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ON THE INTEGRAL MODULUS OF CONTINUITY OF FOURIER SERIES

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Abstract

For a wide class of sine trigonometric series we obtain an estimate for the integral modulus of continuity.

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1. Introduction

Let $F(x)$ be a function of period 2π in L_p $(1 \leqslant p < \infty)$. Then the integral modulus of continuity of order k of F in L_p is defined by

$$\omega_p^k(h; F) = \sup_{0<|t|\leqslant h} \|\Delta_t^k F(x)\|_{L_p},$$

where

$$\Delta_t^k F(x) = \sum_{\alpha=0}^{k} (-1)^{k-\alpha} \binom{k}{\alpha} F(x+\alpha t)$$

and $\|\cdot\|_{L_p}$ denotes the norm in L_p.

Concerning the integral modulus of continuity of order 1 of a sine series whose coefficients form a quasiconvex null sequence, Izumi [2] and Teljakovskiĭ [5] have obtained some interesting estimates. The class of quasiconvex null sequence has further been extended by Teljakovskiĭ [6] in the following form.

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Let

$$\sum_{k=1}^{\infty} a_k \sin kx \tag{1.1}$$

be a sine series satisfying $a_k = o(1)$, $k \to \infty$. If there exists a sequence $\langle A_k \rangle$ such that

$$A_k \downarrow 0, \qquad k \to \infty, \tag{1.2}$$

$$\sum_{k=0}^{\infty} A_k < \infty, \tag{1.3}$$

$$|a_k - a_{k+1}| = |\Delta a_k| \leqslant A_k \quad \text{for all } k, \tag{1.4}$$

then we say that (1.1) belongs to the class S.

Setting $A_k = \sum_{m=k}^{\infty} |\Delta^2 a_m|$, we observe that every quasiconvex null sequence satisfies the condition S.

Let $g(x)$ be the sum of the sine series (1.1) belonging to the class S. Teljakovskiĭ [6] showed that the condition

$$\sum_{k=1}^{\infty} \frac{|a_k|}{k} < \infty \tag{1.5}$$

is sufficient for the integration of the series (1.1) belonging to the class S.

The aim of this paper is to find an estimate for the integral modulus of continuity of order k of the series (1.1) belonging to the class S.

2. Results

We establish the following

THEOREM. *If* (1.1) *belongs to the class S and* (1.5) *holds, then*

$$\omega_1^k\left(\frac{1}{n}; g\right) \leqslant B_k n^{-k} \log n \sum_{v=1}^{n} (v+1)^{k+1} \Delta A_v + B_k \sum_{v=n+1}^{\infty} (v+1)\left(1 + \log \frac{v}{n}\right) \Delta A_v,$$

where B_k is a constant depending upon k and not necessarily the same at each occurrence.

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Letting $A_v = \sum_{m=v}^{\infty} |\Delta^2 a_m|$, the case $k = 1$ of our theorem yields

COROLLARY. *If $\langle a_k \rangle$ is quasiconvex null sequence satisfying* (1.5), *then*

$$\omega_1\left(\frac{1}{n}; g\right) \leqslant Bn^{-1}\log n \sum_{v=1}^{n} (v+1)^2 |\Delta^2 a_v|$$
$$+ B \sum_{v=n+1}^{\infty} (v+1)\left(1 + \log \frac{v}{n}\right) |\Delta^2 a_v|.$$

This result corresponds to a theorem of Izumi [2] as stated in Teljakovskiĭ [5].

3. Proof of the theorem

Under the assumed hypothesis, g is integrable. Since the symmetry of the function implies $|\Delta_t^k g(-x)| = |\Delta_{-t}^k g(x)|$, therefore

$$\int_{-\pi}^{\pi} |\Delta_t^k g(x)|\,dx = \int_0^{\pi} |\Delta_{-t}^k g(x)|\,dx + \int_0^{\pi} |\Delta_t^k g(x)|\,dx.$$

Hence, to prove the theorem, it is sufficient to evaluate

$$\int_0^{\pi} |\Delta_{\pm t}^k g(x)|\,dx, \qquad \text{for } 0 < t \leqslant \pi/n.$$

We write

$$\int_0^{\pi} |\Delta_{\pm t}^k g(x)|\,dx = \int_0^{(k+1)\pi/n} + \int_{(k+1)\pi/n}^{\pi} \tag{3.1}$$
$$= I_1 + I_2, \text{ say.}$$

We first estimate I_1. Denoting by $\tilde{D}_v(x)$ the kernel conjugate to the Dirichlet kernel, the use of partial summation yields

$$g(x) = \sum_{v=1}^{\infty} \Delta a_v \tilde{D}_v(x)$$
$$= \sum_{v=1}^{\infty} A_v \frac{\Delta a_v}{A_v} \tilde{D}_v(x)$$
$$= \sum_{v=1}^{\infty} \Delta A_v \sum_{i=0}^{v} \frac{\Delta a_i}{A_i} \tilde{D}_i(x).$$

Then

$$I_1 \leqslant \sum_{v=1}^{n}\left[\Delta A_v \int_0^{(k+1)\pi/n} \sum_{i=0}^{v} |\Delta^k_{\pm t}\tilde{D}_i(x)|\,dx\right]$$
$$+\int_0^{(k+1)\pi/n}\left|\Delta^k_{\pm t}\sum_{v=n+1}^{\infty}\Delta A_v \sum_{i=0}^{v}\tilde{D}_i(x)\right|dx$$
$$= I_{11} + I_{12}.$$

If $\tilde{D}_i^{(k)}(x)$ denotes the kth derivative of $\tilde{D}_i(x)$, then to estimate I_{11} we use the equality (Aljančić [1], Ram [3])

$$(3.2)\qquad |\tilde{D}_i^{(k)}(x)| = \begin{cases} B_k i^{k+1}, & 0 \leqslant x \leqslant \pi \\ B_k i^k x^{-1}, & 0 < x \leqslant \pi \end{cases} \quad (k = 1, 2, \ldots)$$

and obtain

$$I_{11} \leqslant B_k t^k \sum_{v=1}^{n}\Delta A_v \int_0^{(k+1)\pi/n}\left(\sum_{i=0}^{v}|\tilde{D}_i^{(k)}(x \pm \theta_i t)|\right)dx$$
$$(0 < \theta_i < k)$$
$$\leqslant B_k n^{-k}\sum_{v=1}^{n}\Delta A_v (v+1)^{k+1}.$$

To estimate I_{12}, we use the inequality (Timan [7])

$$\frac{1}{\pi}\int_0^{c/n}|\tilde{D}_v(x)|\,dx \leqslant \frac{2}{\pi}\log\frac{v}{n} + o(1), \qquad c > 0,\ v \geqslant n,$$

and obtain

$$I_{12} \leqslant B_k\left(\sum_{v=n+1}^{\infty}\Delta A_v \sum_{i=2}^{v}\left[\log\frac{i}{n} + o(1)\right]\right)$$
$$= B_k\left(\sum_{v=n+1}^{\infty}\Delta A_v\left[(v+1)\log\frac{v}{n} + (v+1)\right]\right).$$

It follows therefore that

$$(3.3)\qquad I_1 \leqslant B_k n^{-k}\sum_{v=1}^{n}(v+1)^{k+1}\Delta A_v$$
$$+B_k\left[\sum_{v=n+1}^{\infty}(v+1)\left(1+\log\frac{v}{n}\right)\Delta A_v\right].$$

To estimate I_2, we have

$$I_2 = \int_{(k+1)\pi/n}^{\pi} \left|\Delta^k_{\pm t} g(x)\right| dx$$
$$\leqslant \int_{(k+1)\pi/n}^{\pi} \left| \sum_{v=1}^{n} \Delta a_v \Delta^k_{\pm t} \tilde{D}_v(x) \right| dx$$
$$+ \int_{(k+1)\pi/n}^{\pi} \left| \Delta^k_{\pm t} \sum_{v=n+1}^{\infty} \Delta a_v \tilde{D}_v(x) \right| dx$$
$$= I_{21} + I_{22}.$$

We now write

$$I_{21} \leqslant \sum_{m=1}^{n-1} \int_{(k+1)\pi/(m+1)}^{(k+1)\pi/m} \left| \sum_{v=1}^{n} \Delta a_v \Delta^k_{\pm t} \tilde{D}_v(x) \right| dx.$$

By virtue of $t \leqslant \pi/n$ and $x \geqslant (k+1)\pi/(m+1)$, it follows that

$$x - kt \geqslant \frac{k+1}{m+1}\pi - \frac{k}{n}\pi = \frac{\pi}{m+1} + k\pi\left(\frac{1}{m+1} - \frac{1}{n}\right) \geqslant \frac{\pi}{m+1}.$$

Therefore in the subinterval $[(k+1)\pi/(m+1), (k+1)\pi/m]$, using (3.2), we have

$$\left| \sum_{v=1}^{n} \Delta a_v \Delta^k_{\pm t} \tilde{D}_v(x) \right| \leqslant B_k t^k \sum_{v=1}^{n} \left| A_v \frac{\Delta a_v}{A_v} \right| \max_{x-kt \leqslant \xi \leqslant x+kt} \left| \tilde{D}_v^{(k)}(\xi) \right|$$
$$\leqslant B_k t^k \sum_{v=1}^{m} v^{k+1} \left| A_v \frac{\Delta a_v}{A_v} \right| + \frac{B_k t^k}{x - kt} \sum_{v=m+1}^{n} v^k \left| A_v \frac{\Delta a_v}{A_v} \right|$$
$$\leqslant B_k t^k \sum_{v=1}^{m} v^{k+1} A_v + B_k t^k m \sum_{v=m+1}^{n} v^k A_v.$$

But

$$\sum_{v=1}^{m} v^{k+1} A_v = \sum_{v=1}^{m} \Delta A_v \sum_{i=0}^{v} i^{k+1} + A_{m+1} \sum_{i=0}^{m} i^{k+1}$$
$$\leqslant \sum_{v=1}^{m} (v+1)^{k+2} \Delta A_v + m^{k+2} A_{m+1},$$

and

$$\sum_{v=m+1}^{n} v^k A_v = \sum_{v=m+1}^{n} \Delta A_v \sum_{i=0}^{v} i^k + A_{n+1} \sum_{i=0}^{n} i^k - A_{m+1} \sum_{i=0}^{m} i^k$$
$$\leqslant \sum_{v=m+1}^{n} (v+1)^{k+1} \Delta A_v + n^{k+1} A_{n+1}.$$

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Therefore

$$I_{21} \leqslant B_k n^{-k}\left[\sum_{m=1}^{n-1} m^{-2}\left(\sum_{v=1}^{m}(v+1)^{k+2}\Delta A_v + m^{k+2}A_{m+1}\right)\right]$$
$$+B_k n^{-k}\left[\sum_{m=1}^{n-1} m^{-1}\left(\sum_{v=m+1}^{n}(v+1)^{k+1}\Delta A_v + n^{k+1}A_{n+1}\right)\right]$$
$$= B_k n^{-k}\left[\sum_{m=1}^{n-1} m^{-2}\sum_{v=1}^{m}(v+1)^{k+2}\Delta A_v + \sum_{m=1}^{n-1} m^k A_{m+1}\right.$$
$$\left. + \sum_{m=1}^{n-1} m^{-1}\sum_{v=m+1}^{n}(v+1)^{k+1}\Delta A_v + \sum_{m=1}^{n-1} m^{-1}n^{k+1}A_{n+1}\right].$$

The first term in the square bracket is

$$\sum_{v=1}^{n-1}(v+1)^{k+2}\Delta A_v\left(\sum_{m=v}^{n-1} m^{-2}\right) \leqslant \sum_{v=1}^{n-1}(v+1)^{k+2}\Delta A_v\left(\sum_{m=v}^{\infty} m^{-2}\right)$$
$$\leqslant B_k\sum_{v=1}^{n-1}(v+1)^{k+1}\Delta A_v,$$

the second term is

$$\sum_{m=1}^{n-1} m^k A_{m+1} = \sum_{m=1}^{n-1}\Delta A_{m+1}\sum_{i=0}^{m} i^k + A_n\sum_{i=0}^{n} i^k$$
$$\leqslant \sum_{m=1}^{n-1} m^{k+1}\Delta A_{m+1} + n^{k+1}A_n,$$

and the third term is

$$\sum_{m=1}^{n-1} m^{-1}\sum_{v=m+1}^{n}(v+1)^{k+1}\Delta A_v = \sum_{v=1}^{n-1}(v+1)^{k+1}\Delta A_v\sum_{m=1}^{v-1} m^{-1}$$
$$\leqslant B_k\sum_{v=2}^{n-1}(v+1)^{k+1}\Delta A_v\log v$$
$$\leqslant B_k\log n\sum_{v=1}^{n-1}(v+1)^{k+1}\Delta A_v.$$

Therefore

$$I_{21} \leqslant B_k n^{-k}\log n\sum_{v=1}^{n}(v+1)^{k+1}\Delta A_v.$$

Lastly, making use of Abel's transformation and Fomin's Lemma (Ram [4], Lemma 1), we have

$$I_{22} \leqslant \sum_{\alpha=0}^{k} \binom{k}{\alpha} \int_{(k+1)\pi/n \pm \alpha t}^{\pi \pm \alpha t} \left| \sum_{v=n+1}^{\infty} \Delta a_v \tilde{D}_v(x) \right| dx$$

$$\leqslant B_k \int_{\pi/n}^{\pi + k\pi/n} \left| \sum_{v=n+1}^{\infty} \Delta a_v \tilde{D}_v(x) \right| dx$$

$$\leqslant B_k \int_{\pi/n}^{(k+1)\pi} \left| \sum_{v=n+1}^{\infty} A_v \frac{\Delta a_v}{A_v} \tilde{D}_v(x) \right| dx$$

$$= B_k \int_{\pi/n}^{(k+1)\pi} \left[\left| \sum_{v=n+1}^{\infty} \Delta A_v \sum_{i=0}^{v} \alpha_i \tilde{D}_i(x) \right| + A_{n+1} \left| \sum_{i=0}^{n} \alpha_i \tilde{D}_i(x) \right| \right] dx \quad \left(\alpha_i = \frac{\Delta a_i}{A_i} \right)$$

$$\leqslant B_k \left[\sum_{v=n+1}^{\infty} \Delta A_v \int_0^{(k+1)\pi} \left| \sum_{i=0}^{v} \alpha_i \tilde{D}_i(x) \right| dx + A_{n+1} \int_0^{(k+1)\pi} \left| \sum_{i=0}^{n} \alpha_i \tilde{D}_i(x) \right| dx \right]$$

$$\leqslant B_k \left[\sum_{v=n+1}^{\infty} (v+1) \Delta A_v + (n+1) A_{n+1} \right]$$

$$\leqslant B_k \sum_{v=n+1}^{\infty} (v+1) \Delta A_v.$$

Hence

$$I_2 \leqslant B_k \left[n^{-k} \log n \sum_{v=1}^{n} (v+1)^{k+1} \Delta A_v + \sum_{v=n+1}^{\infty} (v+1) \Delta A_v \right]. \tag{3.4}$$

The conclusion of the Theorem follows from (3.1), (3.3), and (3.4).

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