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ON THE DIMENSIONLESSNESS OF INVARIANT SETS

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Abstract. Let M be a subset of $\mathbb{R}$ with the following two invariance properties: (1) $M + k \subseteq M$ for all integers k, and (2) there exists a positive integer $l \geq 2$ such that $\frac{1}{l}M \subseteq M$. (For example, the set of Liouville numbers and the Besicovitch-Eggleston set of non-normal numbers satisfy these conditions.) We prove that if h is a dimension function that is strongly concave at 0, then the h-dimensional Hausdorff measure $\mathcal{H}^h(M)$ of M equals 0 or infinity.

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1. Introduction and statement of results. A *dimension function* h is an increasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$. If h is a dimension function, we shall denote the h-dimensional Hausdorff measure of a subset E of $\mathbb{R}$ by $\mathcal{H}^h(E)$; the reader is referred to [**5**] for the definition of $\mathcal{H}^h(E)$. If $t > 0$ and h equals the power function $h(r) = r^t$, then we shall write $\mathcal{H}^h(E) = \mathcal{H}^t(E)$. Recall that a dimension function h is called an *exact dimension function* of a subset E of $\mathbb{R}$ if $\mathcal{H}^h(E)$ is positive and finite. Finally, a subset E of $\mathbb{R}$ is called *dimensionless* if it does not have an exact dimension function, i.e. if $\mathcal{H}^h(E)$ equals 0 or infinity for all dimension functions h.

In this paper we show that if a subset M of the real line satisfies two rather weak invariance conditions shared by many naturally occurring sets (for example, the set of Liouville numbers and the Besicovitch-Eggleston set of non-normal numbers satisfy these invariance conditions), then the h-dimensional Hausdorff measure of M equals 0 or infinity for a large class of dimension functions h.

Observe that if a dimension function h is concave in a neighbourhood of 0, then

$$\liminf_{r \searrow 0} \frac{h(\lambda r)}{\lambda h(r)} \geq 1$$

for all $\lambda \in (0, 1)$. In this paper we consider dimension functions which satisfy a slightly stronger condition. We shall say that a dimension function h is *strongly concave at* 0 if

$$\liminf_{r \searrow 0} \frac{h(\lambda r)}{\lambda h(r)} > 1,$$

for all $\lambda \in (0, 1)$. We shall now give some examples of dimension functions that are strongly concave at 0.

(1) Power functions $h(r) = r^t$ with $t \in (0, 1)$ are strongly concave at 0.

(2) Recall that a continuous function $L : [0, \infty) \to [0, \infty)$ with $L(r) > 0$ for all $r > 0$ is called *slowly varying* if $\lim_{r \searrow 0} \frac{L(\lambda r)}{L(r)} = 1$ for all $\lambda > 0$. Functions of the form

$h(r) = r^t L(r)$, where $t \in (0, 1)$ and $L : [0, \infty) \to [0, \infty)$ is a slowly varying function, are strongly concave at 0.

(3) The dimension function h defined by $h(r) = \frac{1}{\log \frac{1}{r}}$ for $r \in (0, 1)$ and $h(0) = 0$ is strongly concave at 0.

We can now state the main result of this paper.

THEOREM 1. *Let M be a subset of $\mathbb{R}$ satisfying the following two invariance conditions:*

(1) *$M + k \subseteq M$, for all integers k;*
(2) *there exists a positive integer $l \geq 2$ such that $\frac{1}{l}M \subseteq M$.*

Then $\mathcal{H}^h(M) = 0$ or $\mathcal{H}^h(M) = \infty$, for all dimension functions h that are strongly concave at 0.

The proof of Theorem 1 is given in Section 2.

REMARK 1. If a subset M of $\mathbb{R}$ satisfies condition (1) in Theorem 1; i.e. if $M + k \subseteq M$ for all integers k, then in fact $M + k = M$, for all integers k. Indeed, for all integers k we have $M = (M - k) + k \subseteq M + k$ since $M - k \subseteq M$.

REMARK 2. If a non-empty subset M of $\mathbb{R}$ satisfies conditions (1) and (2) in Theorem 1, i.e. if $M + k \subseteq M$ for all integers k and there exists a positive integer $l \geq 2$ such that $\frac{1}{l}M \subseteq M$, then M is dense in $\mathbb{R}$. Indeed, let $x \in \mathbb{R}$ and $r > 0$. Since M is non-empty there exists $t \in M$. Next, choose integers p and q with $q \geq 1$ such that $|x - \frac{p}{l^q}| \leq \frac{r}{2}$ and $\frac{|t|}{l^q} \leq \frac{r}{2}$. Then clearly $\frac{p+t}{l^q} \in \frac{1}{l^q}(p + M) \subseteq \frac{1}{l^q}M \subseteq \frac{1}{l^{q-1}}M \subseteq \ldots \subseteq M$ and $|x - \frac{p+t}{l^q}| \leq |x - \frac{p}{l^q}| + \frac{|t|}{l^q} \leq \frac{r}{2} + \frac{r}{2} = r$. This shows that M is dense in $\mathbb{R}$.

Many naturally occurring sets of numbers satisfy the conditions in Theorem 1. We shall now consider two examples.

EXAMPLE. The Liouville numbers. Let $\mathbb{L}$ denote the set of Liouville numbers, i.e.

$$\mathbb{L} = \left\{ x \in \mathbb{R} \setminus \mathbb{Q} \;\middle|\; \begin{array}{l} \text{for all } n \in \mathbb{N} \text{ there exist integers } p \text{ and } q \\ \text{with } q > 1 \text{ such that } \left| x - \frac{p}{q} \right| < \frac{1}{q^n} \end{array} \right\}.$$

It is well known that the Hausdorff dimension of $\mathbb{L}$ is 0, cf. for example Oxtoby's book [**6**, Theorem 2.4] for a simple direct proof or [**1**, p. 69] for a proof based on Jarník's theorem. In particular, this implies that the t-dimensional Hausdorff measure $\mathcal{H}^t(\mathbb{L})$ of $\mathbb{L}$ equals 0, for all $t > 0$. It is therefore natural to ask whether or not $\mathbb{L}$ is dimensionless. It follows easily from the definition of the Liouville numbers that $\mathbb{L} + k \subseteq \mathbb{L}$ and $\frac{1}{k}\mathbb{L} \subseteq \mathbb{L}$, for all non-zero integers k, and, by applying Theorem 1 to $\mathbb{L}$, we obtain the following result.

THEOREM 2. *Let h be a dimension function that is strongly concave at* 0. *Then $\mathcal{H}^h(\mathbb{L}) = 0$ or $\mathcal{H}^h(\mathbb{L}) = \infty$.*

EXAMPLE. The Besicovitch-Eggleston set of non-normal numbers. Let $N \geq 2$ be a fixed positive integer, and for $x \in \mathbb{R}$ let $x = [x] + \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{N^n}$, where $[x] \in \mathbb{Z}$ and

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$\varepsilon_n(x) \in \{0, 1, \ldots, N-1\}$, denote the unique non-terminating N-adic expansion of x. For each digit $i \in \{0, 1, \ldots, N-1\}$, we write

$$\Pi_i(x; n) = \frac{|\{1 \leq k \leq n \mid \varepsilon_k(x) = i\}|}{n};$$

so $\Pi_i(x; n)$ denotes the frequency of the digit i among the first n digits in the N-adic expansion of x. For a given probability vector $\mathbf{p} = (p_0, p_1, \ldots, p_{N-1})$, the Besicovitch-Eggleston set $B(\mathbf{p})$ is defined by

$$B(\mathbf{p}) = \{x \in \mathbb{R} \mid \Pi_i(x; n) \to p_i \text{ as } n \to \infty \text{ for all } i\}.$$

Besicovitch [**2**] and Eggleston [**4**] computed the Hausdorff dimension, $\dim B(\mathbf{p})$, of $B(\mathbf{p})$. In fact, they proved that $\dim B(\mathbf{p}) = -\frac{\sum_i p_i \log p_i}{\log N}$; the reader is referred to the textbook [**3**, p. 142] for a contemporary proof of this result based on the ergodic theorem. It is natural to ask whether or not the Besicovitch-Eggleston set $B(\mathbf{p})$ is dimensionless. Since clearly $B(\mathbf{p}) + k \subseteq B(\mathbf{p})$ for all integers k and $\frac{1}{N} B(\mathbf{p}) \subseteq B(\mathbf{p})$, an application of Theorem 1 gives the following result.

THEOREM 3. *Let h be a dimension function that is strongly concave at* 0. *Then* $\mathcal{H}^h(B(\mathbf{p})) = 0$ *or* $\mathcal{H}^h(B(\mathbf{p})) = \infty$.

In fact, using the law of the iterated logarithm (rather than relying on the invariance properties of the set $B(\mathbf{p})$), Smorodinsky [**7**] proved the following stronger version of Theorem 3: $\mathcal{H}^h(B(\mathbf{p})) = 0$ or $\mathcal{H}^h(B(\mathbf{p})) = \infty$ for all concave dimension functions.

2. Proof of Theorem 1. We shall now prove Theorem 1. We first state and prove an auxiliary result. For a dimension function h and a positive real number s write

$$\underline{d}_h(s) = \liminf_{r \searrow 0} \frac{h(sr)}{h(r)}$$

and

$$\overline{d}_h(s) = \limsup_{r \searrow 0} \frac{h(sr)}{h(r)}.$$

PROPOSITION 4. *Let h be a dimension function and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a similarity with ratio equal to $s > 0$; i.e. $|f(x) - f(y)| = s|x - y|$, for all $x, y \in \mathbb{R}^n$. Then*

$$\underline{d}_h(s) \mathcal{H}^h(E) \leq \mathcal{H}^h(f(E)) \leq \overline{d}_h(s) \mathcal{H}^h(E),$$

for all $E \subseteq \mathbb{R}^n$.

Proof. We write $|B|$ for the diameter of a subset B of $\mathbb{R}^n$. For a positive real number δ, we denote the δ approximative h-dimensional Hausdorff measure by $\mathcal{H}^h_\delta$; the reader is referred to [**5**] for the definition of $\mathcal{H}^h_\delta$.

Part 1. We first prove that $\mathcal{H}^h(f(E)) \geq \underline{d}_h(s) \mathcal{H}^h(E)$. Let $\varepsilon > 0$, and choose $r_\varepsilon > 0$ such that $\frac{h(sr)}{h(r)} \geq \underline{d}_h(s) - \varepsilon$ for all $0 < r < r_\varepsilon$. Next, fix $0 < \delta < r_\varepsilon$, and let $(B_i)_i$ be an $s\delta$-cover of $f(E)$. Since $|f^{-1}(B_i)| = \frac{1}{s}|B_i|$, we conclude that $(f^{-1}(B_i))_i$ is a δ-cover of E.

Hence

$$\sum_i h(|B_i|) = \sum_i \frac{h(|B_i|)}{h(\frac{1}{s}|B_i|)} h(|f^{-1}(B_i)|) \geq \sum_i (\underline{d}_h(s) - \varepsilon)\, h(|f^{-1}(B_i)|)$$
$$\geq (\underline{d}_h(s) - \varepsilon)\, \mathcal{H}^h_\delta(E).$$

This implies that $\mathcal{H}^h_{s\delta}(f(E)) \geq (\underline{d}_h(s) - \varepsilon)\mathcal{H}^h_\delta(E)$ for all $0 < \delta < r_\varepsilon$. Letting first $\delta \searrow 0$ and then letting $\varepsilon \searrow 0$ gives $\mathcal{H}^h(f(E)) \geq \underline{d}_h(s)\mathcal{H}^h(E)$.

Part 2. Next we prove that $\mathcal{H}^h(f(E)) \leq \overline{d}_h(s)\mathcal{H}^h(E)$. Let $\varepsilon > 0$, and choose $r_\varepsilon > 0$ such that $\frac{h(sr)}{h(r)} \leq \overline{d}_h(s) + \varepsilon$ for all $0 < r < r_\varepsilon$. Next, fix $0 < \delta < r_\varepsilon$, and let $(B_i)_i$ be an δ-cover of E. Since $|f(B_i)| = s|B_i|$, we conclude that $(f(B_i))_i$ is an $s\delta$-cover of $f(E)$. Hence

$$\sum_i h(|B_i|) = \sum_i \frac{h(|B_i|)}{h(s|B_i|)} h(|f(B_i)|) \geq \sum_i \frac{1}{\overline{d}_h(s) + \varepsilon}\, h(|f(B_i)|)$$
$$\geq \frac{1}{\overline{d}_h(s) + \varepsilon}\, \mathcal{H}^h_{s\delta}(f(E))\,.$$

This implies that $(\overline{d}_h(s) + \varepsilon)\, \mathcal{H}^h_\delta(E) \geq \mathcal{H}^h_{s\delta}(f(E))$ for all $0 < \delta < r_\varepsilon$. Letting first $\delta \searrow 0$ and then letting $\varepsilon \searrow 0$ gives $\mathcal{H}^h(f(E)) \leq \overline{d}_h(s)\mathcal{H}^h(E)$. □

We can now prove Theorem 1.

Proof of Theorem 1. It follows from the assumptions on M that $M \cap [0, 1) = (M \cap [k, k+1)) - k$, for all integers k, and Proposition 4 therefore implies that

$$\mathcal{H}^h(M \cap [0, 1)) = \mathcal{H}^h(M \cap [k, k+1)).$$

Hence, it suffices to prove that $\mathcal{H}^h(M \cap [0, 1))$ equals 0 or infinity. Write $I = [0, 1)$, and for $k = 0, 1, \dots, l-1$ put $I_k = [\frac{k}{l}, \frac{k+1}{l})$. Also define maps $f_k : I \to I_k$ by $f_k(x) = \frac{x+k}{l}$. Since $M + k \subseteq M$, for all integers k, and $\frac{1}{l}M \subseteq M$, we conclude that $f_k(M \cap I) \subseteq M \cap I_k$, whence $\cup_{k=0}^{l-1} f_k(M \cap I) \subseteq M \cap I$. This implies that

$$\begin{aligned}
\mathcal{H}^h(M \cap I) &\geq \mathcal{H}^h\left(\bigcup_{k=0}^{l-1} f_k(M \cap I)\right) \\
&= \sum_{k=0}^{l-1} \mathcal{H}^h(f_k(M \cap I)) \\
&\geq \sum_{k=0}^{l-1} \underline{d}_h\left(\frac{1}{l}\right) \mathcal{H}^h(M \cap I) \\
&= l\underline{d}_h\left(\frac{1}{l}\right) \mathcal{H}^h(M \cap I). \qquad (1)
\end{aligned}$$

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However, since h is strongly concave at 0, we have

$$l\underline{d}_h\left(\tfrac{1}{l}\right) = \liminf_{r\searrow 0} \frac{h\left(\frac{1}{l}r\right)}{\frac{1}{l}h(r)} > 1\,. \tag{2}$$

It now follows from (1) and (2) that $\mathcal{H}^h(M \cap I)$ equals 0 or infinity. □

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