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Varieties of topological groups

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We define a variety \underline{V} of topological groups to be a family of topological groups closed under cartesian products, quotient groups and subgroups. The class of groups, which with some topology appear in \underline{V} , is a variety of groups. A topological group $F(X,\underline{V})$ with generating space X is a free topological group of \underline{V} on X if any continuous mapping of X into any G in \underline{V} can be extended to a continuous homomorphism of $F(X,\underline{V})$ into G. Algebraically $F(X,\underline{V})$ is the free group on the set X of the underlying algebraic variety.

If \underline{V} is the variety of all (all abelian) topological groups then $F(X,\underline{V})$ is the free (free abelian) topological group defined by Markov. We generalize many of Markov's results; in particular, a necessary and sufficient condition for $F(X,\underline{V})$ to exist is found and uniqueness is proved. Further, if \underline{V} is non-indiscrete, $F(X,\underline{V})$ is disconnected. Any topological group Y in \underline{V} is a quotient group of $F(Y,\underline{V})$.

A full variety \underline{V} is one which contains every topological group algebraically isomorphic to a group in \underline{V} . If X is any Tychonoff space then $F(X,\underline{V})$ exists, is Hausdorff and has X as a closed subset. Algebraically fully invariant subgroups of $F(X,\underline{V})$ are closed. A topological group $F_M(X)$ is moderately free if its topology is maximal (with respect to X) and it is algebraically relatively free on X. We see that $F(X,\underline{V})$ is moderately free on X. If $F_M(X)$ is in any variety \underline{W} , then it is a quotient group of $F(X,\underline{W})$. Further, $F_M(X)$ is a free topological group of the variety it generates. We prove that finitely generated subgroups of $F_M(X)$ have the discrete topology. If X

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is compact Hausdorff, then we show $F_M(X)$ is a normal space. In fact we explicitly determine the topology of $F_M(X)$ in this case.

A β -variety \underline{V} is one for which $F(X,\underline{V})$ exists and is Hausdorff for each compact Hausdorff space X. For any β -variety \underline{V} and Tychonoff space X, $F(X,\underline{V})$ exists, is Hausdorff and has X as a closed subset. We find a necessary and sufficient condition for a variety to be a β -variety. Whilst every full variety is a β -variety we show that the converse is false.

We prove that the family of all topological varieties with a given underlying algebraic variety is not a set. In fact the family of all β -varieties with a given underlying algebraic variety is not a set. We also prove the striking result that a variety generated by a family of topological groups of bounded cardinal is not full.

Finally, the varieties $\underline{V}(R)$ and $\underline{V}(T)$ generated by the additive group of reals and the circle group respectively, each with its usual topology, are examined. Each is shown to be a β -variety but not a full variety. Further, it is shown that a locally compact Hausdorff abelian group is in $\underline{V}(T)$ if and only if it is compact. Thus R is not in $\underline{V}(T)$ and consequently $\underline{V}(R)$ properly contains $\underline{V}(T)$.

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