CORRECTION TO "TRANSITIVITIES IN PROJECTIVE PLANES"

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For basic definitions of terms and symbols, see **(3)**. When we refer to theorems by number, it is to be understood that these are theorems of the basic paper.[1] Professor Pickert has pointed out an error in the proof of Theorem 16 (ii). As stated, the theorem is false. Case IV of Theorem 4 shows that the nearfield plane of order 9 is a counter-example. The dual nearfield plane of order 9 is also a counter-example.

We shall now state and prove a correct version of this Theorem.

THEOREM 16 (ii). (*Given a projective plane which is* p_1-L_1 *transitive and* p_2-L_2 *transitive, where* $p_1 \neq p_2$ *and* $L_1 \neq L_2$.) *If* p_1 *is on neither* L_1 *nor* L_2, p_2 *is on neither* L_1 *nor* L_2, *and* p_1 *and* p_2 *are not collinear with the intersection* r *of* L_1 *with* L_2, *then the plane is Desarguesian unless* $n = 9$.

Proof: The theorem differs from the original theorem only in excepting the case where $n = 9$. The error in the original proof arose out of the assumption that the p_1-L_1 and p_2-L_2 perspectivities generate a group which is doubly transitive on the points of the line p_1p_2. If this collineation group is indeed doubly transitive on the points of p_1p_2, then the original proof goes through. Hence we proceed to investigate the permutation group on p_1p_2.

Let G denote the group of collineations generated by the p_1-L_1 perspectivities and the p_2-L_2 perspectivities. Let the line p_1p_2 be denoted by L_∞ and let $L_1 \cap L_\infty = q_1$, $L_2 \cap L_\infty = q_2$. Let G_1 be the permutation group on L_∞ induced by G.

Now, it follows from the hypotheses that p_1, q_1, p_2, and q_2 are four distinct points. If $n = 3$, the plane is Desarguesian. If n is greater than 3, there is at least one other point t on L_∞. Under the p_2-L_2 perspectivities, t can be carried into every point on L_∞ except p_2 and q_2. Under the p_1-L_1 perspectivities, t can be carried into every point on L_∞ except p_1 or q_1.

It follows that G_1 is at least simply transitive on the points of L_∞. Let $G_1(p_i)$ be the subgroup of G_1 which fixes p_i. G_1 will be doubly transitive if and only if $G_1(p_1)$ is transitive on all of the points of L_∞ other than p_1. Now the subgroup of $G_1(p_1)$ induced by the p_1-L_1 perspectivities is transitive on the

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[1]Although the author was not aware of this fact when **(3)** was written, most of the theorems in Part 2 are included in **(4)**.

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points of L_∞ other than p_1 and q_1. Hence, a necessary condition for G_1 to fail to be doubly transitive is that q_1 is fixed by $G_1(p_1)$. Since G_1 is at least simply transitive, we can generalize this condition so that for each $p_i \in L_\infty$ there is a unique point $q_i \in L_\infty$ such that $G_1(p_i)$ fixes q_i. Thus, $G_1(p_i)$ is included in $G_1(q_i)$ and is transitive on points of L_∞ other than p_i and q_i. $G_1(q_i)$ must fix some point on L_∞ other than q_i. This point can be none other than p_i. Hence $G_1(p_i)$ and $G_1(q_i)$ include each other, and $G_1(p_i) = G_1(q_i)$. (The proof that $G_1(p_i) = G_1(q_i)$ was first made by the author in a form which applied only to finite planes; the author is indebted to Professor Pickert for pointing out that finiteness is not required.)

Thus, either G_1 is doubly transitive (and the original proof goes through) or the set of points on L_∞ can be divided into pairs (p_i, q_i) such that every collineation of G which fixes one point of a pair also fixes the other point. Following Andre **(1)**, let us call such pairs of points "admissible pairs." We shall assume from here on that G_1 is not doubly transitive.

The image of an admissible pair under any collineation of G is an admissible pair. Now (p_1, q_1) is an admissible pair and the plane is p_1-L_1 transitive, where $L_1 = rq_1$. It follows that the plane is p_i-L_i transitive for each $p_i \in L_\infty$, where L_i is the line rq_i, since the collineation which carries p_1 into p_i transforms the p_1-L_1 group of perspectivities into the p_i-L_i group of perspectivities. In each case, $G_1(p_i)$ is transitive on the points of L_∞ other than q_i. Thus, every point on L_∞ belongs to exactly one admissible pair. This will be impossible if n is even; we shall henceforth assume that n is odd.

Now the p_i-L_i group of perspectivities is of order $n-1$. Let (p_j, q_j) be an admissible pair, where $i \neq j$. By the p_i-L_i transitive property, there is a perspectivity ρ_i with centre p_i and axis L_i which carries p_j into q_j. But the image of an admissible pair must be an admissible pair; the collineation which carries p_j into q_j must carry q_j into p_j. Thus ρ_i is of order two. The roles of p_i and q_i are interchangeable; thus, there is a perspectivity σ_i of order two with q_i as centre and $p_i r$ as axis. The product of two perspectivities of order two in which the centre of each is on the axis of the other is a perspectivity of order two which fixes all of the points on the line of centres. **(2**, Lemma 6**)** Hence, every perspectivity of order two with centre p_i and axis L_i produces the same permutation of points on L_∞ as does σ_i.

We had previously established that, for every admissible pair (p_j, q_j) there was a perspectivity of order two with centre p_i and axis L_i $(i \neq j)$ which interchanged p_j with q_j. The uniqueness property just established then implies that ρ_i interchanges the points of every admissible pair except p_i and q_i. In other words, for each admissible pair (p_i, q_i) the perspectivity of order two with centre p_i and axis L_i interchanges the points within each admissible pair other than the pair (p_i, q_i).

Now let us set up a co-ordinate system. Take the point r as the origin 0, and choose some admissible pair as the points A and B (the centres of the pencils $x =$ constant and $y =$ constant, respectively). It can be readily verified that the perspectivity with A as centre and the line $y = 0$ as axis which carries the point $(1, 1)$ into $(1, a)$ also carries (c, d) into (c, da) and (m) into (ma), where (c, d) represents any point not on L_∞, and (m) represents the common point on L_∞ for all lines of slope m. Likewise, the perspectivity with B as centre and the line $x = 0$ as axis which carries $(1, 1)$ into $(a, 1)$ also carries (c, d) into (ca, d) and (m) into (am).

The co-ordinate system will then have the following properties:

(i) *The co-ordinatisation is linear.*

(ii) *Multiplication is associative.*

(iii) $(c + b)a = ca + ba$.

Properties (i) and (ii) follow from Theorem 6. Property (iii) follows from an argument similar to that used in Theorem 15.

The uniqueness property of involutions on L_∞ implies that there is exactly one element i of multiplicative order two. Consider the following two perspectivities:

$$\rho\colon (c, d) \to (c, di),\ (m) \to (mi)$$
$$\sigma\colon (c, d) \to (ci, d),\ (m) \to (im).$$

The image of (m) under $\rho\sigma$ will be (imi). But, as previously remarked, $\rho\sigma$ is a perspectivity of order two fixing every point on L_∞. Thus, $m = imi$, and i commutes with every element in the multiplicative group.

(iv) *There is a unique element i of multiplicative order two, and $im = mi$ for every m.*

Now multiplication by i must interchange the points within each admissible pair except the pair (A, B). Hence, for each (m), (m) and (mi) are the points of an admissible pair.

Let us consider the perspectivity of order two with axis $y = x$, centre (i). We will have:

$$A \leftrightarrow B$$
$$(c, c) \text{ is fixed}$$
$$x = c \leftrightarrow y = c$$
$$(c, d) \leftrightarrow (d, c)$$
$$(0, b) \leftrightarrow (b, 0).$$

The point (1) is fixed and, since $(0, b) \in y = x + b$, $(b, 0)$ must be on the image of $y = x + b$. Hence

$$y = x + b \leftrightarrow y = x + (-b), \text{ where } b + (-b) = 0.$$

Moreover, $(c, c + b) \leftrightarrow (c + b, c)$ so that $(c + b, c)$ must be on the line $y = x + (-b)$. This implies

(v) $$(c + b) + (-b) = c, \text{ where } b + (-b) = 0.$$

Also, the fact that $(1, m) \leftrightarrow (m, 1)$ implies that lines of slope (m) go into lines of slope (m^{-1}). But our collineation must interchange the points of admissible pairs. Hence $mi = m^{-1}$ and

$$\text{(vi)} \qquad m^2 = i \quad \text{for} \quad m \neq 1, i, 0.$$

Next, we shall establish that i must be -1. We shall then show that $1 + 1 = -1$, and, finally, that $n = 9$. In what follows, we have obtained a number of very helpful ideas from **(1)**. (The reader should note the use of parentheses in the equations on one hand, and the indication of points on L_∞ by a single element within parentheses.)

It follows from the right distributive law that $(-1)a = -a$, that is, that $a + (-1)a = 0$ for every a in the co-ordinate system. Moreover, it follows from (v) that $(-a + a) + (-a) = -a$ and hence, $-a + a = 0$.

In particular, $-i + i = 0$. But $0 = -i + (-1)(-i) = -i + (-1)^2 i = -i + i^2 = -i + 1$ (unless $-1 = i$). This implies that $i = 1$. Since i was of multiplicative order two, we have a contradiction unless $i = -1$.

Thus, we have established that $i = -1$, and -1 has the following special properties:

$$\text{(vii)} \qquad (-1)^2 = 1, \quad (-1)b = b(-1), \quad b^2 = -1 \text{ if } b \neq 0, \pm 1.$$

Furthermore, if $a, b, ab \neq \pm 1$, $(ab)^2 = -1$, $a^{-1} = -a$, $b^{-1} = -b$. Hence, $ab = -(-b)(-a) = -ba$.

We can now characterize the admissible pairs other than A and B as pairs (m) and $(-m)$.

Now, $(1 + 1)^2 = (1 + 1) + (1 + 1)$. But, either $1 + 1 = -1$ or $(1 + 1)^2 = -1$. Thus, either $1 + 1 = -1$ or $(1 + 1) + (1 + 1) = -1$.

Let us assume, for the moment, that $(1 + 1) + (1 + 1) = -1$. The points (1) and (-1) form an admissible pair. Hence there is a perspectivity with axis $y = x$ and centre (-1) which carries A into the point $(1 + a)$, B into $(-1-a)$, where a may be any element of the co-ordinate system such that $1 + a \neq 0, \pm 1$. (The existence of this perspectivity follows from the fact that the plane was $p_i - L_i$ transitive for each $p_i \in L_\infty$ and that the image of an admissible pair must be an admissible pair.)

The point $(1, 1)$ is fixed under this perspectivity. Hence, the line $x = 1$ maps into the line of slope $(1 + a)$ which goes through $(1, 1)$. It is readily verified that this line has the equation $y = x(1 + a) - a$. The line $y = 0$ will map into the line $y = -x(1 + a)$. Hence $(1, 0)$ must map into the intersection of $y = x(1 + a) - a$ and $y = -x(1 + a)$.

Moreover, every line of slope -1 is fixed. In particular, the line $y = -x + 1$ is fixed. The image of $(1, 0)$ must also be on this line.

Now $(-1, 1 + 1)$ satisfies the equations $y = -x + 1$ and $y = -x(1 + 1)$. In the particular case where $a = 1$, we have that $(1, 0)$ must map into $(-1, 1 + 1)$; it follows that $(-1, 1 + 1)$ must satisfy the equation $y = x(1 + 1) - 1$. That is:

$$1 + 1 = (-1 - 1) - 1.$$

and

$$c + c = (-c - c) - c \text{ for every } c.$$

Using the fact that $(1 + a)^2 = -1$, it follows that $x = (a + a)(1 + a)$, $y = a + a$, are the simultaneous solutions of the equations $y = x(1 + a) - a$ and $y = -x(1 + a)$. This pair of values for x and y are the co-ordinates of the image of $(1, 0)$ under the perspectivity with axis $y = x$, centre (-1) which carries A into $(1 + a)$.

But this pair of values for x and y must also satisfy the equation $y = -x + 1$ and

$$\begin{aligned} a + a &= -(a + a)(1 + a) + 1 \qquad \text{if } 1 + a \neq 0, \pm 1 \\ &= (1 + a)(a + a) + 1, \text{if } a + a \neq \pm 1, \pm(1 + a) \text{ and } 1 + a \neq \pm 1 \\ &= [(a + a) + a(a + a)] + 1 \\ &= [(a + a) - (a + a)a] + 1, a \neq \pm 1, a + a \neq \pm 1, a(a + a) \neq \pm 1 \\ &= [(a + a) + (1 + 1)] + 1, \qquad a \neq \pm 1, 0. \end{aligned}$$

This last equation, and the right inverse law for addition, imply that $1 + 1 = -1$, unless the only values of a that can occur are those included in the exceptions noted. Re-examining the exceptions, we find that there are at most six distinct cases: $a = \pm 1$, $a = \pm(1 + 1)$, $a = 0$ and the value of a such that $1 + a = -1$. That is, the assumption that $1 + 1 \neq -1$ leads to the conclusion that $1 + 1 = -1$ if our co-ordinate system contains more than six distinct elements. Since all planes of order 8 or less are Desarguesian, we can without loss of generality assume that our co-ordinate system contains at least nine distinct elements.

Thus we can, without loss of generality, assume that $1 + 1 = -1$ and, multiplying on the right, $c + c = -c$, for every c.

Again consider the perspectivity with axis $y = x$, centre (-1) which carries A into $(1 + a)$, B into $(-1-a)$, where now a is to be fixed but $a \neq 0, \pm 1$. As before, the point (c, c) is fixed, and the line $x = c$ maps into the line of slope $(1 + a)$ which goes through (c, c), that is,

$$x = c \rightarrow y = x(1 + a) + c^*, \text{ where } c = c(1 + a) + c^*.$$

Also, $y = 0 \rightarrow y = -x(1 + a)$ and $y = -x + c$ is fixed. The simultaneous solution of the equations $y = x(1 + a) + c^*$, $y = -x(1 + a)$ is readily verified to be $x = -c^*(1 + a)$, $y = -c^*$, using $(1 + a)^2 = -1$, $c^* + c^* = -c^*$. This pair of values of x and y must satisfy the equation $y = -x + c$. Hence

$$-c^* = c^*(1 + a) + c.$$

Now, if $c^* \neq 0, \pm 1, \pm(1 + a)$, this can be written

$$-c^* = -(1 + a)c^* + c = (-c^* - ac^*) + c.$$

This implies that $c = ac^*$ and $-ac = c^*$ provided that $c^* \neq 0, \pm 1, \pm(1 + a)$. (Recall that $a \neq 0, \pm 1$.) If we substitute $c^* = -ac$ into $c = c(1 + a) + c^*$, we get

$$c = c(1 + a) - ac.$$

If $c \neq 0, \pm 1, \pm(1 + a)$, this may be written

$$c = -(1 + a)c - ac = (-c - ac) - ac.$$

Adding ac to both sides and using the right inverse law,

$$c + ac = -(c + ac).$$

But, since -1 is of multiplicative order two, $-1 \neq 1$ and $c + ac \neq -(c + ac)$ unless $c + ac = 0$; that is, $(1 + a)c = 0$. With $a \neq -1$, this implies that $c = 0$.

Thus, if $c^* \neq 0, \pm 1, \pm(1 + a)$, the only possible values of c are $c = 0$, ± 1, $\pm(1 + a)$ and, for these values of c, $c^* = -ac$. We have only nine distinct possible values for c^*:

$$0, \pm 1, \pm(1 + a), -a(\pm 1), -a(1 + a), \text{and } -a(-1 - a).$$

But there is a value of c^* for each value of c and $c^*_1 = c^*_2$ if and only if $c_1 = c_2$. Hence, our co-ordinate system contains only nine distinct elements, and $n = 9$. Thus, the assumption that G_1 is not doubly transitive and the plane is not Desarguesian lead to the conclusion that $n = 9$ and the theorem is proved.

References

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2. T. G. Ostrom, *Double transitivity in finite projective planes*, Can. J. Math., *8* (1956), 563–567.

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4. G. Pickert, *Projektive Ebenen* (Berlin, 1955).

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