of the elements in a generating set [8, p. 712].

THEOREM A [8]. Let $G$ be a finite group with generators $z_1, \ldots, z_c$, where $o(z_i) = m_i$. Then

$$\rho(G) \leq 1 + o(G) \left[ c - 1 - \sum_{i=1}^{c} \frac{1}{m_i} \right]. \quad (2.1)$$

Group actions on Klein surfaces have often been investigated using non-euclidean crystallographic (NEC) groups; here see the monograph [2], an excellent reference for the work on Klein surfaces. Let $\mathcal{V}$ denote the group of all dianalytic automorphisms of the open upper half-plane $U$. An NEC group is a discrete subgroup $\Gamma$ of $\mathcal{V}$ (with the quotient space $U/\Gamma$ compact). Associated with the NEC group $\Gamma$ is its signature, which has the form

$$\langle \rho; \pm; [\lambda_1, \ldots, \lambda_r]; \{(v_{11}, \ldots, v_{1k_1}), \ldots, (v_{k1}, \ldots, v_{rk})\}\rangle. \quad (2.2)$$

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The quotient space $X = U/\Gamma$ is a surface with topological genus $p$ and $k$ holes. The surface is orientable if the plus sign is used and non-orientable otherwise. The ordinary periods $\lambda_1, \ldots, \lambda_r$ are the ramification indices of the natural quotient mapping from $U$ to $X$ in fibers above interior points of $X$. The link periods $v_{i_1}, \ldots, v_{i_k}$, are the ramification indices in fibers above points on the $i$th boundary component of $X$. Associated with the signature (2.2) is a presentation for the NEC group $\Gamma$. For more information about signatures, see [14] and [2].

Let $\Gamma$ be an NEC group with signature (2.2) and assume $k \geq 1$ so that the quotient space $U/\Gamma$ is a bordered surface. Then the non-euclidean area $\mu(\Gamma)$ of a fundamental region for $\Gamma$ can be calculated directly from its signature [14, p. 235]:

$$\mu(\Gamma)/2\pi = \gamma - 1 + \sum_{i=1}^{r} \left( 1 - \frac{1}{\lambda_i} \right) + \sum_{i=1}^{k} \sum_{j=1}^{\gamma_i} \frac{1}{2} \left( 1 - \frac{1}{v_{ij}} \right),$$

(2.3)

where $\gamma$ is the algebraic genus of the quotient space $U/\Gamma$. If $\Lambda$ is a subgroup of finite index in $\Gamma$, then

$$[\Gamma : \Lambda] = \mu(\Lambda)/\mu(\Gamma).$$

(2.4)

An NEC group $K$ is called a surface group if the quotient map from $U$ to $U/K$ is unramified. If the quotient space $U/K$ has a non-empty boundary, then $K$ is called a bordered surface group. Bordered surface groups contain reflections but no other elements of finite order.

Let $X$ be a bordered Klein surface of algebraic genus $g \geq 2$, and let $G$ be a group of dianalytic automorphisms of $X$. Then $X$ can be represented as $U/K$, where $K$ is a bordered surface group with $\mu(K) = 2\pi(g - 1)$. Further, there exist an NEC group $\Gamma$ and a homomorphism $\phi: \Gamma \to G$ onto $G$ such that kernel $\phi = K$ [7]. The group $G \cong \Gamma/K$, so that from (2.4) the algebraic genus $g$ of the bordered surface $X$ on which $G$ acts is given by

$$g = 1 + o(G) \cdot \mu(\Gamma)/2\pi.$$  

(2.5)

Thus (2.5) and (2.3) give the relationship between the algebraic genera $g$ and $\gamma$ of $X$ and $U/\Gamma$, respectively. This relationship is sometimes given as the Riemann-Hurwitz formula for the quotient mapping $X \to X/G = U/\Gamma$; see [6], for example. In the following we will consistently use $g$ and $\gamma$ for the algebraic genera of the surfaces $X$ and $U/\Gamma$, respectively.

Minimizing the algebraic genus $g$ for a particular group $G$ is therefore equivalent to minimizing $\mu(\Gamma)$. Among the NEC groups $\Gamma$ for which $G$ is a quotient of $\Gamma$ by a bordered surface group, then, we want to identify one for which $\mu(\Gamma)$ is as small as possible.

3. A lower bound. Here we establish a useful lower bound for the real genus of a finite 3-group. We specialize the approach in [11, §3] to obtain a bound that is much better for 3-groups. The approach in [11] considered generators as having order 2, order 3 or “high” order, and the ones of high order were all treated as having order 4 to produce the lower bound. In a 3-group there are no generators of order 2, of course, and high order generators must have order at least 9.

Let $G$ be a finitely presented 3-group and $S$ a generating set for $G$. Let $t_3(S)$ denote the number of generators in $S$ of order 3, and let $t_h(S)$ be the number of generators of order
larger than 3 (and thus at least 9). We will write simply $t_3$ and $t_h$ if the generating set is obvious. Then $|S| = t_3 + t_h$. We define

$$\theta(G) = \min \{8t_h(S) + 6t_3(S) \mid S \text{ a generating set for } G\}.$$ 

A generating set for which $\theta(G)$ is attained is said to be $\theta$-minimal. The quantity $\theta(G)/9$ arises naturally when obtaining a lower bound for (2.3) by treating all generators of high order as having order 9; see the proof of Theorem 1. The following result is basic.

**Lemma 1.** Let $G'$ be a quotient group of the finitely presented group $G$. Then

$$\theta(G) \geq \theta(G').$$

**Proof.** Let $\pi : G \to G'$ denote the quotient map. Then let $S$ be a $\theta$-minimal generating set for $G$, and let $S'$ be the induced generating set for $G'$. Write $t_3 = t_3(S)$ and $t'_3 = t_3(S')$ and so forth. Clearly $|S| \geq |S'|$ so that

$$t_h + t_3 \geq t'_h + t'_3.$$

If $y \in S'$ with $o(y) > 3$, then there is at least one generator $x$ in $S$ such that $\pi(x) = y$. But $o(x) > 3$ also. Hence we have

$$t_h \geq t'_h,$$

Now since $S$ is $\theta$-minimal,

$$\theta(G) = 8t_h + 6t_3 = 6(t_h + t_3) + 2t_h \geq 6(t'_h + t'_3) + 2t'_h \geq \theta(G'),$$

whether or not the generating set $S'$ is $\theta$-minimal.

Let $G$ be a finite 3-group, and suppose there exist an NEC group $\Gamma$ and a homomorphism $\phi : \Gamma \to G$ onto $G$ such that $K = \ker \phi$ is a bordered surface group. Since the surface group $K$ contains no analytic elements of finite order, each ordinary period of $\Gamma$ must be a power of 3. We need an upper bound for $\theta(G)$.

**Lemma 2.** Let $G$ be a finite 3-group. Suppose there exist an NEC group $\Gamma$ with signature (2.2) and a homomorphism $\phi : \Gamma \to G$ onto $G$ such that $K = \ker \phi$ is a bordered surface group. Let $r_3$ denote the number of ordinary periods of $\Gamma$ equal to 3 and $r_h$ the number greater than 3. Then

$$\theta(G) \leq 8(\gamma + r_h) + 6r_3,$$

where $\gamma$ is the algebraic genus of the quotient space $U/\Gamma$.

**Proof.** Simplify the canonical presentation for $\Gamma$ as in [11, §2]; the simplified presentation has $\gamma + r$ generators (that are elliptic, hyperbolic or glide reflections) plus some additional reflections. Let $S$ be the induced generating set for $G$. Since the order of $G$ is odd, all reflections in $\Gamma$ are in the bordered surface group $K = \ker \phi$. Therefore the generating set $S$ has at most $\gamma + r$ elements. Of the elements in $S$, clearly at most $\gamma + r_h$ can have order larger than three. Now apply the definition of $\theta(G)$.

Now we establish our general lower bound.

**Theorem 1.** Let $G$ be a finite 3-group. Then

$$\rho(G) \geq 1 + o(G)[\theta(G) - 9]/9.$$  

(3.1)
The only 3-groups with $\rho \leq 1$ are cyclic [8]. For a cyclic group $G$, $\rho(G) = 0$, and it is a simple matter to check that the inequality holds. Assume then that $\rho(G) \geq 2$, and let $G$ act on the bordered surface $X$ of algebraic genus $g \geq 2$. Then represent $X$ as $U/K$ where $K$ is a bordered surface group, and obtain an NEC group $\Gamma$ and a homomorphism $\phi: \Gamma \to G$ onto $G$ such that kernel $\phi = K$. We use the notation of Lemma 2. In particular, $\gamma$ denotes the algebraic genus of the quotient space $U/\Gamma$. Each ordinary period of $\Gamma$ must be a power of 3. Using (2.3) we obtain

$$\mu(\Gamma)/2\pi \geq \gamma - 1 + r_3 \cdot 2/3 + r_h \cdot 8/9.$$ 

Therefore

$$9[\mu(\Gamma)/2\pi] \geq 9\gamma + 8r_h + 6r_3 - 9.$$ 

Since $\gamma \geq 0$, applying Lemma 2 yields

$$9[\mu(\Gamma)/2\pi] \geq \theta(G) - 9.$$ 

Now from (2.5) we have $g \geq 1 + o(G)[\theta(G) - 9]/9$. Thus $\rho(G) \geq 1 + o(G)[\theta(G) - 9]/9$.

We believe the lower bound (3.1) is quite useful, in general. We shall see examples of infinite families of 3-groups for which the lower bound gives the real genus. Indeed, the lower bound of Theorem 1 always gives the genus of a 3-group $G$ if $G$ has a $\theta$-minimal generating set that only contains elements of orders 3 and 9.

**Theorem 2.** Let $G$ be a finite 3-group. If a $\theta$-minimal generating set for $G$ contains $t$ elements of order 3, $n$ elements of order 9, and no other elements, then

$$\rho(G) = 1 + o(G)(6t + 8n - 9)/9. \quad (3.2)$$

**Proof.** We have $\theta(G) = 6t + 8n$ and (3.1) holds. But from the general upper bound (2.1), we also obtain

$$\rho(G) \leq 1 + o(G)(n + t - 1 - t/3 - n/9) = 1 + o(G)(6t + 8n - 9)/9.$$ 

**4. General results.** Next we use our lower bound to obtain some general results about 3-groups acting on bordered surfaces.

Let $X$ be a bordered Klein surface of algebraic genus $g \geq 2$. Then the automorphism group $G$ of $X$ has order at most $12(g - 1)[6]$. This general upper bound can be improved, of course, in special cases. (See [2] for a survey of these results.) For example, there is a basic upper bound for $p$-groups [3], where $p$ is an odd prime. In particular, if the automorphism group $G$ is a 3-group, then $o(G) \leq 3(g - 1)$. An immediate consequence of this result is a lower bound for the real genus of a 3-group. Here we obtain this lower bound as a simple consequence of Theorem 1.

**Theorem 3.** Let $G$ be a finite 3-group that is not cyclic. Then

$$\rho(G) \geq 1 + o(G)/3.$$ 

**Proof.** Any generating set for $G$ must have at least two elements, and obviously $\theta(G) \geq 2 \cdot 6$. Now (3.1) gives the result.
There are infinite families of 3-groups for which these bounds are attained. See [3, §5], [2, pp. 130, 131], and [11, §5]. The bound of Theorem 3 can be improved in special cases, of course.

**Theorem 4.** Let $G$ be a finite 3-group with $\rho(G) \geq 2$. If $G$ is not generated by elements of order 3, then

$$\rho(G) \geq 1 + 5\sigma(G)/9.$$  

**Proof.** Since $\rho(G) \neq 0$, the rank of $G$ is at least two, and a generating set for $G$ must contain at least one element of high order. Hence $\theta(G) \geq 8 + 6$, and (3.1) gives the lower bound.

Let $G$ be a finite 3-group. It may well be that in any presentation for $G$, there are at least two generators of high order. Let $\Omega$ be the subgroup of $G$ generated by the elements of order 3. Then $\Omega$ is a characteristic subgroup of $G$.

**Theorem 5.** Let $G$ be a finite 3-group. If $G/\Omega$ is not cyclic, then

$$\rho(G) \geq 1 + 7\sigma(G)/9.$$  

**Proof.** Since $G/\Omega$ is not cyclic, any generating set for $G$ must have at least two elements of order larger than 3. Hence $\theta(G) \geq 2 - 8$, and the result follows from (3.1).

The lower bound for the real genus is even better for 3-groups that have rank three or more.

**Theorem 6.** Let $G$ be a finite 3-group. If $\text{rank}(G) > 2$, then

$$\rho(G) \geq \sigma(G) + 1.$$  

**Proof.** Since $\text{rank}(G) \geq 3$, obviously $\theta(G) \geq 3 - 6$.

This series of results about a 3-group $G$ can also be obtained by considering the possible signatures for an NEC group $\Gamma$ such that $G$ is a quotient of $\Gamma$ by a bordered surface group and the non-euclidean area $\mu(\Gamma)$ is small. For instance, if $G$ is a 3-group of the maximum possible order for the value of the genus, then $G$ must be a quotient of an NEC group with signature $(0; +; [3, 3]; \{(\})$. For examples of this approach, see [9, §3] and [12, §3].

5. Genus formulas for particular families. We begin with an easy application of (3.2) to abelian groups that only have factors of $\mathbb{Z}_3$ and $\mathbb{Z}_9$.

**Theorem 7.**

$$\rho((\mathbb{Z}_3)^t \times (\mathbb{Z}_9)^n) = 1 + 3^{t+2n^2}(6t + 8n - 9).$$  

**Proof.** Let $G = (\mathbb{Z}_3)^t \times (\mathbb{Z}_9)^n$. Clearly $\text{rank}(G) = t + n$, with a $\theta$-minimal generating set for $G$ containing $t$ elements of order 3 and $n$ elements of order 9. Now (3.2) gives

$$\rho(G) = 1 + 3^t 9^n(6t + 8n - 9)/9.$$  

The formula of Theorem 7 can also be obtained from the general results in [13], although it does not appear there explicitly. The approach in [13] utilizes graphs of groups and is quite different, however. In addition, see [11, p. 1284], where elementary abelian 3-groups are considered.
Next let $K$ be the nonabelian group of order 27 with no element of order 9. The group $K$ has presentation

$$R^3 = S^3 = (RS)^3 = (R^{-1}S)^3 = 1.$$  

The group $K$ is a semi-direct product $(Z_3)^2 \times \phi Z_3$.

**Theorem 8.** $\rho(K^n) = 1 + 3^{m-1}(4n - 3)$.

**Proof.** The group $K$ is generated by two elements of order 3, and it is clear that $\theta(K) = 12$ and $\theta(K^n) = 12n$, with a $\theta$-minimal generating set for $K^n$ containing $2n$ elements of order 3. Now (3.2) yields $\rho(K^n) = 1 + 3^{m-1}(6\cdot 2n - 9)/9$.

In particular, $\rho(K) = 10$ [11, p. 1282].

For $m \geq 3$, let $M_m$ be the group with generators $X, Y$ and defining relations

$$X^{2m-1} = Y^3 = 1, \quad Y^{-1}XY = X^{1+3^{m-2}}.$$  

(5.1)

The group $M_m$ is a nonabelian group of order $3^m$ [5, p. 190]. The properties of these groups are well-known, of course [5, pp. 190–194]. Each possesses a maximal cyclic subgroup of order $3^{m-1}$. In fact, these groups are characterized among all nonabelian 3-groups by this property [5, p. 193].

The group $M_m$ is not generated by elements of order 3 and 9 (if $m > 3$), and the lower bounds of Theorems 1 and 4 are not attained. To obtain the lower bound for the real genus of $M_m$, we modify the proof of Proposition 1 of [12].

**Theorem 9.** $\rho(M_m) = 2(3^{m-1} - 1)$ for $m \geq 3$.

**Proof.** Write $G = M_m$. We know $\rho(G) \geq 2$. Let $G$ act on a bordered Klein surface $X$ of algebraic genus $g \geq 2$. Then represent $X$ as $U/K$, where $K$ is a bordered surface group, and obtain an NEC group $\Gamma$ with signature (2.2) and a homomorphism $\alpha : \Gamma \to G$ onto $G$ such that kernel $\alpha = K$. Let $\gamma$ be the algebraic genus of the quotient space $U/\Gamma$. Since the order of $G$ is odd, it is basic that all period cycles of $\Gamma$ are empty (Each reflection is in the kernel $K$, but the surface group $K$ contains no analytic elements of finite order).

Simplify the canonical presentation for $\Gamma$ as in [11, §2]. In this simplified presentation, there must be at least two elements with order 3 or more, since $\Gamma/K \cong G$. The number of generators of $\Gamma$ with order larger than two is at most $\gamma + r$, where $r$ is the number of ordinary periods. Therefore $\gamma + r \geq 2$. Let $A = \mu(\Gamma)/2\pi$, which is given by (2.3). We obtain a lower bound for $A$. Again, each ordinary period of $\Gamma$ must be a power of 3. If $\gamma \geq 2$, then obviously $A \geq 1$. If $\gamma = 1$, then $r \geq 1$ and $A \geq 2/3$.

Suppose $\gamma = 0$ so that $r \geq 2$. If $r \geq 3$, then $A \geq -1 + 3 \cdot 2/3 = 1$. Assume $r = 2$. Then the group $\Gamma$ has signature $(0; +; [\lambda_1, \lambda_2]; \{(\}})$, where we may take $\lambda_1 \leq \lambda_2$. From (2.3)

$$A = 1 - \frac{1}{\lambda_1} - \frac{1}{\lambda_2}.$$  

If $\lambda_2 \geq \lambda_1 \geq 9$, then $A \geq 1 - 2 \cdot (1/9) = 7/9 > 2/3$.

Assume, then, that $\gamma = 0, r = 2$, and $\lambda_1 = 3$. The group $\Gamma$ has presentation

$$x^3 = y^{\lambda_2} = c^3 = [c, e] = xye = 1.$$
But the only generating reflection \( e \) must be in the bordered surface group \( K \), and \( e \) is redundant. Thus the quotient group \( G \cong \Gamma/K \) is generated by the two elements \( \alpha(x) \) and \( \alpha(y) \). In any presentation for \( G \), there must be at least one element of order \( 3^{m-1} \), since the subgroup generated by elements of order dividing \( 3^{m-2} \) has index 3 in \( G \) [5, Th. 4.3(i)(c), p. 191]. Therefore \( \lambda_2 = 3^{m-1} \), and
\[
A = 1 - \frac{1}{3} - \frac{1}{3^{m-1}}.
\]
In this case, then, \( A < 2/3 \) and from (2.5) \( \rho = 1 + 3^m \cdot A = 1 + 3^m\left(\frac{2}{3} - \frac{1}{3^{m-1}}\right) = 2(3^{m-1} - 1) \). In all other cases, \( A \geq 2/3 \). Thus \( \rho(G) \geq 2(3^{m-1} - 1) \).

The upper bound for \( \rho(G) \) is provided by (2.1) applied to the defining presentation (5.1).

In particular, the group \( M_3 \) of order 27 has real genus 16 [12, p. 405]. Write \( M = M_3 \); \( M \) is a semi-direct product \( Z_9 \times Z_3 \).

**Theorem 10.** \( \rho(M^n) = 1 + 3^{3n-2}(14n - 9) \).

**Proof.** The group \( M \) is generated by an element of order 9 and one of order 3, of course. Clearly \( \theta(M) = 14 \) and \( \theta(M^n) = 14n \), with a \( \theta \)-minimal generating set for \( M^n \) containing \( n \) elements of order 3 and \( n \) of order 9. Now (3.2) gives \( \rho(M^n) = 1 + 3^n(8n + 6n - 9)/9 \).

We also briefly consider direct products of elementary abelian 3-groups and the groups \( K \) and \( M \).

**Theorem 11.** \( \rho((Z_3)^n \times K) = 1 + 3^{n+2}(2n + 1) \).

**Proof.** A \( \theta \)-minimal generating set for \((Z_3)^n \times K \) contains \( n + 2 \) elements of order 3.

**Theorem 12.** \( \rho((Z_3)^n \times M) = 1 + 3^{n+1}(6n + 5) \).

**Proof.** A \( \theta \)-minimal generating set for \((Z_3)^n \times M \) contains \( n + 1 \) elements of order 3 and one of order 9.

The general lower bound of Theorem 1 is attained for the groups of Theorems 7, 8, 10, 11 and 12.

### 6. The groups of order 81

The real genus of each group with order less than 32 has been determined [12]. For 3-groups the next order of interest is 81. There are 15 groups of order 81; five of these are abelian. These groups are listed in Burnside's classic book [4, pp. 145, 146], and our notation \( G_n \) refers to the \( n \)-th group in Burnside's list. The real genus of each abelian group has been determined [13]. Interestingly, all the nonabelian groups of order 81 yield to the methods of §§3–5. The group \( G_9 \) is the direct product \( Z_3 \times M \), and \( G_{14} \) is the direct product \( Z_3 \times K \). Also \( G_6 \) is the group \( M_4 \). We consider each remaining group as an extension of a large normal subgroup.

The groups \( G_7, G_{10}, \) and \( G_{15} \) are extensions of \( Z_3 \times Z_9 \). The abelian group \( A = Z_3 \times Z_9 \) has presentation
\[
P^9 = Q^3 = 1, \ PQ = QP.
\]
(6.1)
To obtain the group \( G_{15} \), adjoin to \( A \) an element \( R \) of order 3 that transforms the elements of \( A \) according to the automorphism \( P \to PQ, Q \to P^{-3}Q \). Then the group \( G_{15} \) [4, p. 146] has generators \( P, Q, R \) and defining relations (6.1) together with

\[
R^3 = 1, \quad R^{-1}PR = PQ, \quad R^{-1}QR = P^{-3}Q.
\]

Then the element \( R^{-1}P \) has order 3, and clearly \( G_{15} = \langle R, R^{-1}P \rangle \). Thus \( G_{15} \) is generated by two elements of order 3. Now \( \theta(G_{15}) = 12 \) and \( \rho(G_{15}) = 28 \) by (3.2).

To obtain the group \( G_{10} \), adjoin to \( A = Z_3 \times Z_9 \) an element \( S \) of order 3 that transforms the elements of \( A \) according to the automorphism \( P \to PQ, Q \to Q \). Then the group \( G_{10} \) [4, p. 145] is generated by \( P, Q, S \) with defining relations (6.1) and

\[
S^3 = 1, \quad S^{-1}PS = PQ, \quad QS = SQ.
\]

Then the subgroup \( \Omega(G_{10}) \cong (Z_3)^3 \), so that \( G_{10} \) is not generated by elements of order 3. But obviously \( G_{10} = \langle S, P \rangle \), and \( \theta(G_{10}) = 14 \). Hence \( \rho(G_{10}) = 46 \).

The group \( G_7 \) is a third extension of \( Z_3 \times Z_9 \). The group \( G_7 \) has presentation [4, p. 145]

\[
P^9 = Q^3 = R^3 = 1, \quad PQ = QP, \quad PR = RP, \quad R^{-1}QR = QP^3.
\]

The Frattini subgroup \( \Phi(G_7) = \langle P^3 \rangle \cong Z_3 \), so that \( G_7/\Phi \cong (Z_3)^3 \) and \( G_7 \) has rank 3. Also \( \Omega(G_7) \) is a nonabelian subgroup of order 27, so that \( G_7 \) is not generated by elements of order 3. But \( G_7 = \langle P, Q, R \rangle \), of course, and a \( \theta \)-minimal generating set for \( G_7 \) has two elements of order 3 and one of order 9. Thus \( \rho(G_7) = 100 \), using (3.2).

The groups \( G_{11}, G_{12}, \) and \( G_{13} \) are extensions of the nonabelian group \( M = M_3 \) with presentation (from (5.1))

\[
X^3 = Y^3 = 1, \quad Y^{-1}XY = X^4.
\]

First, to obtain the group \( G_{11} \), adjoin to \( M \) an element \( W \) of order 3 that transforms the elements of \( M \) according to the automorphism \( X \to XY, Y \to Y \). The group \( G_{11} \) [4, p. 145] has generators \( X, Y, W \) and defining relations (6.2) plus

\[
W^3 = 1, \quad W^{-1}XW = XY, \quad YW = WY.
\]

Then the element \( W^{-1}X \) has order 3, and \( G_{11} = \langle W, W^{-1}X \rangle \). Thus \( G_{11} \) is generated by two elements of order 3 and \( \theta(G_{11}) = 12 \). Hence \( \rho(G_{11}) = 28 \).

The group \( G_{12} \) [4, p. 145] has generators \( X, Y, W \) and defining relations (6.2) and

\[
W^{-1}XW = XY, \quad YW = WY, \quad W^3 = X^3.
\]

Then \( \Omega(G_{12}) \) is a nonabelian subgroup of order 27, so that \( G_{12} \) is not generated by elements of order 3. But the element \( WX \) has order 3 and \( G_{12} = \langle X, WX \rangle \). Now \( \theta(G_{12}) = 14 \) and \( \rho(G_{12}) = 46 \).

The group \( G_{13} \) is a third extension of \( M \). The group \( G_{13} \) [4, p. 145] has presentation (6.2) together with

\[
W^{-1}XW = XY, \quad YW = WY, \quad W^3 = X^6.
\]
The element $W$ has order 9. Here we have $\Phi(G_{13}) = \Omega(G_{13}) = \langle Q, P^3 \rangle \cong (Z_3)^2$ and $G_{13}/\Phi \cong (Z_3)^2$. Thus any generating set for $G_{13}$ must have at least two elements of order larger than 3. Since $G_{13} = \langle X, W \rangle$, we have $\theta(G_{13}) = 16$ and $\rho(G_{13}) = 64$ by (3.2).

Finally consider $G_8$, the group with presentation [4, p. 145]

$$P^9 = Q^9 = 1, Q^{-1}PQ = P^4.$$ 

This group is a semi-direct product $Z_9 \times_\phi Z_9$. For this group $\Phi(G_8) = \Omega(G_8) = \langle Q^3, P^3 \rangle \cong (Z_3)^2$, and $G_8/\Phi \cong (Z_3)^2$. Thus the two elements $P$ and $Q$ of order 9 form a $\theta$-minimal generating set for $G$. Hence $\theta(G_8) = 16$ and $\rho(G_8) = 64$.

The following table gives $\rho(G)$ for each nonabelian group $G$ of order 81.

<table>
<thead>
<tr>
<th>Group</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_4$</td>
<td>52</td>
</tr>
<tr>
<td>$G_7$</td>
<td>100</td>
</tr>
<tr>
<td>$G_8$</td>
<td>64</td>
</tr>
<tr>
<td>$Z_3 \times M$</td>
<td>100</td>
</tr>
<tr>
<td>$G_{10}$</td>
<td>46</td>
</tr>
</tbody>
</table>

The nonabelian groups of order 81 provide examples of groups for which the bounds of §4 are attained. For instance, each of the groups with $\rho = 46$ is generated by an element of order 3 and one of order 9, and the bound of Theorem 4 is realized.

REFERENCES


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