The Linear Complexes belonging to the Invariant System of Three Quadrics.

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INTRODUCTION.

In the *Proceedings of the London Mathematical Society*, Ser. 2, Vol. 20 (1921), pp. 465–489, Professor H. W. Turnbull has studied the projective invariant theory of three quadrics. The following paper is based on this work and develops one definite section of the theory. From the geometrical point of view the linear complex is now seen to be fundamental in the study of three arbitrary quadrics; particularly when their (2, 2, 2) invariant ϕ_{123} vanishes.*

I give a complete system of the linear complexes belonging to the concomitants of three quadrics. A complete list of the linear complexes fifty in number and a list of some of their invariants expressed in terms of the invariants of the three quadrics is given, §§ 1–4. In §§ 7–11, the linear complexes are determined, and in §§ 12–16 a list of the identities used in the reduction of the linear complexes and in the calculation of their invariants is given, along with typical examples of both processes.

§ 1. *List of Linear Complexes of Three Quadrics shewing Degree in coefficients.*

THE K_2 GROUP.

(1) $(abp)\,a_\gamma\, b_\gamma$	(1, 1, 3)	$(\alpha\beta p)\,c_a\, c_\beta$	(3, 3, 1)
(2) $(abp)\,(a_\beta\, c_a\, b)$	(4, 4, 1)	$(\alpha\beta p)\,(\alpha_b\gamma_a\beta)$	(4, 4, 3)
(3) $(BC)''\,(BC)$	(1, 2, 2)	$(BC)'''\,(BC)$	(3, 2, 2)
(4) $(BC)''\,(BA)\,(AC)$	(3, 2, 2)	$(BC)'''\,(BA)\,(AC)$	(5, 2, 2)

* $\phi_{123}=0$ when the three quadrics can be expressed as the sum of the same five squares (Toeplitz, *Math. Annal.*, XI.)

THE K_3 GROUP.

(5)	$(Bac)(Bp)(c_\beta a)$	(1, 5, 1)	$(B\alpha\gamma)(Bp)(\gamma_b \alpha)$	(3, 3, 3)
(6)	,, $(Bp)(c_a b_\gamma a)$	(4, 3, 4)	,, $(Bp)(\gamma_a \beta_c \alpha)$	(4, 5, 4)
(7)	,, $(Ap)(AB)(c_\beta a)$	(3, 5, 1)	,, $(Ap)(AB)(\gamma_b \alpha)$	(5, 3, 3)
(8)	,, ,, $(AC)(BC)(c_\beta a)$	(3, 5, 3)	,, ,, $(AC)(BC)(\gamma_b \alpha)$	(5, 3, 5)
(9)	,, ,, $(AB)(c_a b_\gamma a)$	(6, 3, 4)	,, ,, $(AB)(\gamma_a \beta_c \alpha)$	(6, 5, 4)
(10)	,, ,, $(BC)(CA)(c_a b_\gamma a)$	(6, 3, 6)	,, ,, $(BC)(CA)(\gamma_a \beta_c \alpha)$	(6, 5, 6)
(11)	,, $(C\alpha\beta)(BC)'' a_\beta c_a$	(5, 5, 3)	,, $(Cab)(BC)''' b_a a_\gamma$	(7, 3, 5)
(12)	,, $(B\alpha\gamma)(acp) b_a b_\gamma$	(4, 3, 4)	,, $(Bac)(\alpha\gamma p)(a_\beta c)$	(4, 5, 4)
(13)	,, $(Cab)(bcp)(BC)$	(1, 3, 3)	,, $(C\alpha\beta)(\beta\gamma p)(BC)$	(3, 5, 5)
(14)	,, ,, $(bcp)(BA)(AC)$	(3, 3, 3)	,, ,, $(\beta\gamma p)(BA)(AC)$	(5, 5, 5)
(15)	,, ,, $(cap)(BC)(a_\gamma b)$	(2, 3, 6)	,, ,, $(\gamma\alpha p)(BC)(\alpha_c \beta)$	(6, 5, 6)
(16)	,, ,, $(cap)(BA)(AC)(a_\gamma b)$	(4, 3, 6)	,, ,, $(\gamma\alpha p)(BA)(AC)(\alpha_c \beta)$	(8, 5, 6)
(17)	,, ,, $(\gamma\alpha p)(BC) c_a b_\gamma$	(4, 3, 6)	,, ,, $(cap)(BC) a_\gamma c_\beta$	(4, 5, 6)
(18)	,, ,, $(\gamma\alpha p)(BA)(AC) c_a b_\gamma$	(6, 3, 6)	,, ,, $(cap)(BA)(AC) a_\gamma c_\beta$	(6, 5, 6)
(19)	,, ,, $(BC)'' b_a c_a$	(5, 3, 3)	,, ,, $(BC)''' a_\beta a_\gamma$	(7, 5, 5)
(20)	,, ,, $(BC)''(b_\gamma a_\beta c)$	(5, 6, 6)	,, ,, $(BC)'''(\beta_c \alpha_b \gamma)$	(7, 6, 6)
(21)	$F_{12}(Cab)(AC) a_\beta$	(3, 4, 2)	$F_{12}(C\alpha\beta)(AC) b_a$	(5, 4, 2)
(22)	$F_{12}(Cab)(AB)(BC) a_\beta$	(3, 6, 2)	$F_{12}(C\alpha\beta)(AB)(BC) b_a$	(5, 6, 2)
(23)	$F_{12}(Abc) c_\beta$	(2, 4, 1)	$F_{12}(A\beta\gamma) b_\gamma$	(2, 4, 3)

THE K_4 GROUP.

(24)	$F_4(Cab)(Bp) b_a$	(4, 3, 2)	$F_4(C\alpha\beta)(Bp) a_\beta$	(4, 5, 2)
(25)	$F_4(Cab)(BC)(Cp) b_a$	(4, 3, 4)	$F_4(C\alpha\beta)(BC)(Cp) a_\beta$	(4, 5, 4)

§ 2. Let the invariants * of the three quadrics be represented by :—

$a_\alpha^2 = \Delta_1$, $(BC)^2 = \phi_{23}$, $b_\gamma^2 = \Theta_{23}$, $(BC)(CA)(AB) = \phi_{123}$,

$(a_\beta c_\alpha b_\gamma a) = \Omega$, $(BCa\alpha)^2 = F_4^2$, $(Abc)(Bca)(AB)(a_\gamma b) = \Sigma_3$,

$(Abc)(Bca)(AB)(a_\beta c_\alpha b) = T_3$, $(Abc)(A\beta\gamma)\, b_\gamma c_\beta = \pi_1$

$(Abc)(A\beta\gamma)(b_\alpha c)({}_\beta a_\gamma) = Q_1$,

§ 3. *Invariants of the Complexes.*

Let I_n denote the invariant of the complex numbered "n" in the table, and I_n^1 the invariant of its dual.

$$I_1 = -\tfrac{3}{8}\Delta_3\phi_{123}, \qquad I_1^1 = \tfrac{3}{2}\Delta_1\Delta_2\Delta_3^{-1} I_1,$$
$$I_2 = -\tfrac{9}{64}\Delta_1\Delta_2\left[\phi_{12}\phi_{123} - \tfrac{4}{9}\pi_3\right], \qquad I_2^1 = \tfrac{3}{2}\Delta_3 I_2,$$
$$I_3 = -\left[\phi_{23}\phi_{123} - \tfrac{4}{9}\pi_1\right], \qquad I_3^1 = \tfrac{3}{2}\Delta_1 I_3,$$
$$I_4 = -\phi_{123}\left[\phi_{12}\phi_{13} + \tfrac{1}{6}\Delta_1\phi_{23} - \tfrac{2}{3}F_4^2\right] - \tfrac{8}{27}Q_1, \qquad I_4^1 = \tfrac{3}{2}\Delta_1 I_4,$$
$$I_5 = -\tfrac{1}{16}\Delta_2^2\phi_{123}, \qquad I_5^1 = \tfrac{3}{2}\Delta_1\Delta_2\Delta_3 I_5,$$
$$I_6 = -\tfrac{3}{128}\Delta_1\Delta_2\Delta_3\left[\phi_{13}\phi_{123} - \tfrac{4}{9}\pi_2\right], \qquad I_6^1 = \tfrac{3}{2}\Delta_2 I_6,$$
$$I_7 = -\tfrac{1}{96}\Delta_1\Delta_2^2\phi_{123}, \qquad I_7^1 = \tfrac{3}{2}\Delta_1\Delta_2^{-1}\Delta_3 I_7,$$
$$I_8 = -\tfrac{1}{576}\Delta_1\Delta_2^2\Delta_3\phi_{123}, \qquad I_8^1 = \tfrac{3}{2}\Delta_1\Delta_2^{-1}\Delta_3 I_8,$$
$$I_9 = -\tfrac{1}{256}\Delta_1^2\Delta_2\Delta_3\left[\phi_{123}\phi_{13} - \tfrac{4}{9}\pi_2\right], \qquad I_9^1 = \tfrac{3}{2}\Delta_2 I_9,$$
$$I_{10} = -\tfrac{1}{1536}\Delta_1^2\Delta_2\Delta_3^2\left[\phi_{123}\phi_{13} - \tfrac{4}{9}\pi_2\right], \qquad I_{10}^1 = \tfrac{3}{2}\Delta_2 I_{10},$$
$$I_{23} = -\tfrac{3}{8}\Delta_2\left[\phi_{12}\phi_{123} - \tfrac{4}{9}\pi_3\right], \qquad I_{23}^1 = \tfrac{3}{2}\Delta_3 I_{23},$$
$$I_{24} = -\tfrac{1}{16}\Delta_1\Delta_2\left[\phi_{13}\phi_{123} - \tfrac{4}{9}\pi_2\right], \qquad I_{24}^1 = \tfrac{3}{2}\Delta_2 I_{24},$$
$$I_{25} = -\tfrac{1}{96}\Delta_1\Delta_2\Delta_3\left[\phi_{13}\phi_{123} - \tfrac{4}{9}\pi_2\right], \qquad I_{25}^1 = \tfrac{3}{2}\Delta_2 I_{25},$$

§ 4. Let K_{rs} denote the simultaneous invariant of the complex (r) and the complex (s); ${}_rK_s$ the invariant of the complex (s) and the dual complex (r); and ${}_{rs}K$ the invariant of the dual complexes (r) and (s).

$$K_{13} = \Sigma_3, \qquad {}_{13}K = \Sigma_3^1,$$
$$K_{23} = T_3 - \tfrac{1}{16}\Delta_1\Delta_2\phi_{123}, \qquad {}_{23}K = T_3^1 - \tfrac{3}{32}\Delta_1\Delta_2\Delta_3\phi_{123},$$
$${}_1K_1 = \tfrac{1}{16}\Delta_1\Delta_2\Delta_3 - \Omega, \qquad {}_5K_5 = \tfrac{1}{6}\Delta_2\left[\tfrac{1}{16}\Delta_1\Delta_2\Delta_3 - \Omega\right],$$
$${}_7K_7 = \tfrac{1}{36}\Delta_1\Delta_2\left[\tfrac{1}{16}\Delta_1\Delta_2\Delta_3 - \Omega\right], \qquad {}_8K_8 = \tfrac{1}{216}\Delta_1\Delta_2\Delta_3\left[\tfrac{1}{16}\Delta_1\Delta_2\Delta_3 - \Omega\right],$$

where Σ_3^1 and T_3^1 are the duals of Σ_3 and T_3.

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* *Cf. Proc. Lond. Math. Soc., loc. cit.*, p. 483. Type 9 on this table is reducible. *Proc. Lond. Math. Soc.* Vol. 22. Series 2. Records p. iii. (1923).

§ 5. *Notation.*

In symbolic form let the point, plane, and line equations of the three quadrics be :—

$$f_1 = a_x^2 = a'^2_x = \dots , \qquad \phi_1 = u_\alpha^2 = u_{\alpha'}^2 = \dots ,$$
$$f_2 = b_x^2 = b'^2_x = \dots , \qquad \phi_2 = u_\beta^2 = u_{\beta'}^2 = \dots ,$$
$$f_3 = c_x^2 = c'^2_x = \dots , \qquad \phi_3 = u_\gamma^2 = u_{\gamma'}^2 = \dots ,$$

and

$$\pi_1 = (A_{12}p_{34} + A_{13}p_{42} + A_{14}p_{23} + A_{34}p_{12} + A_{42}p_{13} + A_{23}p_{14})^2 = (Ap)^2 = (A'p)^2,$$
$$\pi_2 = (B_{12}p_{34} + B_{13}p_{42} + B_{14}p_{23} + B_{34}p_{12} + B_{42}p_{13} + B_{23}p_{14})^2 = (Bp)^2 = (B'p)^2,$$
$$\pi_3 = (C_{12}p_{34} + C_{13}p_{42} + C_{14}p_{23} + C_{34}p_{12} + C_{42}p_{13} + C_{23}p_{14})^2 = (Cp)^2 = (C'p)^2,$$

where $a_x = \Sigma_i a_i x_i$, $u_a = \Sigma_i u_i \alpha_i = (u\alpha)$: $i = 1, 2, 3, 4$,

$$A_{ij} = (a_1 a_2)_{ij} = (a_{1i} a_{2j} - a_{1j} a_{2i}), \text{ or briefly } A = (a_1 a_2),$$
$$\alpha_{ijk} = (a_1 a_2 a_3)_{ijk} = \mid a_{1i} a_{2j} a_{3k} \mid \text{ or briefly } \alpha = (a_1 a_2 a_3),$$
$$\alpha = (A\, a_3) = (a_3\, A) \text{ etc.}$$

with similar meanings for B, β, C, γ and dashed letters.

§ 6. We shall require four types of factors, F_1, F_2, F_3, and F_4 :—

(i) Six of type F_1 : (Ap), (Bp), (Cp), a_α, b_β, c_γ.

(ii) Twenty-one of type F_2 : a_β, a_γ, b_γ, b_α, c_α, c_β, (BC), (CA), (AB), (bcp), (cap), (abp), $(\beta\gamma p)$, $(\gamma\alpha p)$, $(\alpha\beta p)$, $(BC)''$, $(CA)''$, $(AB)''$, $(BC)'''$, $(CA)'''$, $(AB)'''$.

(iii) Eighteen of type F_3 : (Abc), (Bca), (Cab), $(A\beta\gamma)$, $(B\gamma\alpha)$, $(C\alpha\beta)$, F_{ij} and G_{ij}, $i \neq j : i, j = 1, 2, 3$.

(iv) Three of type F_4: $(BCa\alpha) = F_4$, $(CAb\beta) = F'_4$, $(ABc\gamma) = F''_4$,

where $\quad (BC)'' = (BCaap) = (B,\ Ca,\ ap) = \Omega_B\, (b_1\, Ca)\, (b_2\, ap)$

$$= (b_1\, Ca)(b_2\, ap) - (b_2\, Ca)\, (b_1\, ap) ;$$

$$(BC)''' = (BC\alpha\alpha p) = \Omega_{BC} (b_1 c_1 p)\, b_{2\alpha}\, c_{2\alpha} = (b_1 c_1 p)\, b_{2\alpha}\, c_{2\alpha}$$
$$- (b_2 c_1 p)\, b_{1\alpha}\, c_{2\alpha} + (b_2 c_2 p)\, b_{1\alpha}\, c_{1\alpha} - (b_1 c_2 p)\, b_{2\alpha}\, c_{1\alpha},$$

$$F_{12} = (Apb\beta) = \Omega_A\, (a_1\, bp)\, a_{2\beta},$$
$$G_{12} = (Apb\gamma) = \Omega_A\, (a_1\, bp)\, a_{2\gamma},$$
$$F_4 = (BCa\alpha) = \Omega_B\, (b_1\, Ca)\, b_{2\alpha}.$$

where also $(BCa\alpha) + (CBa\alpha) = (BC)\, a_\alpha$; the rest being of type $(abcd)$.

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The symbol $(a_\beta\, c_{\,\alpha} b) = a_\beta\, c_\beta\, c_\alpha\, b_\alpha$ and is called a chain. If the two end letters are the same it is called a closed chain.

The notation $F \equiv \phi$ means that $F - \phi$ is reducible, *i.e.* can be expressed in terms of simpler forms.

§ 7. *General Proposition I.*

If the product MN represents a linear complex where M contains the p factor and N is the product of F_1 and F_2 factors then M must contain an even number of capital letters.

For the only factors involving capital letters which appear in N are of the type (BC), and therefore N must contain an even number of capital letters, and since in any invariant all letters appear in pairs M must also contain an even number of capital letters.

§ 8. K_1 *Group.*

Since the only available factors are of the types (Ap) and a_α there are obviously no linear complexes in this group.

§ 9. K_2 *Group.*

The p factor must be one of the types :—

(1) (abp) with its dual $(\alpha\beta p)$,

(2) $(BC)''$ with its dual $(BC)'''$.

According to proposition I. the p factor (Ap) is inadmissible. Linear Complexes in this group are then of types :—

(1) $(abp)\,(a_\gamma\, b)$ and $(abp)\,(a_\beta\, c_\alpha\, b)$,

(2) $(BC)''\,(BC)$ and $(BC)''\,(BA)\,(AC)$,

and their duals.

§ 10. K_3 *Group.*

From the table (*Proc. Lond. Math. Soc.*, *loc. cit.* p. 484) we see that linear complexes belonging to this group must contain F_3 factors of the types

I. (Abc) and its dual $(A\beta\gamma)$,

II. F_{ij},

III. G_{ij}, $ij = 1, 2, 3$ $i \neq j$.

These shall be considered in order.

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I. This gives one of four types (*loc. cit.*, p. 485)—

(α) $(Bac)\,N$ with its dual $(B\alpha\gamma)\,N$,

(β) $(Bac)\,(C\alpha\beta)\,N$,

(γ) $(Bac)\,(B\alpha\gamma)\,N$,

(δ) $(Bac)\,(Cab)\,N$ with its dual $(B\alpha\gamma)\,(C\alpha\beta)\,N$,

where N consists of F_2 and F_1 factors.

Case (α) $(Bac)\,N$. By proposition I. N cannot contain any factors of the types $(BC)''$, $(BC)'''$ or (abp): N must therefore contain one factor of the types (Bp) or (Ap).

Linear complexes in this group (α) are then of types:—

(1) $(Bac)\,(Bp)\,(c_\beta\, a)$ and $(Bac)\,(Bp)\,(c_\alpha\, b_\gamma\, a)$,

(2) $(Bac)\,(Ap)\,(AB)\,(c_\beta\, a)$ and $(Bac)\,(Ap)\,(AB)\,(c_\alpha\, b_\gamma\, a)$,

$(Bac)\,(Ap)\,(BC)\,(CA)\,(c_\beta\, a)$ and $(Bac)\,(Ap)\,(BC)\,(CA)\,(c_\alpha\, b_\gamma\, a)$,

and their duals.

Case (β) $(Bac)\,(C\alpha\beta)\,N$. Possible p factors are of types:—

(1) (Ap), (Bp), (Cp); (2) (bcp); (3) (abp); (4) (cap); (5) $(BC)''$; (6) (CA).''

From the Reduction System (*loc. cit.*, pp. 476–478) we see that in cases (2), (3), (4), (5), and (6) identities exist which may reduce any product containing those factors. Since the dual of $(Bac)\,(C\alpha\beta)$ is of the same type we do not require to consider separately the cases $(\beta\gamma p)$, $(BC)'''$ and (AB).'''

These cases shall now be considered in order: and since the only factors in N which do not involve p are of types (BC) and b_γ, the complements of the letters unpaired in the product $(Bac)\,(C\alpha\beta)$ and the p factors are separated into two groups, one of capital letters and one of other letters: giving

(1) By proposition I. no linear complex of this type exists,

(2) $(Bac)\,(C\alpha\beta)\,(bcp)\,[B,\ C]\,[a,\ b,\ \alpha,\ \beta]$,

(3) $(Bac)\,(C\alpha\beta)\,(abp)\,[B,\ C]\,[b,\ c,\ \alpha,\ \beta]$,

(4) $(Bac)\,(C\alpha\beta)\,(cap)\,[B,\ C]\,[\alpha,\ \beta]$,

(5) $(Bac)\,(C\alpha\beta)\,(BC)''\,[a,\ c,\ \alpha,\ \beta]$,

(6) $(Bac)\,(C\alpha\beta)\,(CA)''\,[A,\ B]\,[a,\ c,\ \alpha,\ \beta]$.

Since $(Bac)(C\alpha\beta)c_\beta$ is reducible (*loc. cit.*, p. 481) we need only retain

(2) $(Bac)(C\alpha\beta)(bcp)(BC)a_\beta b_a$*,

and $(Bac)(C\alpha\beta)(bcp)(BA)(AC)a_\beta b_a$,*

(4) $(Bac)(C\alpha\beta)(cap)(BC)(_\beta a_\gamma b_a)$,*

and $(Bac)(C\alpha\beta)(cap)(BA)(AC)(_\beta a_\gamma b_a)$,*

(5) $(Bac)(C\alpha\beta)(BC)'' a_\beta c_a$,

(6) $(Bac)(C\alpha\beta)(CA)''(AB)a_\beta c_a$,*

Case (γ) $(Bac)(B\alpha\gamma)N$ which we notice is a self dual form.

According to the Reduction System (*loc. cit.*, pp 476–478) the only F_1 and F_2 factors which are irreducible when combined with (Bac) and $(B\alpha\gamma)$ are :—

(1) (Ap), (Bp), (Cp), a_β, a_γ, b_γ, b_a, c_a, c_β, (AB), (BC), (CA),

and (2) (bcp), (cap), (abp), $(\beta\gamma p)$, $(\gamma\alpha p)$, $(\alpha\beta p)$, $(BC)''$, $(AB)''$, $(BC)'''$, $(AB)'''$, where in any of cases (2) identities exist which may reduce the products.

Now since N must contain one p factor and since a and c, α and γ may be interchanged in the product $(Bac)(B\alpha\gamma)$, we need only consider these types :—

(1) (Ap) or (Bp), (2) (bcp), (3) (cap), (4) (BC),'' (5) $(BC)'''$.

Factors such as $(\beta\gamma p)$ merely come as duals of cases (2) and (3).

Let us consider each case in turn :—

(1) By proposition I. there are no linear complexes of this type,

(2) $(Bac)(B\alpha\gamma)(bcp)[a, b, \alpha, \gamma]$,

(3) $(Bac)(B\alpha\gamma)(cap)[\alpha, \gamma]$,

(4) $(Bac)(B\alpha\gamma)(BC)''[B, C][a, c, \alpha, \gamma]$,

(5) $(Bac)(B\alpha\gamma)(BC)'''[B, C][a, c, \alpha, \gamma]$,

In case (2) $[a, b, \alpha, \gamma]$ must be $b_a a_\gamma$.

In case (3) (α, γ) must be $b_a b_\gamma$ as any other chain would combine with (cap) to give a linear complex as a factor.

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* This denotes that the form is reducible.

In case (4) there is a factor $(BC)''$ $[B, C]$ and similarly in case (5) there is a factor $(BC)'''$ $[B, C]$.

Linear complexes then in this group are of types:—

(2) $(Bac)\,(B\alpha\gamma)\,(bcp)\,b_{\alpha}\,a_{\gamma}$,

(3) $(Bac)\,(B\alpha\gamma)\,(cap)\,b_{\alpha}\,b_{\gamma}$ and their duals.

Types (2) and (3) are equivalent.

Case (δ) $(Bac)\,(Cab)\,N$. From the Reduction System (*loc. cit.*, pp. 476–478) we see that the F_1 and F_2 factors which are irreducible when combined with $(Bac)\,(Cab)$ are,

(1) (Ap), (Bp), (Cp), a_{β}, a_{γ}, b_{γ}, b_{α}, c_{α}, c_{β}, (bcp), (cap), (abp), (BC), (CA), (AB), and $(BC)''$

(2) $(\beta\gamma p)$, $(\gamma\alpha p)$, $(\alpha\beta p)$, $(CA)''$, $(AB)''$, and $(BC)'''$,

where in any of cases (2) identities exist which may possibly reduce the products.

Now N must contain one p factor and since $(Bac)\,(Cab)$ is symmetrical in the symbols of the second and third quadrics we need only consider the cases in which the p factors are

(1) (Ap) or (Bp), (2) (bcp), (3) (cap), (4) $(\beta\gamma p)$,
(5) $(\gamma\alpha p)$, (6) $(BC)''$, (7) $(BC)'''$, (8) $(CA)''$.

These shall be considered in order, as in previous cases.

(1) By Proposition I. there are no linear complexes of this type,

(2) $(Bac)\,(Cab)\,(bcp)\,[B, C]$,

(3) $(Bac)\,(Cab)\,(cap)\,[B, C]\,[a, b]$,

(4) $(Bac)\,(Cab)\,(\beta\gamma p)\,[B, C]\,[b, c, \gamma, \beta]$,

(5) $(Bac)\,(Cab)\,(\gamma\alpha p)\,[B, C]\,[b, c, \gamma, \alpha]$,

(6) $(Bac)\,(Cab)\,(BC)''\,[b, c]$,

(7) $(Bac)\,(Cab)\,(BC)'''\,[b, c]$,

(8) $(Bac)\,(Cab)\,(CA)''\,[A, B]\,[b, c]$.

In cases (2), (3), (4), and (5) $[B, C]$ may either be (BC) or $(BA)\,(CA)$. In case (8) $[A, B]$ must be (AB) since $(AC)\,(CB)$ would involve the factor $(AC)\,(AC)''$.

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In case (3) $[a, b]$ must be $(a_\gamma b)$, since $(a_\beta c_\alpha b)$ would involve the factor $(cap)(c_\beta a)$.

In case (4) (b, c, γ, β) must be $b_\gamma c_\beta$, since $(\gamma_\alpha \beta)(c_\alpha b)$ would involve the factor $(\beta\gamma p)\, a_\gamma\, a_\beta$.

In case (5) $[b, c, \gamma, \alpha]$ is $b_\gamma c_\alpha$ or $(c_\beta a_\gamma) b_\alpha$.

In cases (6), (7), and (8) (b, c) is either $(b_\alpha c)$ or $(b_\gamma a_\beta c)$.

Thus in this group we may have linear complexes of the following types and of their duals :—

$$(2) \begin{cases} (Bac)(Cab)(bcp)(BC), \\ (Bac)(Cab)(bcp)(BA)(AC), \end{cases}$$

$$(3) \begin{cases} (Bac)(Cab)(cap)(BC)(a_\gamma b), \\ (Bac)(Cab)(cap)(BA)(AC)(a_\gamma b), \end{cases}$$

$$(4) \begin{cases} (Bac)(Cab)(\beta\gamma p)(BC)\, b_\gamma\, c_\beta,^* \\ (Bac)(Cab)(\beta\gamma p)(BA)(AC)\, b_\gamma\, c_\beta,^* \end{cases}$$

$$(5) \begin{cases} (Bac)(Cab)(\gamma\alpha p)(BC)\, b_\gamma\, c_\alpha, \\ (Bac)(Cab)(\gamma\alpha p)(BA)(AC)\, b_\gamma\, c_\alpha, \\ (Bac)(Cab)(\gamma\alpha p)(BC)(c_\beta a_\gamma)\, b_\alpha,^* \\ (Bac)(Cab)(\gamma\alpha p)(BA)(AC)(c_\beta a_\gamma)\, b_\alpha,^* \end{cases}$$

$$(6) \begin{cases} (Bac)(Cab)(BC)''(b_\alpha c), \\ (Bac)(Cab)(BC)''(b_\gamma a_\beta c),^* \end{cases}$$

$$(7) \begin{cases} (Bac)(Cab)(BC)'''(b_\alpha c),^* \\ (Bac)(Cab)(BC)'''(b_\gamma a_\beta c), \end{cases}$$

$$(8) \begin{cases} (Bac)(Cab)(AB)''(AC)(b_\alpha c),^* \\ (Bac)(Cab)(AB)''(AC)(b_\gamma a_\beta c),^* \end{cases}$$

II. F_{ij}. Factors containing F_{ij} where $F_{12} = (Apb\beta)$.

From table "C" (*loc. cit.*, p. 486) we see that the admissible types are :—

(α) $F_{12}\,(Cab)\,N$ with its dual,

(β) $F_{12}\,(Abc)(A\beta\gamma)\,N$,

(γ) $F_{12}\,(Abc)\,N$,

(δ) $F_{12}\,N$ with its dual,

where N consists of F_1 and F_2 factors.

By proposition I. types (β) and (δ) are impossible. Possible types thus are:—

$$(\alpha)\begin{cases} F_{12}(Cab)(AC)\,a_\beta, \text{ and } F_{12}(Cab)(AC)(a_\gamma\, b_\alpha\, c_\beta)\,,^* \\ F_{12}(Cab)(AB)(BC)\,a_\beta, \text{ and } F_{12}(Cab)(AB)(BC)(a_\gamma\, b_\alpha\, c_\beta),^* \end{cases}$$

(γ) $F_{12}(Abc)\,c_\beta$, and $F_{12}(Abc)(c_\alpha\, b_\gamma\, a_\beta)\,.^*$

III G_{ij}. Factors containing G_{ij} where $G_{12}=(Ap b\gamma)$.

From table D (*loc. cit.*, p 487) we see that possible types linear in p are:—

$$(\alpha)\ G_{12}(Abc)(A\beta\gamma)\,c_\beta\,[A],$$
$$(\beta)\ G_{12}\,N.$$

But by proposition I. neither of these types can give a linear complex.

§ 11. *K_4 Group.* F_4 factors are of the type $F_4''=(Abc\gamma)$.

From list G (*loc. cit.*, p. 489) we see that possible linear complexes of this type are:—

$$(\alpha)\ F_4''\,F_{13}\,[B],$$
$$(\beta)\ F_4''\,(Abc)\,b_\gamma\,[B],$$

By proposition I. there are no linear complexes of type (α) but there are two of type (β)

$$(\beta)\ F_4''\,(Abc)\,b_\gamma\,(Bp), \text{ and } F_4''\,(Abc)(BA)\,b_\gamma\,(Ap)\,,$$

There are no others in this group since from the Reduction System (*loc. cit.*, p. 477) we see that $F_4''(Cp)$ is reducible.

By means of the following identities some of the complexes (*) were reduced until only fifty were left, twenty-five pairs, each member of a pair being the dual of the other member.

§ 12. *Identities used in the reduction of the complexes.*

$$\begin{aligned}
&\text{I.} & F_4\,c_\beta &\equiv (BC)\,a_\beta\,c_\alpha - (Bac)(C\alpha\beta),\\
&\text{II.} & F_4\,(cap) &\equiv (Bac)\,F_{31}\,,\\
&\text{III.} & F_4\,(CA)'' &\equiv (AB)(Cab)\,G_{32}\,,\\
&\text{IV.} & F_4\,(bcp) &\equiv (Bca)\,G_{32} + (Bca)\,b_\alpha\,(Cp) - (Cab)\,c_\alpha\,(Bp),\\
&\text{V.} & G_{32}\,a_\beta &\equiv (Cab)(\alpha\beta p) + (C\alpha\beta)(abp) + G_{31}\,b_\alpha\,,\\
&\text{VI.} & F_{21}\,b_\gamma &\equiv G_{21}\,b_\alpha + (B\alpha\gamma)(abp)\,,\\
&\text{VII.} & F_{31}\,b_\gamma &\equiv G_{32}\,a_\gamma - (Cab)(\gamma\alpha p)\,,\\
&\text{VIII.} & (BC)''\,b_\gamma &\equiv (Cab)\,G_{21} + (BC)(abp)\,a_\gamma\,,\\
&\text{IX.} & (AB)''\,(Cab) &= (Abc)(Bca)(Cp) - (Abc)(acp)(BC)\\
&&& \quad - (Bca)(cbp)(CA),
\end{aligned}$$

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§ 13. *Typical example of a reduction.*

To reduce $(Bac)\,(C\alpha\beta)\,(BC)\,(bcp)\,a_\beta\,b_\alpha = E$ (type β_2)

$$E \equiv F_4\,(BC)\,(bcp)\,c_\beta\,a_\beta\,b_\alpha \quad \text{by I.}$$
$$\equiv (Bca)\,G_{32}\,(BC)\,c_\beta\,a_\beta\,b_\alpha \quad \text{by IV.}$$
$$\equiv (Bca)\,(BC)\,c_\beta\,b_\alpha\,[(Cab)\,(\alpha\beta p) + (C\alpha\beta)\,(abp) + G_{31}\,b_\alpha]$$
$$\equiv (Bca)\,(Cab)\,(BC)\,(\alpha\beta p)\,c_\beta\,b_\alpha \quad \text{(type 17)}$$

since the second term is reducible by I., and the third term has the factor b_α^2.

§ 14. For the purposes of the calculation of their invariants the linear complexes may be divided into three groups :—

A. $(abp)\,a_\delta\,b_\epsilon$, B. $(\alpha\beta p)\,d_\alpha\,e_\beta$, C. $(Bp)\,(BD)$,

where the degrees of δ, ϵ; d, e; and D are respectively 3 mod 4, 1 mod 4, and 2 mod 4, and where δ, ϵ; d, e; and D form closed chains.

$$I_A = (aba'b')\,a_\delta\,b_\epsilon\,a'_{\delta'}\,b'_{\epsilon'} = -\tfrac{1}{4}\,(AB)\,(A\delta\delta')\,(B\epsilon\epsilon'),\ A = aa'$$
$$I_B = (\alpha\beta\alpha'\beta')\,d_\alpha\,e_\beta d'_{\alpha'} e'_{\beta'} = -\tfrac{9}{16}\,\Delta_1\,\Delta_2\,(AB)\,(dd'A)\,(ee'B),\ B = bb'$$
$$I_C = (BB')\,(BD)\,(B'D') = \tfrac{1}{6}\,\Delta_2\,(DD').$$

§ 15. Other identities used in the calculation of the invariants were :—

I. $(abcd)\,e_x = (ebcd)\,a_x + (aecd)\,b_x + (abed)\,c_x + (abce)\,d_x$,

II. $a_\gamma\,a'_{\gamma'}\,b_\gamma\,b'_{\gamma'}\,\{A, B\} = \tfrac{3}{8}\,\Delta_3\,(BC)\,(CA)\,\{A, B\}$
$A = aa'$, $B = bb'$.

III. $(Bac)\,(BB')\,(B'ef) = \tfrac{1}{6}\,\Delta_2\,(acef)$,

IV. $(A\beta\beta')\,(C\beta\beta')\,\{A, C\} = \tfrac{3}{2}\,\Delta_2\,(AB)\,(BC)\,\{A, C\}$,

V. $(AB)\,(A'B)\,(AC)\,(A'C) = \phi_{12}\,\phi_{13} + \tfrac{1}{6}\,\Delta_1\,\phi_{23} - \tfrac{2}{3}F_4^2$,

VI. $(AB)\,(AB')\,(B'C)\,(CA')\,(A'B) = \phi_{12}\,\phi_{123} - \tfrac{4}{9}\,\pi_3$.

§ 16. *Typical example of the calculation of an invariant.*

To calculate I_2. I_2 is of the type A where $\delta = \beta\,.\,c_\beta$ and $\epsilon = \alpha\,.\,c_\alpha$.

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Therefore $I_2 = -\frac{1}{4}(AB)(A\beta\beta')(B\alpha\alpha')\, c_\beta\, c_\alpha\, c'_{\beta'}\, c'_{\alpha'}$

$$= -\tfrac{1}{16}(AB)(A\beta\beta')(B\alpha\alpha')(C\beta\beta')(C\alpha\alpha')$$

$$= -\tfrac{9}{64}\Delta_1\Delta_2(AB)(AB')(CB')(BA')(CA') \quad \text{by IV.}$$

$$= -\tfrac{9}{64}\Delta_1\Delta_2\left[\phi_{12}\,\phi_{123} - \tfrac{4}{9}\pi_2\right] \quad \text{by VI.}$$

§ 17. Thus with the exception of types a^2_β, $(Abc)^2$—*all irreducible invariants of three quadrics are directly interpreted as invariants of the concomitant linear complexes.*

The vanishing of an invariant of the quadrics necessitates the vanishing of one or more of the invariants of these complexes. For example, if $\phi_{123} = 0$, then I_1, I_4, I_5, I_7, I_8 and I', etc., all vanish. This means that the complexes, of which these are the invariants, are special and that their directrices are projective invariants of the three quadrics. This is the simplest case of surfaces for which such invariant straight lines exist, there being, for example, no such invariants belonging to a system of two quadrics.

It is also interesting to notice that some of the invariants of the quadrics never occur as simple factors of the invariants of the complexes; *e.g.*, the invariant of degrees (244) *i.e.* π_1 never occurs as a factor, but $(\phi_{123}\ \phi_{23} - \frac{4}{9}\pi_1)$ is always the factor whenever π_1 appears. This suggests that when considering this geometrical interpretation of the invariants of the three quadrics, the (244) invariant should be taken as $\phi_{123}\,\phi_{23} - \frac{4}{9}\pi_1$ and not the more simple π_1.