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Lefschetz Numbers for C^*-Algebras

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Abstract. Using Poincaré duality, we obtain a formula of Lefschetz type that computes the Lefschetz number of an endomorphism of a separable nuclear C^*-algebra satisfying Poincaré duality and the Kunneth theorem. (The Lefschetz number of an endomorphism is the graded trace of the induced map on K-theory tensored with $\mathbb{C}$, as in the classical case.) We then examine endomorphisms of Cuntz–Krieger algebras O_A. An endomorphism has an invariant, which is a permutation of an infinite set, and the contracting and expanding behavior of this permutation describes the Lefschetz number of the endomorphism. Using this description, we derive a closed polynomial formula for the Lefschetz number depending on the matrix A and the presentation of the endomorphism.

1 Introduction

Suppose A and B are two separable nuclear C^*-algebras. To say that A and B are *Poincaré dual* means that there is given a K-homology class for $A \otimes B$ such that cup-cap product with this class induces an isomorphism between the K-theory of A and the K-*homology* of B. The homology class plays the role of the orientation class of a compact manifold. The idea in this form is due to Alain Connes (see [4]). Since the definition was invented, quite a number of examples of Poincaré dual pairs have appeared in the operator algebra literature connected with dynamical systems, foliations, hyperbolic groups, twisted K-theory, C^*-algebras of discrete groups with finite $B\Gamma$, *etc.*

The object of this note is to propose a simple application of the existence of duality between a pair of algebras, which runs roughly along the lines of a classical argument with de Rham cohomology and differential forms. Suppose $\phi\colon X \to X$ is a smooth self-map of a compact oriented manifold. Assume that ϕ is in general position with regard to fixed-points. Then ϕ induces a map on homology with rational coefficients, and its Lefschetz number is

$$\mathrm{tr}_s(\phi_*) := \mathrm{trace}\big(\phi_*\colon \mathrm{H}_{\mathrm{ev}}(X) \to \mathrm{H}_{\mathrm{ev}}(X)\big) - \mathrm{trace}\big(\phi_*\colon \mathrm{H}_{\mathrm{odd}}(X) \to \mathrm{H}_{\mathrm{odd}}(X)\big).$$

The Lefschetz fixed-point theorem states that this number is equal to the number of fixed points of ϕ counted with appropriate multiplicities. The proof, which can be found in any textbook, involves ideas connected with Poincaré duality in de Rham theory: normal bundles, integration of forms, Thom classes, and so on. The Kunneth formula is a separate additional ingredient. It is sometimes therefore said that the Lefschetz fixed-point formula *follows* from Poincaré duality and the Kunneth formula.

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In this article, we are going to formalize the exact connection between Poincaré duality and the Lefschetz fixed-point theorem in such a way as to apply to the category of C^*-algebras, with K-theory and K-homology playing the role of ordinary homology and cohomology. Part of the proof of the classical Lefschetz theorem is absorbed into our statement, so that the classical theorem can be deduced from ours by a simple, essentially linear, index calculation. We show via essentially straightforward formal calculations with KK-theory, that if one has a Poincaré duality with "fundamental class" $\Delta \in \mathrm{KK}^n(A \otimes B, \mathbb{C})$, and if one has a morphism $f \in \mathrm{KK}(A, A)$, then the trace of the map on K-theory induced from f is equal to the result of a certain index pairing (see Section 2) involving f and Δ. This index pairing can in principle be computed in geometric terms, provided that the cycles underlying f and Δ themselves admit interesting "geometric" descriptions. Thus, to summarize, the Lefschetz number can be realized as a Kasparov product. Of course there is more than one such realization; in [8] we pursue a similar idea to produce other kinds of identities in equivariant KK-theory.

Of course the main merit of the observation is that one now has the possibility of proving analogues of the Lefschetz theorem in many different settings, provided one has available an interesting instance of noncommutative Poincaré duality.

The significance of the classical Lefschetz formula tends to be explained in terms of the equality of a *local* and a *global* invariant. In connection with C^*-algebras, this does not entirely make sense. What kind of Lefschetz formulas can we expect in connection with C^*-algebras? One example, based on the abstract Lefschetz formula presented here, is worked out in [6]. This involves proper actions of discrete groups on manifolds. The primitive ideal space of a cross product $C_0(X) \rtimes G$ in this situation is the extended quotient

$$G \setminus \hat{X}, \text{ where } \hat{X} := \{(x, h) \in X \times G \mid h \in \mathrm{Stab}_G(x)\},$$

where G acts on $\hat{X}$ by $g(x, h) = (gx, ghg^{-1})$, and which, as a set, identifies canonically with the primitive ideal space of $C_0(X) \rtimes G$ and inherits a corresponding hull-kernel topology. It is a bundle over the ordinary space $G \setminus X$ with fibre at Gx the irreducible dual of $\mathrm{Stab}_G(x)$, but it is not Hausdorff. The Lefschetz formula for an automorphism of this situation has the corresponding shape: the geometric side of the formula involves fixed points in the ordinary space $G \setminus X$, and secondly, involves representation theoretic data for the isotropy of these fixed-points.

The second purpose of this note is to consider the case of a pair of *simple* algebras in duality, namely to pairs $A = O_A$, $B = O_{A^T}$ of Cuntz-Krieger algebras (see [11]). Here, in contrast to the example of the previous paragraph, there are no points at all. Given an endomorphism of O_A arising from certain geometric data, we will solve the index problem on the geometric side of the formal Lefschetz formula. The endomorphisms with which we work correspond to n-tuples of continuous partially defined homeomorphisms

$$\varphi \colon Z \subset \Sigma_A^+ \to \Sigma_A^+,$$

where Σ_A^+ is the symbol space of sequences (x_i) such that $A_{x_i, x_{i+1}} = 1$ for all i. The information involved in such an n-tuple can be summarized in a *single* map on the

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countable set of paths in the graph corresponding to A. The geometric computation of the Lefschetz number turns out to depend, roughly, on the difference between the number of strings whose length is shrunk by the map and the number of strings whose length is expanded by the map. This eventually leads to a description of the Lefschetz number roughly in the following terms: if we write $t_i = \sum s_\mu s_\nu^*$ for words μ, ν, for one of the images of the generators of O_A under the endomorphism, then an appearance of (μ, ν) with $|\mu| \leq |\nu|$ contributes +1 to the Lefschetz number and $|\mu| > |\nu|+1$ contributes -1 (and there are no contributions when $|\mu| = |\nu|+1$). As a result of this decription, we can, if we want, write down an explicit, closed formula for the Lefschetz number, which is a polynomial expression in the entries of the matrix A and its powers.

This example thus shows that the "Lefschetz trick" results in interesting formulas even in what one might loosely refer to as a "very" noncommutative situation.

The idea of formalizing the Lefschetz fixed-point theorem's proof using Poincaré duality (and the Kunneth theorem) in order to work in a more general context, is due to André Weil, though not of course in connection with C^*-algebras and K-theory. It was used by him in connection with the so-called Weil conjectures (see the Appendix to [9]). So in this sense, we have rediscovered an old trick. Even so, it seems worth making it explicit in the operator algebraic context in view of the variety of Lefschetz-type formulas one can reasonably hope to achieve by using C^*-algebras and KK-theory, which embrace such a wide selection of geometric situations.

2 The Abstract Lefschetz Theorem

Kasparov's KK-theory is a realization of an additive $\mathbb{Z}/2$-graded category with objects C^*-algebras and morphisms $A \to B$ the elements of $\mathrm{KK}^\bullet(A, B)$, defined as a quotient of a certain set of cycles, by a certain equivalence relation (see [10]).

In addition to its structure of an additive category, KK is a symmetric monoidal category with unit object the C^*-algebra $\mathbb{C}$ and bifunctor given by the tensor product of C^*-algebras on objects and the "external product"

$$\mathrm{KK}^\bullet(D_1, D_1') \times \mathrm{KK}^\bullet(D_2, D_2') \to \mathrm{KK}^\bullet(D_1 \otimes D_2, D_1' \otimes D_2'), \quad (f_1, f_2) \mapsto f_1 \hat{\otimes}_{\mathbb{C}} f_2.$$

on morphisms, and the flip $\Sigma\colon A \otimes B \to B \otimes A$ inducing the braiding.

The interaction between the flip, the monoidal structure, and the grading in KK is summarized by the following diagram, which *graded commutes*, for all D_i, D_i'.

$$\begin{array}{ccc}
\mathrm{KK}^\bullet(D_1, D_1') \times \mathrm{KK}^\bullet(D_2, D_2') & \xrightarrow{\hat{\otimes}_{\mathbb{C}}} & \mathrm{KK}^\bullet(D_1 \otimes D_2, D_1' \otimes D_2') \\
\Big\downarrow{\scriptstyle \mathrm{flip}} & & \Big\downarrow{\scriptstyle \mathrm{flip}} \\
\mathrm{KK}^\bullet(D_2, D_2') \times \mathrm{K}_\bullet(D_1, D_1') & \xrightarrow{\hat{\otimes}_{\mathbb{C}}} & \mathrm{KK}^\bullet(D_2 \otimes D_1, D_2' \otimes D_1')
\end{array}$$

In other words,

$$f_1 \hat{\otimes}_{\mathbb{C}} f_2 = (-1)^{\partial f_1 \partial f_2} [\Sigma] \hat{\otimes}_{D_2 \otimes D_1} (f_2 \hat{\otimes}_{\mathbb{C}} f_1) \hat{\otimes}_{D_2' \otimes D_1'} [\Sigma]$$

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for all $f_1 \in \mathrm{KK}^\bullet(D_1, D_1')$ and $f_2 \in \mathrm{KK}^\bullet(D_2, D_2')$.

Of course a category with similar properties is the category of complex $\mathbb{Z}/2$-graded vector spaces and vector space maps, where for the monoidal structure we use graded tensor product of vector spaces, and for the braiding we use the *graded flip*

$$\Sigma^s(a \hat{\otimes}_{\mathbb{C}} b) := (-1)^{\partial a \partial b} b \hat{\otimes}_{\mathbb{C}} a$$

instead of the ordinary flip. The action of a linear transformation $T_1 \hat{\otimes}_{\mathbb{C}} T_2$ on $V_1 \hat{\otimes}_{\mathbb{C}} V_2$, where $T_i \colon V_i \to V_i'$, V_i, V_i' graded vector spaces, is defined by

$$(T_1 \hat{\otimes}_{\mathbb{C}} T_2)(a \hat{\otimes}_{\mathbb{C}} b) := (-1)^{\partial T_1 \partial b} T_1(a) \hat{\otimes}_{\mathbb{C}} T_2(b).$$

Then a short calculation, depending on the fact that

$$\partial x_1 \partial x_2 + \partial T_1 \partial x_2 + (\partial T_1 + \partial x_1)(\partial T_2 + \partial x_2) = \partial T_1 \partial T_2 + \partial x_1 \partial x_2 \mod (2),$$

shows that the monoidal structure on $\mathbf{V}^s$ is also graded commutative, in the sense described above for $\mathrm{KK}^\bullet$.

These definitions ensure that the K-theory functor $\mathrm{KK} \to \mathbf{V}^s$,

$$A \mapsto \mathrm{K}_\bullet^{\mathbb{C}}(A), \ f \in \mathrm{KK}_{\mathbb{C}}^\bullet(A, B) \mapsto f_* \colon \mathrm{K}_\bullet^{\mathbb{C}}(A) \to \mathrm{K}_\bullet^{\mathbb{C}}(B),$$

which associates with a C^*-algebra A the complex, $\mathbb{Z}/2$-graded vector space $\mathrm{K}_\bullet^{\mathbb{C}}(A) := \mathrm{K}_\bullet(A) \otimes_{\mathbb{Z}} \mathbb{C}$, is compatible with the symmetric monoidal structures on each category, at least on a bootstrap category $\mathcal{N}$ (the Kunneth theorem) where it is an isomorphism (the Universal Coefficient theorem.)

In order to illustrate these facts in a concrete way, we prove the following simple lemma.

Lemma 2.1 *Suppose $c = \sum a_i \hat{\otimes}_{\mathbb{C}} b_i \in \mathrm{K}_\bullet(A \otimes B)$ is written as a sum with a_i, b_i homogeneous.*[1] *Let $f \in \mathrm{KK}^\bullet(A, A')$ and $g \in \mathrm{KK}^\bullet(B, B')$ be homogeneous. Then*

$$c \hat{\otimes}_{A \otimes B} (f \hat{\otimes}_{\mathbb{C}} g) = \sum (-1)^{\partial b_i \partial f} (a_i \hat{\otimes}_A f) \hat{\otimes}_{\mathbb{C}} (b_i \hat{\otimes}_B g) \in \mathrm{K}_\bullet(A' \otimes B').$$

Proof Suppressing subscripts, suppose $a \in \mathrm{KK}_{\mathbb{C}}^\bullet(\mathbb{C}, A)$, $b \in \mathrm{KK}_{\mathbb{C}}^\bullet(\mathbb{C}, B)$, f and g as above. Then

$$\begin{aligned}(a \hat{\otimes}_{\mathbb{C}} b) \hat{\otimes}_{A \otimes B} (f \hat{\otimes}_{\mathbb{C}} g) &= a \hat{\otimes}_A (1_A \hat{\otimes}_{\mathbb{C}} b) \hat{\otimes}_{A \otimes B} (f \hat{\otimes}_{\mathbb{C}} 1_B) \hat{\otimes}_{A' \otimes B} (1_{A'} \hat{\otimes}_{\mathbb{C}} g) \\ &= a \hat{\otimes}_A \Sigma_*(b \hat{\otimes}_{\mathbb{C}} 1_A) \hat{\otimes}_{A \otimes B} (f \hat{\otimes}_{\mathbb{C}} 1_B) \hat{\otimes}_{A' \otimes B} (1_{A'} \hat{\otimes}_{\mathbb{C}} g).\end{aligned}$$

Since $\Sigma_*(f \hat{\otimes}_{\mathbb{C}} 1_B) = \Sigma_*(1_B \hat{\otimes}_{\mathbb{C}} f)$, the above is equal to

$$a \hat{\otimes}_A (b \hat{\otimes}_{\mathbb{C}} 1_A) \hat{\otimes}_{B \otimes A} (1_B \hat{\otimes}_{\mathbb{C}} f) \hat{\otimes}_{B \otimes A'} \Sigma^*(1_{A'} \hat{\otimes}_{\mathbb{C}} g). \tag{2.1}$$

[1] An element of a graded set is *homogeneous* if it has a definite degree.

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Using graded commutativity we have

$$(b \mathbin{\hat{\otimes}_{\mathbb{C}}} 1_A) \mathbin{\hat{\otimes}_{B\otimes A}} (1_B \mathbin{\hat{\otimes}_{\mathbb{C}}} f) = b \mathbin{\hat{\otimes}_{\mathbb{C}}} f = (-1)^{\partial b \partial f} \Sigma_*(f \mathbin{\hat{\otimes}_{\mathbb{C}}} b)$$
$$= (-1)^{\partial b \partial f} \Sigma_*\big(f \mathbin{\hat{\otimes}_{A'}} (1_{A'} \mathbin{\hat{\otimes}_{\mathbb{C}}} b)\big).$$

Putting this into (2.1) and moving the flip across the tensor product, the above is equal to

$$(-1)^{\partial b \partial f} a \mathbin{\hat{\otimes}_A} f \mathbin{\hat{\otimes}_{A'}} (1_{A'} \mathbin{\hat{\otimes}_{\mathbb{C}}} b) \mathbin{\hat{\otimes}_{A' \otimes B}} (1_{A'} \mathbin{\hat{\otimes}_{\mathbb{C}}} g) =$$
$$(-1)^{\partial b \partial f} (a \mathbin{\hat{\otimes}_A} f) \mathbin{\hat{\otimes}_{\mathbb{C}}} (b \mathbin{\hat{\otimes}_B} g),$$

as required. ■

We next state the essential definition of this note (see [1, 4, 5, 11].)

Definition 2.2 Let A and B be C^*-algebras. Then *A and B are dual in* KK (with a dimension shift of n) if there exists $\Delta \in \mathrm{KK}^n(A \otimes B, \mathbb{C})$ such that the composition

$$\text{(2.2)} \quad \mathrm{KK}^\bullet(D_1, D_2 \otimes A) \xrightarrow{-\hat{\otimes}_{\mathbb{C}} 1_B} \mathrm{KK}^\bullet(D_1 \otimes B, D_2 \otimes A \otimes B) \xrightarrow{\hat{\otimes}_{A\otimes B}\Delta} \mathrm{KK}^{\bullet+n}(D_1 \otimes B, D_2)$$

is an isomorphism for every D_1, D_2. We call Δ the *fundamental class* of the duality.

Suppose A and B are dual with class Δ. In the above notation, set $D_1 = \mathbb{C}$ and $D_2 = B$. Then there is a unique class $\widehat{\Delta}' \in \mathrm{KK}^{-n}(\mathbb{C}, B \otimes A)$ such that the isomorphism

$$\mathrm{KK}^{-n}(\mathbb{C}, B \otimes A) \xrightarrow{\cong} \mathrm{KK}^0(B, B)$$

carries $\widehat{\Delta}'$ to 1_B. We call $\widehat{\Delta}'$ the *dual fundamental class*. By definition, we have the equation

$$\text{(2.3)} \qquad (\widehat{\Delta}' \mathbin{\hat{\otimes}_{\mathbb{C}}} 1_B) \mathbin{\hat{\otimes}_{B\otimes A\otimes B}} (1_B \mathbin{\hat{\otimes}_{\mathbb{C}}} \Delta) = 1_B.$$

A simple computation shows that the map

$$\text{(2.4)} \quad \mathrm{KK}^\bullet(D_1 \otimes B, D_2) \xrightarrow{\hat{\otimes}_{\mathbb{C}} 1_A} \mathrm{KK}^\bullet(D_1 \otimes B \otimes A, D_2 \otimes A) \xrightarrow{\widehat{\Delta}' \hat{\otimes}_{B\otimes A}} \mathrm{KK}^{\bullet - n}(D_1, D_2 \otimes A).$$

is an inverse to (2.2). We obtain a second equation

$$\text{(2.5)} \qquad (1_A \mathbin{\hat{\otimes}_{\mathbb{C}}} \widehat{\Delta}') \mathbin{\hat{\otimes}_{A\otimes B\otimes A}} (\Delta \mathbin{\hat{\otimes}_{\mathbb{C}}} 1_A) = 1_A.$$

If one prefers to arrange things in a different logical pattern, one can start with a pair of classes Δ and $\widehat{\Delta}'$ and insist that they satisfy equations (2.3) and (2.5). Then the map as in (2.2) can be shown to be an isomorphism with inverse (2.4).

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Remark 2.3 In the above notation,

$$(2.6)\quad (1_A \hat{\otimes}_{\mathbb{C}} \widehat{\Delta}') \hat{\otimes}_{A\otimes B\otimes A} (\Delta \hat{\otimes}_{\mathbb{C}} 1_A) = (-1)^n \left(\Sigma_*(\widehat{\Delta}') \hat{\otimes}_{\mathbb{C}} 1_A\right) \hat{\otimes}_{A\otimes B\otimes A} \left(1_A \hat{\otimes}_{\mathbb{C}} \Sigma^*(\Delta)\right).$$

In [5], the definition of Poincaré duality involved classes $\widehat{\Delta} \in \mathrm{KK}^{-n}(\mathbb{C}, A \otimes B)$ and $\Delta^n(A \otimes B, \mathbb{C})$ satisfying appropriate equations. To connect our current discussion with that definition, set $\widehat{\Delta} = \Sigma_*(\widehat{\Delta}')$. Then, by (2.6), the analogues of equations (2.3) and (2.5) are

$$(\Sigma_*(\widehat{\Delta}) \hat{\otimes}_{\mathbb{C}} 1_B) \hat{\otimes}_{B\otimes A\otimes B} (1_B \hat{\otimes}_{\mathbb{C}} \widehat{\Delta}) = 1_B,$$

$$(\widehat{\Delta} \hat{\otimes}_{\mathbb{C}} 1_A) \hat{\otimes}_{A\otimes B\otimes A} \left(1_A \hat{\otimes}_{\mathbb{C}} \Sigma^*(\widehat{\Delta})\right) = (-1)^n 1_A,$$

which is as in [5].

Notice also that the roles of A and B are symmetric when n is even and anti-symmetric when n is odd.

Given A and B dual as above, define a $\mathbb{Z}$-bilinear map

$$(2.7)\qquad \mathrm{K}_\bullet(A) \times \mathrm{K}_\bullet(B) \to \mathbb{Z}, \quad (x \mid y) := y \hat{\otimes}_B \hat{x},$$

where $\hat{x}$ denotes the Poincaré dual of x.

Lemma 2.4 *With the Poincaré duality pairing defined in* (2.7),

$$(x \mid y) = (-1)^{\partial x \partial y} (x \hat{\otimes}_{\mathbb{C}} y) \hat{\otimes}_{A\otimes B} \Delta$$

for homogeneous elements $x \in \mathrm{K}_\bullet(A)$, $y \in \mathrm{K}_\bullet(B)$.

Proof Expanding the definitions, we have

$$\begin{aligned} y \hat{\otimes}_B \hat{x} &= y \hat{\otimes}_B (x \hat{\otimes}_{\mathbb{C}} 1_B) \hat{\otimes}_{A\otimes B} \Delta = y \hat{\otimes}_B \Sigma_*(1_B \hat{\otimes}_{\mathbb{C}} x) \hat{\otimes}_{A\otimes B} \Delta \\ &= y \hat{\otimes}_B (1_B \hat{\otimes}_{\mathbb{C}} x) \hat{\otimes}_{A\otimes B} \Sigma^*(\Delta) = (y \hat{\otimes}_{\mathbb{C}} x) \hat{\otimes}_{B\otimes A} \Sigma^*(\Delta) \\ &= (-1)^{\partial x \partial y} \Sigma_*(x \hat{\otimes}_{\mathbb{C}} y) \hat{\otimes}_{B\otimes A} \Sigma^*(\Delta) = (-1)^{\partial x \partial y} (x \hat{\otimes}_{\mathbb{C}} y) \hat{\otimes}_{A\otimes B} \Delta. \quad \blacksquare \end{aligned}$$

Tensoring with the complex numbers, we obtain a duality pairing

$$(\,\cdot \mid \cdot\,)\colon \mathrm{K}^{\mathbb{C}}_\bullet(A) \times \mathrm{K}^{\mathbb{C}}_\bullet(B) \to \mathbb{C}.$$

This pairing is non-degenerate if B satisfies the Universal Coefficient Theorem. It is supported on $\{(x, y) \mid \partial(x) + \partial(y) = n\}$.

Now note that if A and B are Poincaré dual, then $\mathrm{K}_\bullet(A)$ and $\mathrm{K}_\bullet(B)$ are finitely generated abelian groups (and for the same reason, if A and B are dual in $\mathrm{KK}_{\mathbb{C}}$, then $\mathrm{K}^{\mathbb{C}}_\bullet(A)$ and $\mathrm{K}^{\mathbb{C}}_\bullet(B)$ have finite rank).

By elementary methods one can thus find a basis $(x_{\epsilon,i})$ for $\mathrm{K}^{\mathbb{C}}_\bullet(A)$ and a dual basis $(x^*_{n-\epsilon,j})$ for $\mathrm{K}^{\mathbb{C}}_\bullet(B)$ with respect to $(\,\cdot \mid \cdot\,)$, *i.e.*, so that we have

$$(x_{\epsilon,i} \mid x^*_{\eta,j}) = \delta_{\eta,n-\epsilon}\delta_{ij}.$$

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Lemma 2.5 *In terms of the bases* $(x_{\epsilon,i})$ *and* $(x^*_{\eta,j})$*, the class* $\widehat{\Delta}'$ *is given by*

$$\widehat{\Delta}' = \sum_{i,\epsilon}(-1)^{n-\epsilon}x^*_{n-\epsilon,i}\,\hat{\otimes}_{\mathbb{C}}\,x_{\epsilon,i}.$$

Proof It suffices to show that the map

$$\text{(2.8)}\qquad \mathrm{KK}^{-n}_{\mathbb{C}}(\mathbb{C}, B\otimes A) \xrightarrow{\hat{\otimes}_{\mathbb{C}}1_B} \mathrm{KK}^{-n}_{\mathbb{C}}(B, B\otimes A\otimes B) \xrightarrow{\hat{\otimes}_{A\otimes B}\Delta} \mathrm{KK}^{0}_{\mathbb{C}}(B,B)$$

sends $\sum_{i,\epsilon}(-1)^{n-\epsilon}x^*_{n-\epsilon,i}\,\hat{\otimes}_{\mathbb{C}}\,x_{\epsilon,i}$ to the identity in $\mathrm{KK}^{\bullet}_{\mathbb{C}}(B,B)$. Since we are over $\mathbb{C}$, the UCT gives that $\mathrm{KK}^{\bullet}_{\mathbb{C}}(B,B)\cong \mathrm{Hom}_{\mathbb{C}}\big(\mathrm{K}^{\mathbb{C}}_{\bullet}(B),\mathrm{K}^{\mathbb{C}}_{\bullet}(B)\big)$. If $x\in \mathrm{K}^{\mathbb{C}}_{\bullet}(B)$, and denoting our proposed formula for $\widehat{\Delta}$ by $\widehat{\delta}$, then we have

$$x\,\hat{\otimes}_B\,(\widehat{\delta}\,\hat{\otimes}\,1_B) = (-1)^{\partial x\partial\widehat{\delta}}\widehat{\delta}\,\hat{\otimes}_{\mathbb{C}}\,x$$

by Lemma 2.1. Hence the image of $\widehat{\delta}$ under (2.8) sends $x\in \mathrm{K}^{\mathbb{C}}_{\bullet}(B)$ to

$$\begin{aligned}
&(-1)^{n\partial x}\sum_{i,\epsilon}(-1)^{n-\epsilon}\big(x^*_{n-\epsilon,i}\,\hat{\otimes}_{\mathbb{C}}\,x_{\epsilon,i}\,\hat{\otimes}_{\mathbb{C}}\,x\big)\ \hat{\otimes}_{B\otimes A\otimes B}\,(1_B\otimes\Delta)\\
&\quad = (-1)^{n\partial x}\sum(-1)^{n-\epsilon}x^*_{n-\epsilon,i}\cdot\big((x_{\epsilon,i}\,\hat{\otimes}_{\mathbb{C}}\,x)\,\hat{\otimes}_{A\otimes B}\,\Delta\big)\\
&\quad = \sum(-1)^{n\partial x+(n-\epsilon)+\epsilon\partial x}x^*_{n-\epsilon,i}\cdot(x_{\epsilon,i}\mid x).
\end{aligned}$$

Now setting $x = x^*_{\gamma,j}$, each term vanishes save when $\epsilon = n-\gamma$, in which case the sign is $(-1)^{n\gamma+\gamma+(n-\gamma)\gamma} = +1$. ∎

With these preliminaries out of the way, we can now state and prove the formal Lefschetz theorem for Poincaré dual pairs of C^*-algebras alluded to in the introduction.

Suppose we have a duality with fundamental classes $\widehat{\Delta}'\in \mathrm{KK}^{-n}(B\otimes A,\mathbb{C})$ and $\Delta\in\mathrm{KK}^{n}(A\otimes B,\mathbb{C})$. Let $f\in\mathrm{KK}(B,B)$. Define

$$\mathbf{Ind}(\Delta,f) := \widehat{\Delta}'\,\hat{\otimes}_{B\otimes A}\,(f\,\hat{\otimes}_{\mathbb{C}}\,1_A)\,\hat{\otimes}_{B\otimes A}\,\Sigma^*\Delta\in\mathrm{KK}(\mathbb{C},\mathbb{C})\cong\mathbb{Z}.$$

As the notation suggests, this 'index' only depends on f and Δ subject to the condition that Δ implement a Poincaré duality. However, the way **Ind** is defined involves both Δ and the dual class $\widehat{\Delta}'$, so that if one changes the duality, it is easy to check that two cancelling changes are introduced into **Ind**, so that $\mathbf{Ind}(\Delta,f)$ does not depend on the choice of Δ.

Motivated by the classical case, we define the *Lefschetz number* $\mathrm{Lef}(f)$ of $f\in \mathrm{KK}^0(B,B)$ in the standard way as the graded trace of f acting on the complexified K-theory of B:

$$\begin{aligned}
\mathrm{Lef}(f) &:= \mathrm{tr}_s(f_*\colon \mathrm{K}^{\mathbb{C}}_{\bullet}(B)\to\mathrm{K}^{\mathbb{C}}_{\bullet}(B))\\
&:= \mathrm{trace}_{\mathbb{C}}\big(f_*\colon \mathrm{K}^{\mathbb{C}}_{0}(B)\to\mathrm{K}^{\mathbb{C}}_{0}(B)\big) - \mathrm{trace}_{\mathbb{C}}\big(f_*\colon \mathrm{K}^{\mathbb{C}}_{1}(B)\to\mathrm{K}^{\mathbb{C}}_{1}(B)\big).
\end{aligned}$$

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Theorem 2.6 *Let A and B be C^*-algebras satisfying the Universal coefficient theorems and the Kunneth theorem. Suppose that A and B are dual with fundamental class $\Delta \in \mathrm{KK}^n(A \otimes B, \mathbb{C})$ and dual class $\widehat{\Delta}' \in \mathrm{KK}^{-n}(\mathbb{C}, B \otimes A)$. Then for any $f \in \mathrm{KK}^0(B, B)$, the Lefschetz number of f is equal to the index $\mathrm{Lef}(f) = \mathbf{Ind}(\Delta, f)$.*

Proof Let $f \in \mathrm{KK}^0(B, B)$. We can write $f_*(x^*_{\epsilon,i}) = \sum_r f^{\epsilon}_{ir} x^*_{\epsilon,r}$. Hence,

$$(1_A \hat{\otimes}_{\mathbb{C}} f)_*(\widehat{\Delta}) = \sum (-1)^{n-\epsilon} f^{n-\epsilon}_{ir}\, x^*_{n-\epsilon,r} \hat{\otimes}_{\mathbb{C}} x_{\epsilon,i}.$$

Applying the flip gives

$$\Sigma_*(1_A \hat{\otimes}_{\mathbb{C}} f)_*(\widehat{\Delta}) = \sum (-1)^{n-\epsilon+\epsilon(n-\epsilon)}\, f^{n-\epsilon}_{ir}\, x_{\epsilon,i} \hat{\otimes}_{\mathbb{C}} x^*_{n-\epsilon,r}.$$

Finally, pairing this expression using Δ gives

$$\begin{aligned} \langle \Sigma_*(1_A \hat{\otimes}_{\mathbb{C}} f)_*(\widehat{\Delta}), \Delta \rangle &= \sum (-1)^{n-\epsilon+\epsilon(n-\epsilon)} f^{n-\epsilon}_{ir}\, (x_{\epsilon,i} \hat{\otimes}_{\mathbb{C}} x^*_{n-\epsilon,r}) \hat{\otimes}_{A\otimes B} \Delta \\ &= \sum (-1)^{n-\epsilon} f^{n-\epsilon}_{ir} (x^*_{n-\epsilon,r} \,|\, x_{\epsilon,i}) \\ &= \sum (-1)^{n-\epsilon} f^{n-\epsilon}_{ii} = \mathrm{tr}(f^0_*) - \mathrm{tr}(f^1_*) = \mathrm{tr}_s(f_*), \end{aligned}$$

as required. ■

Finally, we note that it is rather natural to call the Lefschetz number of the identity morphism $1_B \in \mathrm{KK}^\bullet(B, B)$ the *Euler characteristic* of B; it is the difference in ranks of $\mathrm{K}_0(B)$ and $\mathrm{K}_1(B)$, and by our formal Lefschetz theorem it is the index

$$\mathrm{Eul}_B = \langle \widehat{\Delta}, \Delta \rangle, \tag{2.9}$$

which is a sort of formal Gauss–Bonnet theorem.

Example 2.7 Let A be the C^*-algebra of sections $C_\tau(X)$ of the Clifford algebra of a compact manifold X, and $B = C(X)$. The best known example of K-theoretic Poincaré duality is in this situation. The class Δ is represented by the unbounded self-adjoint operator $D := d + d^*$ acting on the bundle $\Lambda^*_{\mathbb{C}}(X)$ of differential forms on X, where d is the de Rham differential, and the additional datum of the Clifford multiplication

$$C_\tau(X) \otimes C(X) \to \mathbb{B}(L^2(\Lambda^*_{\mathbb{C}}(X)))\,.$$

The class $\widehat{\Delta}$ involves a Clifford multiplication by an appropriate vector field on $X \times X$, acting on a submodule of $C(X) \otimes C_\tau(X)$, but the important point is that this vector field vanishes on the diagonal. It is immediate that when we take the Kasparov product of Δ and $\widehat{\Delta}$, we get simply the operator D acting on forms $L^2(\Lambda^*_{\mathbb{C}}(X))$. Hence (2.9) says that $\mathrm{Eul}_X = \mathrm{Index}(D_{\mathrm{dR}})$, with D_{dR} the de Rham operator on X. See [7] for a closely related computation.

It is also a simple matter to deduce the classical Lefschetz fixed-point formula in the same way. The fact that the class $\widehat{\Delta}$ is supported in a neighbourhood of the diagonal in $X \times X$ means that if we twist $\widehat{\Delta}$ by a smooth map whose graph $X \to$

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$X \times X$ is transverse to the diagonal, and pair with Δ, the result is supported in a small neighbourhood of the fixed-point set of the map. The latter is a discrete set. Thus, the formal Lefschetz theorem gets us this far, and to finish the computation we need to carry out a local index computation. (See [6] for the details in a more general context.)

The reader wanting other simple examples may wish to consider an automorphism of a finite group. A pleasant noncommutative Lefschetz formula for this situation can be deduced using the formal Lefschetz formula. This formula gives the well-known relationship between the number of fixed points of the induced map $\hat{\zeta}\colon \widehat{G} \to \widehat{G}$ on the (finite) space of irreducible representations, and the number of "ζ-twisted conjugacy classes" in G. The reference [6] also contains this result.

In the remainder of this note, we are going to work out a highly noncommutative example (the algebras A and B are *simple*). The merit of considering an example like this is that we do get a genuinely new equality of invariants – a genuinely new Lefschetz theorem. The difficulty with the example is that it is not so easy to see what its meaning is. It is helpful to have the classical examples at hand for comparison.

3 Example: Endomorphisms of Cuntz-Krieger Algebras

Let O_A (see [2, 3]) be the Cuntz–Krieger algebra with (irreducible) matrix A, the universal C^*-algebra generated by n nonzero partial isometries $s_1, \dots, s_n$ such that

$$\sum A_{ij} s_j s_j^* = s_i^* s_i.$$

We are going to illustrate the formal Lefshetz theorem given in the previous section by proving an analogue of the Lefschetz fixed-point theorem for (certain) endomorphisms of O_A.

We first remind the reader of the following theorem of Cuntz and Krieger.

Lemma 3.1 (see [2]) *The group* $\mathrm{K}_0(O_A)$ *is* $\mathbb{Z}/(1 - A^t)\mathbb{Z}$*, and the group* $\mathrm{K}_1(O_A)$ *is the quotient of* $\mathbb{Z}/(1 - A^t)\mathbb{Z}$ *by its torsion subgroup.*

To compute the Lefschetz number of an endomorphism of O_A, we must therefore split off the free part of the K_0 group and compute the images of a set of generators, and similarily, find free generators for K_1 and compute their generators.

Example 3.2 A standing numerical example will be the case

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

By Lemma 3.1, K-theory of O_A is infinite cylic in each of dimensions 0 and 1, with free generator $[s_1 s_1^*]$ the class of the projection $s_1 s_1^*$ in even degree, and free generator the class $[s_1 + s_3^*]$ of the unitary $s_1 + s_3^*$ in odd degree. Note that $[s_2 s_2^*] = 0$ and $[s_3 s_3^*] = -[s_1 s_1^*]$ in K_0.

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Any (unital) endomorphism $\alpha\colon O_A \to O_A$ maps each generator s_i to a partial isometry $t_i \in O_A$ such that $t_1, \dots, t_n$ satisfy the same relations. Conversely, by the universal property of O_A, any choice of $t_1, \dots, t_n$ satisfying the Cuntz–Krieger relations gives rise to a 'symmetry' of O_A, *i.e.*, an endomorphism. A well-known family of is the periodic 1-parameter family given by the circle action

$$s_i \mapsto z s_i, \quad \text{where} \quad z \in S^1, \quad i = 1, 2, \dots, n.$$

These endomorphisms are, however, obviously homotopic to the identity, whence we cannot expect very interesting Lefschetz numbers (they will all be zero, since the Euler characteristic of O_A is zero). Instead we are interested in more combinatorially defined endomorphisms.

The following definition is vaguely analogous to the assumption that one has an infinitely differentiable map, in the setting of the classical Lefschetz theorem. Let Σ_A^+ denote the Cantor set of sequences (x_i) in the graph Λ determined by the matrix A.

Definition 3.3 Let $Z \subset \Sigma_A^+$ be an open subset and $\varphi\colon Z \to \Sigma_A^+$ be a continuous map with domain Z. Then φ is *smooth* if

- Z is a cylinder set;
- there exists $k \in \mathbb{N}$ and a map $\psi'\colon \mathbf{P}_{\leq k} \to \mathbf{P}$ such that

$$\psi(x_1, x_2, x_3, \dots) = (\psi'(x_1, \dots, x_k), x_{k+1}, x_{k+2}, \dots),$$

for all $x = (x_1, x_2, x_3, \dots) \in Z$,

where $\mathbf{P} := \{(x_1, \dots, x_m) \mid A_{x_i, x_{i+1}} = 1, m \geq 0\}$ is the set of finite allowable strings in the alphabet determined by A, and $\mathbf{P}_{\leq k}$ is the set of strings of length at most k.

We allow the empty string $\varnothing$. With this convention, the left shift

$$\sigma_A\colon Z := \Sigma_A^+ \to \Sigma_A^+$$

is smooth, since $\sigma_A(x_1, x_2, \dots) = (\sigma_A'(x_1), x_2, x_3, \dots)$, where $\sigma_A'(x) = \varnothing$ for every string of length 1.

Our definition is actually closer to the idea of a quasi-conformal map. Note that $\mathbf{P}$ is the vertex set of the tree $\tilde{\Lambda}$, which is the universal cover of Λ. As such, it admits a canonical path metric. It is "Gromov hyperbolic" as a metric space, and so has a Gromov boundary.

Lemma 3.4 *A smooth map $\varphi\colon Z \to \Sigma_A^+$ is the boundary value of a quasi-isometry $\varphi'\colon Z' \to \tilde{\Lambda}$, where Z' is a subset of* $\mathbf{P}$.

Proof Suppose we are given a smooth map $\psi\colon Z \to \Sigma_A^+$ in the above sense, with k and ψ' as in the definition. We can take the cylinder set Z to be of the form $Z = \{x \in \Sigma_A^+ \mid \pi_N(x) \in F\}$, where $\pi_N\colon \Sigma^+ \to \mathbf{P}_N$ is the projection, and F is a finite subset of $\mathbf{P}_N$, and where N is larger than k. Now whether an infinite string is or is not in the domain of φ only depends on the first N letters. Geometrically, the set of infinite

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strings x with first N letters belonging to a given fixed finite set of finite strings is a clopen set of Σ_A^+, and is the closure in the compactification of the tree, of the set of *finite* strings of length at least N, and with first N letters in the given set. Hence the set Z is the boundary values of a subset $Z' \subset \mathbf{P}$, *i.e.*, $Z = \overline{Z'} \cap \partial\tilde{\Lambda}$, where Z' is the set of finite strings of length at least N with first N letters in F.

Assuming now that the finite string $(x_1, \dots, x_m)$ is in Z', whence that any boundary point $x = (x_1, \dots, x_m, x_{m+1}, \dots)$ is in Z, we have

$$\psi(x) = (\psi'(x_1, \dots, x_k), x_{k+1}, \dots, x_m, \dots),$$

which says that the last letter of $\varphi'(x_1, \dots, x_k)$ is allowed to be followed by x_{k+1}. Therefore, we can define $\psi'(x_1, \dots, x_m) := (\psi'(x_1, \dots, x_k), x_{k+1}, \dots, x_m)$. It is clear that ψ' is a quasi-isometry of the tree. It therefore extends to the boundary $\partial\tilde{\Lambda} = \Lambda$ and it is clear that its boundary values give precisely $\psi \colon Z \to \Sigma_A^+$. ■

A typical "geometric" endomorphism of O_A will be specified by the following definition.

Definition 3.5 A *geometric endomorphism* of O_A, where A is n-by-n, shall refer to the data of a partition $\Sigma_A^+ = Z_1 \cup \cdots \cup Z_n$ of the symbol space and an n-tuple $\Psi = (\psi_1, \dots, \psi_n)$ of smooth homeomorphisms $\psi_i \colon W_i \xrightarrow{\cong} Z_i \subset \Sigma_A^+$, such that $W_i = \bigcup_{A_{ij}=1} Z_i$.

It is clear that such partially defined maps determine elements of O_A; we define

$$t_i := \sum_{\mu \in W_i', |\mu| = k} s_{\psi_i'(\mu)} s_\mu^*, \tag{3.1}$$

where the summation is over the words of length k in W_i', with $W_i' \subset \mathbf{P}$ with $\overline{W'} \cap \partial\tilde{\Lambda} = W_i$ as explained above, and where ψ_i' are the extensions of the ψ_i to $\mathbf{P}$, and, of course, where k is sufficiently large.

Then the range projection of t_i identifies, in the obvious sense, with the image Z_i of ψ_i, and the cokernel projection identifies with the domain of definition W_i of ψ_i. Hence, due to the last condition in Definition 3.5, we get an endomorphism $\alpha_\Psi(s_i) := t_i$ of O_A.

Remark 3.6 The identity endomorphism corresponds to the evident partition with $Z_i = \{x \mid x \text{ begins with } i\}$ and $\psi_i(x) = (i, x)$ for $x \in W_i := \bigcup_{A_{ij}=1} Z_j$.

From now on we will abuse notation and denote by the same letter the partially defined maps $\psi_i \colon \mathbf{P} \to \mathbf{P}$ and the maps $\psi_i \colon \Sigma_A^+ \to \Sigma_A^+$. Of course there is ambiguity in the choice of the lifts $\psi \colon \mathbf{P} \to \mathbf{P}$, but we fix choices once and for all. Similarly, we will write W_i instead of W_i' and Z_i instead of Z_i'.

Example 3.7 A good example of a geometric endomorphism for

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

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is

$$t_1 = s_1^2 s_1^* s_2^* + s_1 s_2 (s_2^*)^2 + s_2 s_3^2 s_3^* s_2^* + s_2 s_3 s_2 s_2^* s_3^* + s_2 s_1^*,$$
$$t_2 = s_3 s_2, t_3 = s_3^2 s_3^*.$$

The corresponding partition and triple of smooth maps is as follows.

(i) Z_1 is all sequences (x_n) beginning with 1 or 2.
(ii) Z_2 is all sequences (x_n) beginning with 32.
(iii) Z_3 is all sequences (x_n) beginning with 33.

$\underline{\psi_1}$ We require $\psi_1 \colon W_1 := Z_1 \cup Z_3 \xrightarrow{\cong} Z_1$. If a sequence begins with 1, then we replace the initial 1 by a 2. If a sequence begins with 2, then we replace the initial 2 by a 1, unless the second coordinate is 3. In that case, we replace the initial 23 by 233. Finally, on sequences beginning with 32, we replace the initial 32 by 232. Observe that the image of ψ_1 is all strings beginning with 2 or 1.

$\underline{\psi_2}$ We require $\psi_2 \colon W_2 := Z_1 \cup Z_2 \cup Z_3 = \Sigma_A^+ \xrightarrow{\cong} Z_2$. We add 32 to the beginning of any sequence.

$\underline{\psi_3}$ We require $\psi_3 \colon W_3 := Z_2 \cup Z_3 \xrightarrow{\cong} Z_3$. To any sequence beginning with 3 we add an additional 3.

Remark 3.8 The endomorphism α_Ψ above sends the range projection of s_1 to the range projection of t_1, which is $s_1 s_1^* + s_2 s_2^*$. Hence $(\alpha_\Psi)_*([s_1 s_1^*]) = [s_1 s_1^*] + [s_2 s_2^*] = [s_1 s_1^*]$, so the induced map $(\alpha_\Psi)_* \colon \mathrm{K}_0(O_A) \to \mathrm{K}_0(O_A)$ is the identity. To see the action on $\mathrm{K}_1(O_A)$, one can check that the map $(t_1 + t_3^* | \cdot) \colon \mathrm{K}_0(O_A) \to \mathbb{Z}$ induced from the Poincaré duality pairing (see the end of this section) vanishes identically. Therefore, $[t_1 + t_3^*] = 0 \in \mathrm{K}_1(O_A)$ and $(\alpha_\Psi)_* \colon \mathrm{K}_1(O_A) \to \mathrm{K}_1(O_A)$ is the zero map. So the Lefschetz number of $\alpha_\Psi \colon O_A \to O_A$ is equal to 1. (In particular, α_Ψ is not an *automorphism*.)

We now describe an invariant of any geometric endomorphism, which will be a *single* partially defined map $\mathbf{P} \to \mathbf{P}$.

Definition 3.9 Let $\Psi = (\psi_1, \dots, \psi_n)$ be a geometric endomorphism. Extend the ψ_i to partially defined self-maps of $\mathbf{P}$. We let $\dot{\Psi} \colon \mathbf{P} \to \mathbf{P}$ be the partially defined map defined by

$$\dot{\Psi}(x_1, \dots, x_n) := \psi_{x_n}(x_1, \dots, x_{n-1})$$

if $(x_1, \dots, x_{n-1}) \in \mathrm{Dom}(\psi_{x_n})$.

Example 3.10 The map $\dot{\Psi}$ of Example 3.7 is defined on paths of length 2 by

$$\dot{\Psi}(11) = 2,\ \dot{\Psi}(21) = 1,\ \dot{\Psi}(12) = (321),\ \dot{\Psi}(22) = (322),$$
$$\dot{\Psi}(32) = (323),\ \dot{\Psi}(33) = (33),$$

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and on paths of length 3 by

$$\dot\Psi(111) = (21), \dot\Psi(121) = (22),\ \dot\Psi(211) = (11),\ \dot\Psi(221) = (12).$$

$$\dot\Psi(\star\star 2) = (32\star\star) \text{ for any } (\star\star) \text{ allowable,}$$

$$\dot\Psi(323) = (33),\ \dot\Psi(333) = (333).$$

Finally, on words of length 4, Ψ is defined by

$$\dot\Psi(1111) = (211),\ \dot\Psi(1121) = (212),\ \dot\Psi(1221) = (222),$$

$$\dot\Psi(2111) = (111),\ \dot\Psi(1211) = (221),\ \dot\Psi(2211) = (121),$$

$$\dot\Psi(3211) = (2321),\ \dot\Psi(1121) = (212),\ \dot\Psi(2121) = (112),$$

$$\dot\Psi(\star\star\star 2) = (32\star\star\star),\ \text{ for any } \star\star\star \text{ allowable,}$$

and

$$\dot\Psi(3223) = (33223),\ \dot\Psi(3233) = (33233),\ \dot\Psi(3323) = (33323),$$

$$\dot\Psi(3333) = (33333).$$

In these formulas, any string not mentioned is not in the domain.

Say that two partially defined maps $\dot\Psi$ and $\dot\Psi$ are equivalent if $\dot\Psi = \dot\Psi'$ on sufficiently long strings. Then it is only the class of $\dot\Psi$ modulo $\sim$ that will matter to us for what is coming. Denote by $[\dot\Psi]$ the class of $\dot\Psi$. We are going to associate a geometric endomorphism with an integer invariant. This invariant will only depend on the equivalence class $[\dot\Psi]$ and not on $\dot\Psi$ itself.

By *formal series* $\sum_{k=1}^\infty a_k$, where a_k are real numbers, we will refer to the sequence of of its terms, modulo the equivalence relation $\sum_{k=1}^\infty a_k \sim \sum_{k=1}^\infty b_k$ if $\sum_{k=1}^m a_k = \sum_{k=1}^m b_k$ for m sufficiently large. For example $1+2+3+4+\cdots \sim 3+0+3+4+\cdots$. The condition $\sim$ implies, obviously, that $a_k = b_k$ for large enough k, but it is stronger.

Definition 3.11 Let $\Xi\colon \mathbf{P} \to \mathbf{P}$ be a partially defined bijection with finite propagation; that is, there exists $N := \mathrm{Prop}(\Xi)$ such that

$$\Xi(\mathbf{P}_k) \subset \bigcup_{|l-k|\le N} \mathbf{P}_l, \quad \text{for all } k.$$

Let $\mathrm{Dom}(\Xi) \subset \mathbf{P}$ be its domain and $\mathrm{Im}(\Xi)$ its range. We set

$$\mathrm{Index}_k(\Xi) := \mathrm{card}\left(\mathbf{P}_k \cap \mathrm{Im}(\Xi)\right) - \left(\mathbf{P}_k \cap \mathrm{Dom}(\Xi)\right).$$

We let $\mathrm{Index}(\Xi)$ be the formal series

$$\mathrm{Index}(\Xi) = \sum_{i=1}^{\infty} \mathrm{Index}_i(\Xi). \tag{3.2}$$

We show below that the index only depends on $[\Xi]$ and converges if $\Xi = \dot\Psi$ for some geometric endomorphism Ψ.

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Lemma 3.12 *The index has the following properties.*

(i) *If* Ξ *and* Ξ' *are two partially defined maps that agree on* $\mathbf{P}_k$ *for all sufficiently large* k*, then* $\operatorname{Index}(\Xi) = \operatorname{Index}(\Xi')$ *as formal series. Hence,* Index *is compatible with* $\sim$.

(ii) *For any geometric endomorphism* Ψ, $\operatorname{Index}_k(\dot{\Psi}) = 0$ *for* k *sufficiently large. Hence the formal series in* (3.2) *converges in this case.*

Proof We prove (ii) first. For simplicity, we assume that a given partially defined map Ξ has propagation at most 1. We start by assuming that Ξ has no strings in its domain of length $\leq m - 1$, for some $m \geq 2$. Now we remove any strings of length m from the domain of Ξ. Let the new partially defined map be called Ξ'. We claim that the index (or more precisely the formal sum (3.2)) has not changed. The index in dimension $m - 1$ has clearly been reduced by the number of elements in dimension m that previously mapped to dimension $m - 1$. Call this $a(m, m - 1)$. Thus,

$$\operatorname{Index}_{m-1}(\Xi') = \operatorname{Index}_{m-1}(\Xi) - a(m, m - 1).$$

On the other hand, the domain in dimension m has been reduced by $\operatorname{card}(\operatorname{Dom}(\Xi) \cap \mathbf{P}_m)$, while the image in dimension m has been reduced by $a(m, m)$. So

$$\operatorname{Index}_m(\Xi') = \operatorname{Index}_m(\Xi) - a(m, m) + \operatorname{Dom}(\Xi) \cap \operatorname{card}(\mathbf{P}_m).$$

Finally, the image in dimension $m+1$ is reduced by $a(m, m+1)$. Meanwhile, the index in dimension $< m - 1$ has not changed, nor has the index in dimensions $> m + 1$, since Ξ changes lengths of strings by at most 1. So

$$\begin{aligned} &\operatorname{Index}(\Xi') \\ &\quad = \operatorname{Index}(\Xi) - a(m, m - 1) - a(m, m) + -a(m, m) + \operatorname{card}(\operatorname{Dom}(\dot{\Psi}) \cap \mathbf{P}_m) \\ &\quad = \operatorname{Index}(\Xi). \end{aligned}$$

This proves the result.

Now this means that any Ξ can have its domain successively shrunk by eliminating strings of length 1, then 2, and so on, without altering its index. The first assertion is now immediate, since without changing the index, we can alter both maps until they agree as partially defined maps.

The second assertion is left to the reader (it follows from the analytic considerations discussed in the last section of the paper). ■

Example 3.13 Consider Examples 3.7 and 3.10. The domain in dimension 1 has 0 elements. The image has 2 elements. So $\operatorname{Index}_1(\dot{\Psi}) = 2$. The domain in dimension 2 has 6 elements, and the image has 6 elements. Hence $\operatorname{Index}_2(\dot{\Psi}) = 0$. In dimension 3 there are 13 elements in the domain and 12 in the image. So, $\operatorname{Index}_3(\dot{\Psi}) = -1$. One checks that $\operatorname{Index}_k(\dot{\Psi}) = 0$ for $k > 3$. Hence

$$\operatorname{Index}(\dot{\Psi}) = 2 + 0 - 1 = 1.$$

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Remark 3.14 Altering the domain of a $\dot{\Psi}$ on a finite piece is analogous to altering a map $f\colon M \to M$ up to homotopy, whilst retaining transversality. The net effect on the fixed points (with signs) is zero.

Remark 3.15 (i) The identity morphism of O_A corresponds to the partially defined map $\dot{\Psi}_{\mathrm{id}}$ with domain of definition the set of paths $(x_1, \dots, x_n)$ such that $A_{x_n,x_1} = 1$, *i.e.*, the set of loops in the graph. The action of $\dot{\Psi}_{\mathrm{id}}$ is by shifting the parameterization of loops. In particular, $\mathrm{Index}(\Psi_{\mathrm{id}}) = 0$.

(ii) If the graph corresponding to A is *complete*, then $\mathrm{Index}(\dot{\Psi}) = 0$ for *every* $\dot{\Psi}$. This follows from the Lefschetz theorem.

The point about the index is that there is a lot of cancellation in the expression (3.2). Taking into account this cancellation, we get a much more computable description of the index

Lemma 3.16 *Let* $\Xi\colon \mathbf{P} \to \mathbf{P}$ *be a partially defined homeomorphism with finite propagation. Let* $m > 0$*. Define*

$$\gamma_m(\Xi) := \sharp\{x \in \mathbf{P} \mid |x| > m,\ |\Xi(x)| \le m\} - \sharp\{x \in \mathbf{P} \mid |x| \le m,\ |\Xi(x)| > m\}.$$

Then

$$\mathrm{Index}_1(\Xi) + \mathrm{Index}_2(\Xi) + \cdots + \mathrm{Index}_m(\Xi) = \gamma_m(\Xi).$$

In particular, if $\Xi = \dot{\Psi}$ *for some geometric endomorphism* Ψ*, then* $\gamma_m = \gamma_{m+1} = \cdots = \mathrm{Index}(\dot{\Psi})$ *for m sufficiently large.*

Proof Let $a(i,j)$ denote the number of strings of length i that are mapped by Ξ to strings of length j. Let $\delta(i,j) := a(i,j) - a(j,i)$. Assume that Ξ alters lengths of strings by at most N. Choose $k > 0$. By definition,

$$\mathrm{Index}_m(\Xi) = \sharp\,(\mathrm{Im}(\Xi) \cap \mathbf{P}_m) - \sharp\,(\mathrm{Dom}(\Xi) \cap \mathbf{P}_m).$$

On the other hand, $\sharp\,\mathrm{Im}(\Xi) \cap \mathbf{P}_m) = \sum_{k=-N}^{N} a(m+k, m)$, while $\sharp\,(\mathrm{Dom}(\Xi) \cap \mathbf{P}) = \sum_{k=-N}^{N} a(m, m+k)$, whence

$$\mathrm{Index}_m(\Xi) = \sum_{k=-N}^{N} \delta(m+k, m).$$

Of course $\delta(i,j) = -\delta(j,i)$. Hence when we take the (formal) sum

$$\mathrm{Index}(\Xi) = \sum_{m=1}^{\infty} \sum_{k=-N}^{N} \delta(m+k, m),$$

a term $\delta(i,j)$ appears exactly twice with opposite signs, if i and j are small enough relative to m. It follows that

$$\sum_{k=1}^{m} \mathrm{Index}_k(\Xi) = \gamma_m(\Xi)$$

because of telescoping.

The last assertion follows from Lemma 3.12. ∎

For instance, in Examples 3.7 and 3.10, $m = 3$ is large enough; note that $\mathrm{Prop}(\dot{\Psi}) \leq 1$. There are 8 strings of length 4 mapped to strings of length 3 and 7 strings of length 3 mapped to strings of length 4, so

$$\mathrm{Index}(\dot{\Psi}) = 8 - 7 = 1.$$

Based on Lemma 3.16, we can give a poynomial formula for the index as follows. Fix m large. Fix j. We count the number of strings of length $m + j$ (for $j = 1, 2, \ldots, N$, which are mapped to strings of length $\leq m$. We refer to the presentation (3.1). Fix i and μ with $|\mu| = k$. Suppose that $|\psi_i(\mu)| \leq |\mu| - j + 1$. Consider a string $w = (\mu, u)$ of length $m + j$, where $|u| = m + j - |\mu|$ is a string (path) from the terminus $t(\mu)$ of μ to i. Then this is mapped under $\dot{\Psi}$ to a string of length $\leq m + j - 1 - j + 1 = m$. Hence for each such i, μ, and u we get a positive contribution to the index. For fixed μ and i the number of possible u's is equal to the number of paths of length $m + j - k$ from $t(\mu)$ to i, which equals $A^{m+j-k}_{t(\mu)i}$. Hence the total positive contribution to the index is

$$\sum_{i=1}^{n}\sum_{j=1}^{N}\sum_{\mu\in W_i,\, |\psi_i(\mu)|\leq|\mu|-j+1} A^{m+j-k}_{t(\mu),i}.$$

For the negative contributions, for $j = 0, 1, \ldots, N - 1$ fix i and μ such that $|\psi_i(\mu)| \geq |\mu| + j + 2$. Then for each $w = (\mu, u)$ of length $m - j$, so that u is a string from $t(\mu)$ to i of length $m - j - k$, the length of $\dot{\Psi}(w)$ is $\geq m - j - 1 + j + 2 = m + 1$. Hence we get a negative contribution to the index. Therefore the total negative contributions is

$$\sum_{i=1}^{n}\sum_{j=0}^{N-1}\sum_{\mu\in W_i\, |\psi_i(\mu)|\geq|\mu|+j+2} A^{m-j-k}_{t(\mu),i}.$$

Therefore we get the following curious, completely explicit, polynomial formula for the index (which is equal to the Lefschetz number of the induced endomorphism of O_A).

Theorem 3.17 *The index of $\dot{\Psi}$, where $\Psi \in G_A$ has presentation* (3.1)*, is given explicitly by the formula*

$$\sum_{i=1}^{n}\sum_{j=1}^{N}\sum_{\mu\in W_i,\, |\psi_i(\mu)|\leq|\mu|-j+1} A^{m+j-k}_{t(\mu),i} - \sum_{i=1}^{n}\sum_{j=0}^{N-1}\sum_{\mu\in W_i\, |\psi_i(\mu)|\geq|\mu|+j+2} A^{m-j-k}_{t(\mu),i}$$

for any $N > \mathrm{Prop}(\dot{\Psi}) = \max_{i,\mu}\big(|\mu| - |\psi_i(\mu)| + 1\big)$ for any m large enough.

For instance, in our main example the above formula with $k = 2$, $m = 3$, $N = 1$ gives

$$\begin{aligned}\mathrm{Index}(\dot{\Psi}) = (A^2_{11} + A^2_{21} + A^2_{11} + A^2_{21}) \\ - (A_{12} + A_{22} + A_{12} + A_{22} + A_{32} + A_{22} + A_{32}) = 8 - 7 = 1.\end{aligned}$$

We can now state our Lefschetz formula for Cuntz–Krieger algebra endomorphisms, at least those coming from simple combinatorics of generators and relations.

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Theorem 3.18 *Let $\Psi \in G_A$ and $\alpha_\Psi \colon O_A \to O_A$ be the corresponding endomorphism. Then the Lefschetz number of α equals the index of Ψ:*

$$\mathrm{Lef}([\alpha_\Psi]) = \mathrm{Index}(\dot{\Psi}).$$

To prove this, we need to show that $\mathrm{Index}(\dot{\Psi}) = \mathbf{Ind}(\Delta, [\alpha_\Psi])$ for an appropriate Δ inducing a duality. Kaminker and Putnam proved such a duality in [11]. We refer the reader to their paper for further details and merely sketch the computation here. Let $s_1, \dots, s_n$ denote the generators for O_A and $t_1, \dots, t_n$ the generators for O_{A^t}.

Define $H_A := \ell^2(\mathbf{P})$, where $\mathbf{P}$ is the set of strings, as above. Let

$$S_i \colon H_A \to H_A,\ S_i(e_w) := A_{i\mathrm{o}(w)} e_{iw},\quad R_j(e_w) := A_{\mathrm{t}(w)j} e_{wj}.$$

Clearly $[S_i, R_j] = 0$, while $S_i^*, R_j] = 0$ modulo finite-rank operators. It is also easy to check that $\sum_j A_{ij} S_j S_j^* = S_i^* S_i$ modulo finite rank operators, and similarly the R_j satisfy the relations for O_{A^t}. We obtain the Busby invariant

$$O_A \otimes O_{A^T} \to \mathbb{B}(H_A)/\mathbb{K}(H_A)$$

of an extension of $O_A \otimes O_{A^t}$ by the compact operators and hence (since O_A is nuclear) a class in $\mathrm{KK}^1(O_A \otimes O_{A^t}, \mathbb{C})$. Kaminker and Putnam prove that Δ induces a duality with dual class the element $w = \sum s_i \otimes t_i^*$. Then $ww^* = w^* w$ and each are projections. Therefore $w + 1 - ww^*$ is a unitary in $O_A \otimes O_A^t$ and so defines an $\widehat{\Delta}$ of $\mathrm{KK}^1(\mathbb{C}, O_A, \otimes O_{A^t})$. Now suppose we have an endomorphism

$$s_i \mapsto t_i := \sum_{\mu \in W_i,\ |\mu| = k} s_{\psi_i(\mu)} s_\mu^*.$$

Then under the endomorphism, $\widehat{\Delta}$ is mapped to

$$\sum_{i,\, \mu \in W_i,\ |\mu| = k} s_{\psi_i(\mu)} s_\mu^* \otimes t_i.$$

To compute the pairing

$$\mathbf{Ind}(\Delta, [\alpha]) = \langle (\alpha_\Psi \otimes 1_{O_{A^t}})_*(\widehat{\Delta}), \Delta \rangle$$

we need to compute the index of the obvious lift Fredholm index of $W_\Psi + (1 - W_\Psi W_\Psi^*)$. However, it is clear that $W_\Psi W_\Psi^*$ is *equal* to $W_\Psi^* W_\Psi$ on $\ell^2(\mathbf{P}_m)$ for $m \geq \mathrm{dmin}(\Psi) + 1$. Hence the sum

$$\sum_{j=1}^{\infty} \dim \ker((W_\Psi)_{|_{\ell^2(\mathbf{P}_j)}}) - \dim \mathrm{ran}(W_\Psi) \cap \ell^2(\mathbf{P}_j))$$

converges, and evidently converges to the analytic index. Now we can regard W_Ψ as the operator induced by the partial permutation $\dot{\Psi}$ of $\mathbf{P}$, in which point masses e_w in the kernel of W_Ψ correspond to words w not in the domain of W_Ψ. We are now in the setting of our earlier discussion of partially defined maps $\mathbf{P} \to \mathbf{P}$, and it is clear that the index of W_Ψ is exactly the same as the index defined in Definition 3.11, and we are done by Section 2.

Remark 3.19 It was mentioned above that the geometric index of an *arbitrary* endomorphism must vanish in the case of a Cuntz algebra. This is of course obvious from the Lefschetz formula since the K-theory of Cuntz algebras vanishes rationally. On the other hand, it does not seem very obvious from a geometric point of view. This sort of thing happens in classical topology of course; one proves the existence of fixed points by homology computations.

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