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Angular momentum and overshooting: two as yet unsolved problems in stellar mixing

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Abstract. Helioseismological data have given us two interesting results: *the differential-to-uniform solar rotation curve and the extent of the overshooting region (OV)*. As of today, no model (including numerical simulations) has been able to reproduce these findings. Here, we first present a new model for the angular momentum. It contains new terms representing *vorticity and buoyancy* that were left out in all previous formulations without a clear justification. It is shown that they extract angular momentum from the stellar core, a welcome feature since the standard angular momentum equation leads to a rotation curve that is considerably higher than what is observed. As for the overshooting extent, all models yield values that are an order of magnitude larger than the helio data of $0.07H_p$. We propose a criterion whose main ingredient is a new flux conservation law that includes new terms, one of which increases the dissipation in the radiative zone and thus lowers the OV extent, a tendency in the desired direction. Since we have not coupled the new models to a solar structure-evolution code, we cannot at this stage carry out a comparison with the helio data. The purpose is to exhibit the fact that in both cases the missing ingredients are of such nature as to improve the previous model predictions. A proper quantification remains to be done.

Keywords. Stars: abundances, convection, turbulence

1. Introduction

Since mixing in stars is a complex interplay of processes as diverse as unstable stratification, stable stratification, differential rotation, gravity waves, double-diffusion, etc., the formalism employed to model mixing should be sufficiently general to account for such a large variety of processes. And yet, the literature shows that this is not the case, the two methodologies being employed being: a) large scale numerical simulations and b) heuristic models. In the first case (e.g., Brummell *et al.*, 2002), the values of several parameters, such as the Prandtl number, are widely different from those in stars. The authors of those studies have however stated that their primary goal was the elucidation of the intertwined physical processes and not to provide stellar studies with tools to model the processes of interest. The consequence is that mixing processes are still modeled using heuristic arguments that have severe limitations, as we discuss in section 5. Our assessment is that heuristic models should be abandoned for lack of physical completeness. As a substitute we suggest, work out and assess models of at most algebraic complexity that avoid the guessing work that heuristic models always entail. It is instructive to point out that an analogous situation existed in geophysics, specifically in modeling atmospheric and oceanic mixing. More than 25 years ago, it was decided to forgo the heuristic approach in favor of a more predictive and flexible tool known as RSM

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(Reynolds Stress Methodology) which is now commonly used. For reasons unclear to the present authors, stellar mixing studies are lagging behind geophysical studies and this paper will thus discuss a new model as well as the limitations of the heuristic models. The RSM is a set of equations for the turbulent correlations of the velocity and temperature fields that follow from the Navier-Stokes equations and the temperature equations. The RMS'main features can be summarized as follows: *mathematical structure*, the relevant equations are a set of linear algebraic equations and thus pose no particular numerical problems; *flexibility*, which is one of the key advantages, means that adding new processes such as rotation, vorticity, double-diffusion, etc, does not require guessing work since the RSM has a well defined set of procedural rules; *assessment*, the results of the RSM can be assessed before being used in a stellar context, an important feature that none of the heuristic model satisfies, raising the justified doubt that these models were constructed and tailored to a specific astrophysical setting, a feature that limits their predictive power. Since the RSM was discussed in detail elsewhere, we refer to that work for more details (Canuto, 2008).

2. The angular momentum problem

The angular momentum equation is given by ($\Gamma = sin\theta$):

$$\frac{\partial}{\partial t}\left(r^2\Omega\right) = -\Gamma^{-1}r^{-2}\frac{\partial}{\partial r}\left(r^3\tau_{r\phi}\right) - \Gamma^{-3}\frac{\partial}{\partial \theta}\left(\Gamma^2\tau_{\theta\phi}\right) + ... \tag{2.1}$$

where $\tau_{ij} = \overline{u_i u_j}$ are the Reynolds stresses. If one assumes that $\tau_{r\phi}$ is governed only by shear S_{ij}, one has:

$$\tau_{r\phi} = -2K_m S_{r\phi}, \quad S_{r\phi} = \frac{1}{2}r\Gamma\frac{\partial\Omega}{\partial r}, \quad 2S_{ij} = \overline{u}_{i,j} + \overline{u}_{j,i} \tag{2.2}$$

where K_m is a momentum diffusivity, (2.1) then becomes (Talon and Zahn, 1998; Talon and Charbonnel, 2003; Palacios *et al.*, 2003, 2006):

$$\frac{\partial}{\partial t}\left(r^2\Omega\right) = r^{-2}\frac{\partial}{\partial r}\left(r^4 K_m\frac{\partial\Omega}{\partial r}\right) + ... \tag{2.3}$$

Thompson *et al.* (2003) have written that (2.3) predicts rotation of the solar interior at a rate several times higher than the surface rate in stark disagreement with helio data of nearly uniform rotation. The first obvious conclusion is that since shear alone does not fully represent the mean flow, one must also include *vorticity*:

$$V_{ij} = \frac{1}{2}\left(\overline{u}_{i,j} - \overline{u}_{j,i}\right), \quad V_{r\phi} = -\frac{1}{2}r^{-1}\Gamma\frac{\partial}{\partial r}\left(r^2\Omega\right) \tag{2.4}$$

which leads to a real "diffuse nature" of the angular momentum. Furthermore, it seems natural that if one wants to explain the different behavior of the solar rotation curve in the *convective and radiative* zones, one must have an "ingredient" capable of differentiating between the two regimes. The obvious candidate is the buoyancy flux that is positive in the first case and negative in the second. On the basis of these qualitative considerations, one concludes that in addition to shear and vorticity, the Reynolds stresses must also depend on buoyancy and thus $\tau_{ij}(S, V, B)$. Finally, one must account for possible radiative losses and thus the formalism must include a Peclet number. We conclude that the final form of the Reynolds stresses must be:

$$\tau_{ij}\left(S, V, B, Pe\right) \tag{2.5}$$

3. Reynolds stresses and heat fluxes

In the presence of shear, vorticity, buoyancy and radiative losses, the general form of the traceless Reynolds stress tensor $b_{ij} = \tau_{ij} - \delta_{ij}2K/3$ is (Canuto and Minotti, 2001):

$$A\tau^{-1}b_{ij} = -\frac{8K}{15}S_{ij} - \frac{1}{2}Z_{ij} + \frac{1}{2}B_{ij} \tag{3.1}$$

where

$$Z_{ij} = b_{ik}V_{jk} + b_{jk}V_{ik}, \quad B_{ij} = \alpha\left(g_iJ_j + g_jJ_i\right) - \frac{2}{3}\delta_{ij}\alpha g_kJ_k \tag{3.2}$$

Here, $A = 5$, $\alpha = -\rho_0^{-1}\partial\overline{\rho}/\partial T$ is the volume expansion coefficient and $J_i = \overline{u_i\theta}$ is the heat flux for which the RSM provides the following equations (Canuto and Minotti, 2001):

$$\tau^{-1}A_{ij}J_j = -\tau_{ij}\frac{\partial T}{\partial x_j}, \quad \tau = 2K\epsilon^{-1} \tag{3.3}$$

$$A_{ij} = \lambda_5\delta_{ij} + \lambda_6\tau S_{ij} + \lambda_7\tau V_{ij} + \lambda_8\tau^2\alpha g_i\frac{\partial T}{\partial x_j} + 2\epsilon_{ipj}\Omega_p^0 \tag{3.4}$$

with $\lambda_{6,7,8} = 0.786, 0.643, 0.547$. Canuto and Dubovikov (1998) and Canuto (2008) showed that the Peclet number dependence enters via the two remaining variables:

$$\lambda_5^{-1} = aPe(1+bPe)^{-1}(1+Ri)^{-1}, \quad \lambda_8 = cPe(1+dPe)^{-1} \tag{3.5}$$

$$a = (4\pi^2)^{-1}, \quad b = 5a(1+\sigma_t^{-1}), \quad c = \frac{8}{3}(7\pi^2)^{-1}, \quad d = 4(7\pi^2)^{-1}\sigma_t^{-1} \tag{3.6}$$

where $\sigma_t = 0.72$ and Ri is the Richardson number which is present only in the stable stratification case.

4. New results of the RSM model

We have numerically solved the set of linear algebraic Eqs.(3) and in figure 1 we present the heat diffusivity as a function of Ri and for different values of Pe; in figure 2 we plot the momentum diffusivity while in Fig.3 we plot the turbulent Prandtl number $\sigma_t(Ri, Pe) \equiv K_m/K_h$, which is the ratio between momentun and heat diffusivities. As one can see, the ratio in not constant but increases with Ri. For large Pe, we have superimposed a variety of LES, DNS, lab and direct measurements (Canuto *et al.*, 2008) that the model reproduces quite well.

An example of the results that are obtained from our method is given in figure 1.

5. Previous heuristic mixing model

Here, we compare the results of Figs.1–3 with those of the heuristic relations used by various authors (Mathis *et al.*, 2004; Palacios *et al.*, 2003, 2006; Charbonnel and Talon, 2005, 2007; Maeder and Meynet, 2001):

$$Pe >> 1: \quad \frac{K_{m,h}}{\chi} = 2\frac{Ri(cr)}{Ri} = \frac{1}{3Ri} \tag{5.1}$$

where $\chi(cm^2s^{-1})$ is the radiative diffusivity. (5.1) is consistent with Fig. 1 if:

$$Pe \approx 10^2 \tag{5.2}$$

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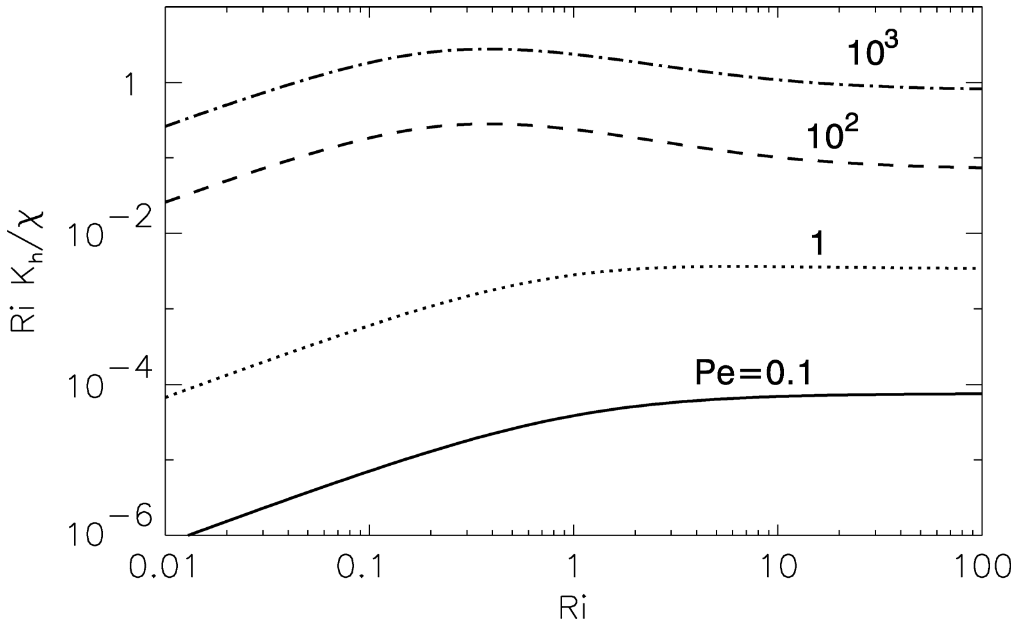

Figure 1. The heat diffusivity K_h in units of the radiative diffusivity vs. the Richardson number Ri for different values of the Peclet number. As expected, the stronger the effect of stable stratification, the larger is the value of Ri and the smaller is the resulting diffusivity. We plot K_h multiplied by Ri in order to allow a direct comparison with the heuristic model (5.1). See the text for details.

As for the momentum diffusivity in Fig. 2, (5.1) is not satisfied. The only possibility is to impose (5.1) but that in turn implies a unique value of Ri equal to:

$$Ri \approx 0.1 \tag{5.3}$$

This means that the heuristic model is valid for only one combination of Pe, Ri given by (5.2) and (5.3) which is quite unusual for any model. Finally, (5.1) implies a turbulent Prandtl number of unity whereas Fig. 3 shows that it is a rather strong function of Ri unless one limits the validity of the model to $Ri << 1$.

6. New angular momentum equation

Using a method of symbolic algebra, we have solved Eqs. (3) without meridional currents. Introducing three dimensionless variables x, X and Z to characterize stratification, shear and vorticity in units of the dynamical time scale $\tau = 2K/\epsilon$:

$$x \equiv (\tau N)^2, \quad X \equiv \tau S_{r\phi}, \quad Z \equiv \tau V_{r\phi} \tag{6.1}$$

the explicit form of the Reynolds stress that enters (2.1) turns out to be:

$$\tau_{r\phi} = -2K_m^{(1)} S_{r\phi} - 2xK_m^{(2)} V_{r\phi} \tag{6.2}$$

which yields the new angular momentum equation to:

$$r^2 \frac{\partial}{\partial t}\left(r^2\Omega\right) = \frac{\partial}{\partial r}\left(K_m^{(1)} r^4 \frac{\partial \Omega}{\partial r}\right) + \frac{\partial}{\partial r}\left(xK_m^{(2)} r^2 \frac{\partial r^2\Omega}{\partial r}\right) \tag{6.3}$$

Several considerations are in order:1) the presence of vorticity has now introduced a new term which has the character of a true diffusion of angular momentum whereas the

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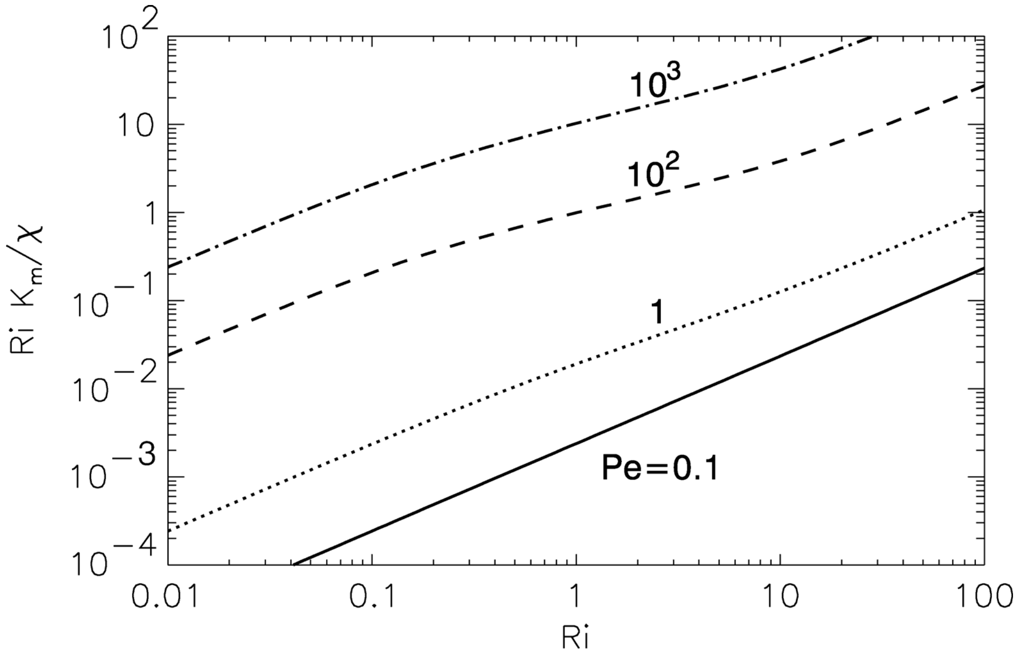

Figure 2. Same as in figure 1 but for the momentum diffusivity K_m.

first traditional term, in spite of being usually called "diffusion", is not, 2) the new term entails vorticity multiplied by buoyancy represented by the variable x which is negative in the convective zone and positive in the radiative, stably stratified, zone, 3) in the latter zone, where we can take $\Omega = constant$ as from the helio data, only the second term in (6.3) survives and since $x > 0$, this implies an *outward transport of angular momentum from the radiative interior*, an "extraction process" that in principle drives it toward a state of uniform rotation, 4) since in the radiative zone turbulence is much weaker than in the CZ, the eddies life time is correspondingly longer and the variable x is an increasing function of Ri making its largest contribution to the second term in (6.3), 5) the two momentum diffusivities are not the same since they themselves depend on x, X, Z but for the purposes of this paper their expressions are not relevant (they can be provided by request to the authors). 6) even if one assumes that the two diffusivities in (6.3) are the same and of the form (5.1), the first term in (6.3) decreases like Ri^{-1} while the combination xK_m decreases with Ri with a lower power.

In summary, *the inclusion of both buoyancy and vorticity leads to a new angular momentum equation which may provide a better model for the helio data since it contains a mechanism to extract angular momentum from the stellar core that is absent in the commonly used formula (2.3).*

7. New equation for the OV extent

As for the OV extent, numerical simulations (Brummel *et al.*, 2002) yield a value of about $0.5H_p$ which is an order of magnitude larger than the helio data of $0.07H_p$. To reconcile model results with the data, we suggest a new criteria for the OV extent. Consider the equation for the turbulent kinetic energy $K(D/Dt \equiv \partial/\partial t + \overline{u}_i \partial/\partial x_i)$:

$$\frac{DK}{Dt} + \frac{\partial F_i^{ke}}{\partial x_i} = P_b + P_m - \epsilon \tag{7.1}$$

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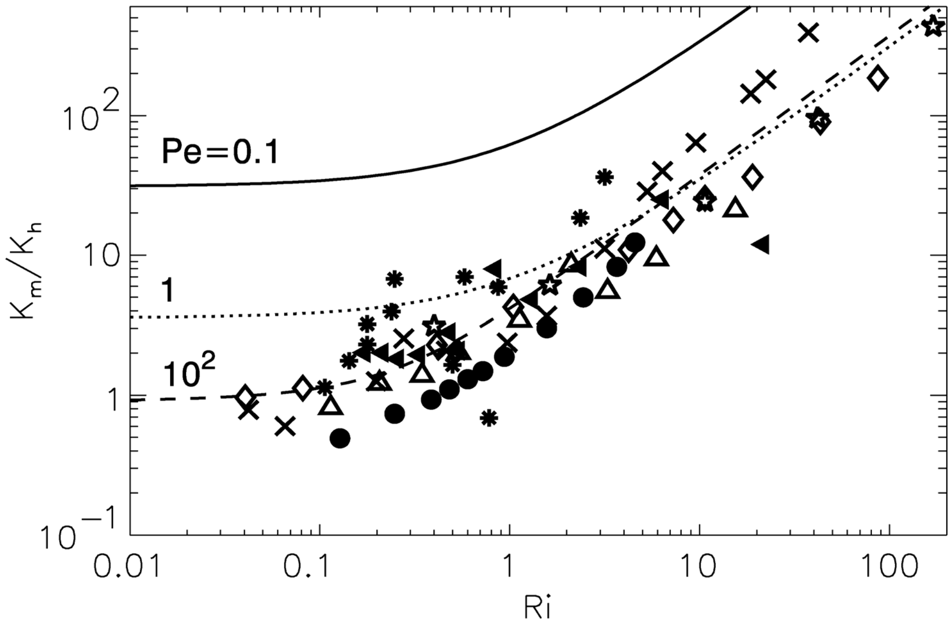

Figure 3. Turbulent Prandtl number $\sigma_t = K_m/K_h$ vs. Ri for different Pe. The data corresponding to the $Pe > 1$ case, are as follows: meteorological observations (Kondo *et al.*, 1978, slanting black triangles; Bertin *et al.*, 1997, snow-flakes), lab experiments (Strang and Fernando, 2001, black circles; Rehmann and Koseff, 2004, slanting crosses; Ohya, 2001, diamonds), LES (Zilitinkevich *et al.*, 2007, 2008, triangles), DNS (Stretch *et al.*, 2001, five-pointed stars).

where F_i^{ke} is the flux of K and $P_{b,m} = (\alpha g_i J_i, -\tau_{ij} S_{ij})$ are the productions of buoyancy and shear. Next, consider the flux conservation law (Canuto, 1997):

$$F_i^{rad} + F_i^{conv} + F_i^{ke} + \overline{u}_j(\mathbf{E}\delta_{ij} + \tau_{ij}) = constant = C \tag{7.2}$$

$$E = c_p T + K + \overline{K} + G, \quad \overline{K} = \frac{1}{2}\overline{\mathbf{u}}^2, \quad g_i \overline{u}_i = \frac{DG}{Dt} \tag{7.3}$$

where $F_i^{conv} = c_p J_i$. While the "traditional" flux conservation law used in stellar models contains only the first two terms in (7.2), one must also account for the flux of K represented by the third term; however, in the presence of mean currents, one has new terms, the first of which represents the flux $\overline{u}_i\mathbf{E}$, where $\mathbf{E}$ is the sum of enthalpy c_pT, turbulent kinetic energy K, mean field kinetic energy $\overline{K}$ and gravitational energy G while the other term is the flux of the Reynolds stresses. Eliminating the heat flux between (7.1) and 7.2), yields in the stationary case the equation for the flux of K:

$$\frac{\partial F_i^{ke}}{\partial x_i} + \frac{\alpha}{c_p} g_i F_i^{ke} = \Phi_{old} + \Phi_{new} \tag{7.4}$$

$$\Phi_{old} = C - \alpha c_p^{-1} g_i F_i^{rad}, \quad \Phi_{new} = -\tau_{ij} S_{ij} - \alpha c_p^{-1} g_i (\tau_{ij}\overline{u}_j + \mathbf{E}\overline{u}_i) - \epsilon \tag{7.5}$$

After some algebra, one obtains that:

$$\Phi_{new} = -\left(1 + T_2 \frac{r}{H_p}\right)\epsilon + (T_0 + T_1)\epsilon - H_p^{-1}(\mathbf{E} + \overline{u_r^2})\overline{u}_r \tag{7.6}$$

The key result is that the first term shows that the dissipation increases with depth thus reducing the extent of the OV compared to the standard criterion without Φ_{new}. This is

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predicated on the fact that:

$$T_2 \sim (\tau N)^2 (\tau\Omega)^2 \tag{7.7}$$

is positive in the radiative zone since $N^2 > 0$. At the same time, the eddies life time is the largest since turbulence is weak and thus $(\tau N)^2 > 1$.

In conclusion, *the new term in the OV equation (7.4) contributes a term that reduces the OV extent which is a desired feature since no model has thus far been able to reproduce the helio data of OV* $\approx 0.07 H_p$.

8. Conclusions

The form of the standard angular momentum equation (2.3) yields results that are not in agreement with helio data since the extraction of angular momentum from the radiative zone is too inefficient. We show that (2.3) is based on a very restricted form of the Reynolds stresses. If one uses the full form of the Reynolds stresses that entails *shear, vorticity and buoyancy*, the combination of the last two ingredients gives rise to a new term that is larger than the canonical one that contains only shear and which leads to extraction of angular momentum from the core, a tendency in the right direction.

As for the OV extent, the key ingredient is the new flux conservation law that entails Reynolds stresses and mean flows. One of the new terms leads to an increased dissipation of the flux of turbulent kinetic energy which in turn entails a smaller OV extent, a welcome feature since thus far all models have yielded an OV extent about an order of magnitude larger than the helio data of $0.07 H_p$.

An interesting aspect of the new models is the relative simplicity of the equations determining Reynolds stresses and heat fluxes since they are given by linear algebraic equations. This is especially relevant if one considers the amount of information they contain: stable stratification, unstable stratification, rigid rotation, shear, and radiative losses (Peclet number). Having established the qualitative behavior of the two models, what is needed next is a specific computation in conjunction with a stellar code.

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Discussion

Langer: For obtaining angular momentum transport in radiative zones, do you assume turbulence to exist?

Canuto: I do assume, as does everybody else, that in the radiative zone there is a variety of instabilities which ultimately will give give rise to a turbulent flow.

Zahn: You mentioned the fact that numerical simulations predict an overshoot which is much too strong. But that is so because such simulations are run with a Peclet number which is too low, owing to the lack of resolution. For a given strength of the convection, the higher thermal diffusion, i.e. the lower the Peclet number, and the deeper is the overshoot, because the buoyancy force is lessened.

Canuto: One thing is what simulations do and another is what the physics of the problem dictates – since the Peclet's number is directly proportional to the rsm turbulent velocity and since the latter is getting smaller as one approaches the bottom of the CZ, so does the Peclet's number – inside the radiative region, such rsm velocity is even smaller and so is Pe – Thus, small Pe, small rsm and small OV distance go together.