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ON FINITELY GENERATED SIMPLE COMPLEMENTED LATTICES

BY

WERNER POGUNTKE

Let L be a lattice, and let P and Q be partially ordered sets. We say that L *is generated by* P if there is an isotone mapping from P into L with its image generating L. P *contains* Q if there is a subset Q' of P which, with the partial ordering inherited from P, gives an isomorphic copy of Q. For an integer $n > 0$, the lattice of partitions of an n-element set will be denoted by $\Pi(n)$; it is well-known that $\Pi(n)$ is simple and complemented (cf. P. Crawley–R. P. Dilworth [1; p. 96]).

The purpose of this note is to prove:

THEOREM. *For a finite partially ordered set P, the following conditions are equivalent:*

(i) *each partition lattice* $\Pi(n)$, with $n \geq 10$, *is generated by* P;
(ii) *there are infinitely many non-isomorphic simple complemented lattices generated by* P;
(iii) *there is a simple complemented lattice generated by* P *which is not isomorphic to* D_1, D_2, M_3, *or* C (*see Figure* 1);
(iv) P *contains* $\mathbb{1}+\mathbb{1}+\mathbb{1}+\mathbb{1}$ *or* $\mathbb{1}+\mathbb{1}+\mathbb{2}$ (*see Figure* 2). ●

The projective plane over the rational numbers is an infinite simple complemented lattice generated by $\mathbb{1}+\mathbb{1}+\mathbb{1}+\mathbb{1}$. An example of an infinite simple lattice generated by $\mathbb{1}+\mathbb{1}+\mathbb{2}$ was given in W. Poguntke [2], but it seems to be unknown whether there is one which is also complemented.

Some remarks and a Lemma. The above Theorem is analogous to the following result of R. Wille [4]: the finite partially ordered sets generating (up to isomorphism) only finitely many simple lattices are precisely those not containing $\mathbb{1}+\mathbb{1}+\mathbb{1}+\mathbb{1}$, $\mathbb{1}+\mathbb{1}+\mathbb{2}$, or $\mathbb{1}+\mathbb{K}_2$ (cf. Figure 2); furthermore, if R is such a partially ordered set, then each simple lattice generated by R is isomorphic to D_1, D_2, or M_3. It was shown in H. Strietz [3], and the proof of our Theorem makes heavy use of this fact, that isomorphic copies of each of the two "critical" partially ordered sets $\mathbb{1}+\mathbb{1}+\mathbb{1}+\mathbb{1}$ and $\mathbb{1}+\mathbb{1}+\mathbb{2}$ are generating sets in

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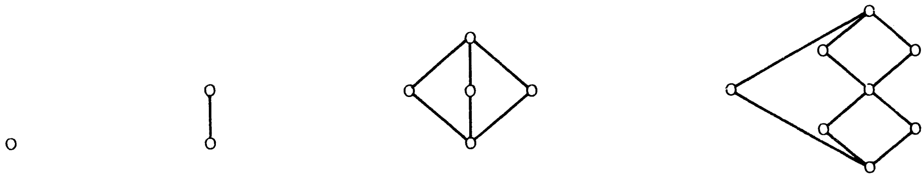


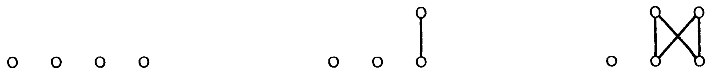
D_1 D_2 M_3 C

Figure 1.

every partition lattice $\Pi(n)$, with $n \geq 10$. The question remained if $\mathbb{1}+\mathbb{K}_2$ has the same property, but our Theorem shows that this is not the case.

The proof of the Theorem uses the following extended version of the D_2-Lemma in R. Wille [4]:

LEMMA. *Let $L \not\cong D_2$ be a simple (or subdirectly irreducible and modular) complemented lattice generated by the union of two finite subsets E_0 and E_1. Then* $\sup E_0 = 1$ *or* $\inf E_1 = 0$. ●



$\mathbb{1}+\mathbb{1}+\mathbb{1}+\mathbb{1}$ $\mathbb{1}+\mathbb{1}+\mathbb{2}$ $\mathbb{1}+\mathbb{K}_2$

Figure 2.

Proof. It follows from the D_2-Lemma that $\sup E_0 \geq \inf E_1$. Assume $\sup E_0 < 1$, and let x be a complement of $\sup E_0$. Since L is the set union of the intervals $[0, \sup E_0]$ and $[\inf E_1, 1]$, and since $x \leq \sup E_0$ is impossible, it follows that $x \geq \inf E_1$ which implies $0 = x \wedge \sup E_0 \geq \inf E_1$, hence $\inf E_1 = 0$. ▲

Proof of the Theorem. The following notation will be used: if S is a partially ordered set and $x \in S$, then $(x] := \{y \in S \mid y \leq x\}$; $[x)$ is defined dually.

Trivially, (i) implies (ii) which implies (iii).

(iii) $\rightarrow$ (iv):

Let us assume that P does not contain $\mathbb{1}+\mathbb{1}+\mathbb{1}+\mathbb{1}$ or $\mathbb{1}+\mathbb{1}+\mathbb{2}$, and let $\psi: P \rightarrow L$ be an isotone mapping from P into a simple complemented lattice L such that ψP generates L. We first observe that ψP, too, does not contain $\mathbb{1}+\mathbb{1}+\mathbb{1}+\mathbb{1}$ or $\mathbb{1}+\mathbb{1}+\mathbb{2}$. In view of the results in R. Wille [4] mentioned above, we may assume that ψP contains a subset $\{u, a, b, c, d\}$ isomorphic to $\mathbb{1}+\mathbb{K}_2$, i.e. u is incomparable with each of a, b, c, d, and $a, b < a \vee b \leq c \wedge d < c, d$. Our aim is to show $L \cong C$.

Note that every $x \in \psi P$ with $x < u$ satisfies $x < a$ or $x < b$; dually, if $y > u$, then $y > c$ or $y > d$. Let U be the set of all elements in ψP that are incomparable with u. Since $\mathbb{1}+\mathbb{1}+\mathbb{1}+\mathbb{1}$ and $\mathbb{1}+\mathbb{1}+\mathbb{2}$ are not contained in ψP, for each element $v \in U$ there is at most one $w \in U$ which is incomparable to v. In

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particular, U has at most two maximal (minimal) elements. Assume U has only one maximal element, $\bar{u}$. Since in this case, $\psi P = (\bar{u}] \cup [u)$, the Lemma yields $\bar{u} = 1$ or $u = 0$, each a contradiction.

Thus, U has precisely two maximal elements $\bar{u}_1$, $\bar{u}_2$, and two minimal elements $\underline{u}_1$, $\underline{u}_2$. Now, we have for each $z \in \psi P$ that

$$z < u \quad \text{if and only if} \quad z < \underline{u}_1 \quad \text{or} \quad z < \underline{u}_2$$

and

$$z > u \quad \text{if and only if} \quad z > \bar{u}_1 \quad \text{or} \quad z > \bar{u}_2.$$

It follows that $\psi P = (\bar{u}_2] \cup [u) \cup [\bar{u}_1)$, and the Lemma (with $E_0 := (\bar{u}_2]$, $E_1 := [u) \cup [\bar{u}_1)$) yields $u \wedge \bar{u}_1 = 0$ (since $\bar{u}_2 \neq 1$). From this, and by symmetry and duality, we get:

$$\{u\} \cup U \subseteq \psi P \subseteq \{u\} \cup U \cup \{0, 1\};$$
$$u \wedge \bar{u}_1 = u \wedge \bar{u}_2 = \underline{u}_1 \wedge u_2 = 0;$$
$$u \vee \underline{u}_1 = u \vee \underline{u}_2 = \bar{u}_1 \vee \bar{u}_2 = 1.$$

It also follows that L consists of the sublattice generated by U plus the element u which covers 0 and is covered by 1.

But the properties of U obviously imply that the sublattice generated by U is a finite linear sum of four-element Boolean lattices and one-element lattices, with at least two copies of four-element Boolean lattices occurring. Now, L cannot be simple unless

$$\{a, b, c, d\} = \{\underline{u}_1, \underline{u}_2, \bar{u}_1, \bar{u}_2\}, \quad e := a \vee b = c \wedge d, \quad \text{and} \quad U \subseteq \{a, b, c, d, e\},$$

which means $L \cong C$.

(iv) → (i):

This turns out to be an easy consequence of the results in H. Strietz [3] mentioned above. Let $P \supseteq \{a, b, c, d\} \cong \mathbb{1} + \mathbb{1} + \mathbb{2}$ (with $c > d$), and let $\{p_1, p_2, p_3, p_4\} \cong \mathbb{1} + \mathbb{1} + \mathbb{2}$ (with $p_3 > p_4$) be a generating set of the partition lattice $\Pi(k)$ ($k \geq 10$). As in R. Wille [4], we define an isotone mapping $\tau : P \to \Pi(k)$ by

$$\tau(a) := p_1, \quad \tau(b) := p_2, \quad \tau(c) := p_3,$$

and

$$\tau(x) := \begin{cases} p_4 & \text{if} \quad x < c,\ x \not< a, b, d \\ 0 & \text{if} \quad x < a, b, \text{or } d \\ 1 & \text{if} \quad x \not\leq a, b, \text{and } c. \end{cases}$$

thus showing that $\Pi(k)$ is generated by P.

Using a generating set $\{q_1, q_2, q_3, q_4\} \cong \mathbb{1} + \mathbb{1} + \mathbb{1} + \mathbb{1}$, the case that P contains $\mathbb{1} + \mathbb{1} + \mathbb{1} + \mathbb{1}$ can be treated in a similar way (see R. Wille [4]). ▲

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Fachbereich Mathematik
der Technischen Hochschule
FB4-AG1
6100 Darmstadt
West Germany.