Elementary Proof that the Arithmetic Mean of any number of Positive Quantities is greater than the Geometric Mean.*

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Def. The arithmetic mean of n quantities $a, b, c, d \ldots$ is $(a+b+c+d\ldots)/n$.

Def. The geometric mean of n quantities $a, b, c, d \ldots$ is $(abcd\ldots)^{1/n}$.

Lemma 1. Both the arithmetic mean and the geometric mean are intermediate in value between the greatest and the least of the n quantities, if they be all positive.

For, if a be the greatest, and b the least,

the arithmetic mean $= (a+b+c+d\ldots)/n$

$< (a+a+\ldots \text{ to } n \text{ terms})/n$

$< na/n$

i.e., $< a$.

and the arithmetic mean $> (b+b\ldots \text{ to } n \text{ terms})/n$

$> nb/n$

i.e., $> b$.

Similarly,

the geometric mean $= (abc\ldots)^{1/n}$

$< (aa\ldots \text{ to } n \text{ factors})^{1/n}$

$< (a^n)^{1/n}$

i.e., $< a$.

and the geometric mean $> (bbb\ldots \text{ to } n \text{ factors})^{1/n}$

$> (b^n)^{1/n}$

i.e., $> b$.

Q.E.D.

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* The above proof is a modification of the elegant proof given by Dr G. H. Bryan in his *Middle Algebra*, and was obviously suggested by it. No third proof on the same lines can be given. Both have this logical advantage referred to by Dr Bryan, that the number of mental steps in the process is finite G. E. C.

https://doi.org/10.1017/S0013091500029230 Published online by Cambridge University Press

Lemma 2. If two positive quantities a and b have the same sum as two other positive quantities x and y, then the greatest and least of the above four quantities occur together in the same pair, and their product is less than that of the other two quantities, which are intermediate to them.

For, say a is the greatest of the four, then since $a+b=x+y$, we have $a-x=y-b$.

But a is $>x$, $\therefore$ y is $>b$.

Again $a-y=x-b$.

But a is $>y$, $\therefore$ x is $>b$.

Hence both x and y are $>b$;

$\therefore$ b is the least of the four quantities, and it occurs in the same pair with a which is the greatest.

$$\begin{aligned} \text{Also} \qquad 4xy &= (x+y)^2 - (x-y)^2 \\ &= (a+b)^2 - (x-y)^2 \\ &> (a+b)^2 - (a-b)^2, \text{ since } a-b \text{ is } > x-y, \\ &> 4ab. \end{aligned}$$

$$\therefore \quad xy \text{ is } > ab.$$

Lemma 3. If one of n quantities be identical with the arithmetic mean of all the n, then the remaining $n-1$ quantities have also the same arithmetic mean.

For let A, b, c, d . . be the n quantities of which the first, A, is also the arithmetic mean of all,

$$\therefore \quad A+b+c+d\ldots=nA;$$

$$\therefore \quad b+c+d\ldots=(n-1)A;$$

$$\therefore \quad (b+c+d\ldots)/(n-1)=A. \qquad \text{Q.E.D.}$$

Prop. Now let A be the arithmetic mean of n positive quantities a, b, c, d... of which a is the greatest and b the least.

$\therefore$ A is intermediate between a and b (Lemma 1).

Choose x so that $A+x=a+b$.

Then A is not the least of the four quantities A, x, a, b, nor is it the greatest;

$\therefore$ A and x are both intermediate to a and b, which are respectively the greatest and least of the four, (Lemma 2), and . $Ax>ab$

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Also the 3 sets of quantities:

$$a,\ b,\ c,\ d\ .\ .\ .\ (n \text{ in number}),$$
$$A,\ x,\ c,\ d\ .\ .\ .\ (\text{,, ,, ,,}),$$
$$x,\ c,\ d\ .\ .\ .\ (n-1 \text{ in number}),$$

all agree in having the same arithmetic mean, viz., A; the first two sets because $a+b=A+x$, and the last two sets by Lemma 3;

$\therefore\ abcd\ldots < Axcd\ldots\ldots$

$\qquad\qquad < A\,.\,Ayd\ldots$

(by application of the same process, y playing the same part now among the $n-1$ quantities $xcd\ldots$ as x did before among the n quantities $abcd\ldots$).

Or, finally, $abcd\ldots < A\,.\,A\,.\,A\ldots$ to n factors,

$\qquad\qquad$ *i.e.*, $< \quad A^n$;

$\therefore$ taking the nth root,

$$A > (abcd\ldots)^{1/n};$$

$\therefore$ the arithmetic mean > the geometric mean.

Q.E.D.

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