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Motivic cohomology spectral sequence and Steenrod operations

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To Victoria

Abstract

For a prime number p, we show that differentials d_n in the motivic cohomology spectral sequence with p-local coefficients vanish unless $p-1$ divides $n-1$. We obtain an explicit formula for the first non-trivial differential d_p, expressing it in terms of motivic Steenrod p-power operations and Bockstein maps. To this end, we compute the algebra of operations of weight $p-1$ with p-local coefficients. Finally, we construct examples of varieties having non-trivial differentials d_p in their motivic cohomology spectral sequences.

1. Introduction

The motivic cohomology spectral sequence (MCSS) is an algebro-geometric analogue of the Atiyah–Hirzebruch spectral sequence in topology. Its second term consists of motivic cohomology groups and the sequence converges to algebraic K-theory.

The spectral sequence was initially constructed for fields by Bloch and Lichtenbaum. Unfortunately, their arguments contained a gap and the construction can now be found only in the unpublished preprint [BL95]. Later, different constructions were built by Grayson [Gra95] and Friedlander and Suslin [FS02]. These two constructions not only globalized the MCSS to the whole category of smooth varieties, but also showed that it is supplied with multiplicative structure. The equivalence of the two approaches was established in [Sus03].

Voevodsky [Voe02a, Voe02b] observed that the slice filtration of the motivic Eilenberg–Mac Lane spectrum leads (modulo some conjectures) to another model of the MCSS. This approach was developed by Levine and he has also shown the equivalence of all three constructions [Lev08]. These steps made it possible to extend the MCSS to the category of Voevodsky's spaces. More historical issues can be found in Weibel's 'K-book' [Wei13, VI.4.4].

The behavior of differentials in the MCSS is quite similar to the topological case. Being taken with rational coefficients, the sequence collapses on its E_2-page (see [GS99]). On the other hand, its structure with integer coefficients becomes too tangled, because of the interrelation of different p-prime effects involved. The purpose of the current paper is to investigate the case of $\mathbb{Z}_{(p)}$-coefficients that allows us to 'distill' the p-prime effects. In this case one gets non-trivial differentials of rather high degree and that makes their computation an interesting question.

Differentials in the Atiyah–Hirzebruch spectral sequence were computed by Buchstaber long ago [Buc69]. In the current paper we establish the parallel result for the MCSS. Philosophically, our approach is quite similar to Buchstaber's one, but the technique is certainly rather different.

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The strategy of the proof is the following. Firstly, we show, using Adams operations, that the first non-trivial differential may appear only on the E_p-page (Proposition 3.1). Then, computing the motivic Steenrod algebra in the corresponding degree, it is possible to show that the differential in question is a scalar multiple of some concrete cohomological operation. Finally, to check that the scalar in question is not zero, we construct examples of varieties such that the differentials d_p in their motivic cohomology spectral sequences are non-trivial (Proposition 6.2 and Example 6.3).

The significant part of our results becomes trivial in the case $p = 2$. So, this case is systematically avoided in the paper. However, we give evidence (see Example 6.1) of non-triviality of the differential d_2 in the MCSS with $\mathbb{Z}_{(2)}$-coefficients over $\mathbb{Q}$.

Let us, finally, mention that the scalar appearing in our result is actually a unit in the field $\mathbb{Z}/p$ and, therefore, plays a negligible role in the spectral sequence structure, so that our theorem gives full control over all differentials up to d_{2p-2}. To compute the differential d_{2p-1} and other possibly non-zero differentials $d_{k(p-1)+1}$, we need a good description of secondary (and higher) cohomological operations. As far as this description is currently not available, this makes studying higher cohomological operations in motivic cohomology an interesting topic.

The computation of the p-local Steenrod algebra is based on Voevodsky's result on the structure of the motivic Steenrod algebra with finite coefficients. Originally, the statement was proven only over fields of characteristic zero, but recent work [HKØ13] extends Voevodsky's construction to fields of characteristic mutually prime to p.

1.1 Notation remarks

We fix a prime number p and denote by $\mathbb{Z}_{(p)}$ the localization of the ring of integers at the prime ideal (p). We also denote by $\mathbb{Z}/p^\infty$ the p-cyclotomic group, i.e. $\varinjlim \mathbb{Z}/p^m\mathbb{Z}$. Unless it is specified, we always assume that $p > 2$.

We always assume the field k to be perfect and $(\operatorname{Char} k, p) = 1$. Here and below by $\operatorname{Char} k$ we denote the characteristic exponent of k. We denote by Sm/k the category of smooth separated schemes of finite type (smooth varieties) over a field k. We also denote by **Spc** the category of pointed Nisnevich sheaves over Sm/k (pointed Voevodsky spaces) and by **Sp** the homotopy category of T-spectra (see below for the definition of T). The reader is referred to [Voe98, § 2] for the constructions of the categories as well as for the description of a closed model category structure on **Spc**. Abusing the notation, we identify smooth varieties with corresponding representable Voevodsky spaces. We denote by $H^{*,*}(-)$ the motivic cohomology [MVW06, 3.4] (cf. also § 4) and by $K_*(-)$ Quillen's K-groups [Qui73, § 7]. We often call the first index of motivic cohomology groups *degree* and the second index *weight*.

- $\mathbf{pt} := \operatorname{Spec} k$.
- $X_+ := X \sqcup \mathbf{pt}$.
- $H^{*,*} := H^{*,*}(\operatorname{Spec} k, \mathbb{Z}/p)$.
- $\mathbb{A}^n$ (respectively $\mathbb{P}^n$) denotes affine (respectively projective) space of dimension n in Sm/k.
- $T := \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$ is the Tate object.
- We denote the T-suspension functor by $T \wedge -$. The natural morphism $X \to T \wedge X$ induces a shifted isomorphism in motivic cohomology. For consistency, we call the inverse map $\tilde{H}^{*,*}(-) \xrightarrow{\cong} \tilde{H}^{*+2,*+1}(T \wedge -)$ the T-suspension isomorphism and it is denoted by Σ_T (cf. (4.2)).
- $\sigma_T := \Sigma_T(1) \in \tilde{H}^{2,1}(T)$ is often called the Tate element.

We often denote by $[X, Y, Z]$ the Bockstein homomorphism $H^{*,*}(-, Z) \to H^{*+1,*}(-, X)$ corresponding to the short exact sequence $0 \to X \to Y \to Z \to 0$ of abelian groups.

Finally, we summarize here some vanishing results, which we will use below.

Statement 1.1. For $X \in Sm/k$, one has $H^{p,q}(X) = 0$ if:

(i) $p > 2q$;
(ii) $p > q + \dim X$;
(iii) $q < 0$;
(iv) $q = 0$ and $p \neq 0$.

Proof. See [MVW06]: Theorem 19.3 for (i), Theorem 3.5 for (ii), Corollary 4.2 for (iii) and (iv). □

2. Main result and outline of the proof

As was shown in [FS02], for any $X \in Sm/k$ there exists the motivic cohomology spectral sequence

$$E_2^{i,j} = H^{i-j,-j}(X) \Rightarrow K_{-i-j}(X), \tag{2.1}$$

starting from the motivic cohomology groups $H^{*,*}(X)$ and converging to the algebraic K-groups of the variety X. The differentials in this spectral sequence are $d_n : E_n^{i,j} \to E_n^{i+n,j-n+1}$ ($n \geqslant 2$).

Theorem 2.1. *Let p be an odd prime and k be a perfect field of characteristic l such that either $l = 0$ or $(l, p) = 1$. For a variety $X \in Sm/k$, the motivic cohomology spectral sequence*

$$E_2^{i,j} = H^{i-j,-j}(X, \mathbb{Z}_{(p)}) \Rightarrow K_{-i-j}(X, \mathbb{Z}_{(p)})$$

has zero differentials d_n for $p-1 \nmid n-1$. The differential d_p coincides with the bistable operation $\mathfrak{B}\alpha P^1 r$, where r denotes the coefficient reduction corresponding to the residue map $\mathbb{Z}_{(p)} \to \mathbb{Z}/p$, the operation P^1 is the first $\mathbb{Z}/p$ motivic Steenrod power, α denotes multiplication of coefficients by an element of $\mathbb{Z}/p^\times$ and $\mathfrak{B} = [\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}, \mathbb{Z}/p]$ is the Bockstein map. Moreover, for any l satisfying the theorem conditions, one can find a field F of characteristic l and a variety $X \in Sm/F$ such that the differential d_p in the corresponding MCSS is non-trivial.

In the next section we prove, following the strategy of Buchstaber, the first statement of the theorem. (Let us also mention that a similar technique was also used by Merkurjev [Mer10] to analyze the structure of the Brown–Gersten–Quillen spectral sequence.) Then, in § 4, the differential d_p is interpreted as a bistable motivic cohomology operation of bidegree $(2p-1, p-1)$, i.e. as an element of the corresponding motivic Steenrod algebra, which is computed in § 5. Finally, in § 6, we construct examples of varieties for which differentials d_p in the MCSS are non-trivial that completes the proof of the theorem.

3. Differentials and Adams operations

The purpose of the current section is to prove the following proposition.

Proposition 3.1. $d_n = 0$ *for* $p - 1 \nmid n - 1$.

Proof. As was shown in [GS99], for every integer k such that $1/k \in \mathbb{Z}_{(p)}$ the Adams operation ψ_k on $K_*(X, \mathbb{Z}_{(p)})$ can be represented as an operation acting on the whole motivic cohomology spectral sequence. Moreover, the action of this operation on the E_2-page is given by the equality $\psi_k(\alpha) = k^{-q}\alpha$ for $\alpha \in H^{*,q}(X)$. Therefore, all topological arguments proposed by Buchstaber [Buc69] work in this case as well. Since Adams operations commute with differentials, for every integer $n > 1$, we get

$$d_n \psi_k = \psi_k d_n : H^{*,*}(X) \to H^{*+2n-1,*+n-1}(X).$$

Hence, one has $(k^{n-1}-1)d_n = 0$. Let us now define the number $M(i)$ as the greatest common divisor of the following sequence:

$$M(i) := \underset{k>1}{\text{g.c.d.}}\{k^N(k^i-1)\}, \tag{3.1}$$

where $N \gg i$. One can easily verify that the numbers $M(i)$ are well defined. The integers $M(i)$ are sometime called Kervaire–Milnor–Adams numbers, probably after the paper [KM60]. Their values are presented in the lemma below. Obviously, $M(n-1)d_n = 0$. Since for $p-1 \nmid n-1$, we have $p \nmid M(n-1)$, the differentials of these degrees vanish. □

LEMMA 3.2. *For a prime p and a positive integer n, denote by $\nu_p(n)$ the greatest dividing p-exponent*[1] *of n. The sequence of Kervaire–Milnor–Adams numbers is determined as follows. For $i \geqslant 1$ and a prime number p, one has $M(2i-1) = 2$ and*

$$\nu_p(M(2i)) = \begin{cases} 1 + \nu_p(4i) & \text{for } (p-1) \mid 2i, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. See [Ada65]. □

COROLLARY 3.3. *The MCSS with $\mathbb{Q}$-coefficients degenerates on its E_2-page.*

Proof. Any differential vanishes after multiplication by an invertible number. □

COROLLARY 3.4. *For $p > 2$, one has $pd_p = 0$ in the MCSS with $\mathbb{Z}_{(p)}$-coefficients.*

Proof. Since, by Lemma 3.2, one has $\nu_p M(p-1) = 1$, the corollary follows. □

Remark 3.5. It is interesting to mention that the sequence $M(2i)$

$$24, 240, 504, 480, 264, 65520, \ldots$$

can be identified with denominators of terms of sequences $\frac{1}{2}\zeta(1-2i)$ or $B_{2i}/4i$.

4. Differentials as cohomology operations

Let us give a brief explanation of the construction of motivic Eilenberg–Mac Lane spaces, following, almost *verbatim*, the exposition of [Voe03].

For a variety $X \in Sm/k$, consider the presheaf $\mathbb{Z}_{\mathrm{tr}}(X)$ of abelian groups on the category Sm/k, which takes a variety U to the free abelian group, generated by all closed integral subvarieties of $X \times U$, which are finite and equidimensional over U. For an abelian group A, we set $A_{\mathrm{tr}} := A \otimes \mathbb{Z}_{\mathrm{tr}}$ and define presheaves of abelian groups:

$$K^{\mathrm{pre}}_{n,A} : U \mapsto A_{\mathrm{tr}}(\mathbb{A}^n)(U)/A_{\mathrm{tr}}(\mathbb{A}^n - \{0\})(U). \tag{4.1}$$

Let $K_n(A)$ be the pointed sheaf (in Nisnevich topology) of sets associated to $K^{\mathrm{pre}}_{n,A}$. The sheaves $K_n(A)$ play the roles of Eilenberg–Mac Lane spaces in the category **Spc**.

Alternatively, one can start from the presheaf $K^{\mathrm{pre}}_{n,\mathbb{Z}}$ and obtain a complex $\mathbb{Z}(n)$ of sheaves of abelian groups on $(Sm/k)_{\mathrm{Nis}}$ (see the construction in [VSF00, ch. 5]). For any $i, j \in \mathbb{Z}$, a smooth scheme X and an abelian group A, one defines motivic cohomology groups as hypercohomology

[1] For example, for any positive integer n, one has $n = 2^{\nu_2(n)}3^{\nu_3(n)}5^{\nu_5(n)}\ldots$.

groups $H^{i,j}(X,A) := \mathbf{H}^i(X_{\mathrm{Nis}}, A(j))$, where $A(j) = A \otimes \mathbb{Z}(j)$. Let $K(i,j,A)$ be the simplicial abelian group sheaf corresponding to the complex $A(j)[i]$. Applying again the forgetful functor, one gets the simplicial sheaf of sets that determines an object (also denoted by $K(i,j,A)$) of the motivic homotopy category of spaces $\mathcal{H}o_{\mathbb{A}^1}$. The sheaves $K(i,j,A)$ are $\mathbb{A}^1$-local [Del09, §§ 2.2–2.4] and, for any smooth scheme X, one has $H^{i,j}(X,A) = \mathrm{Hom}_{\mathcal{H}o_{\mathbb{A}^1}}(X_+, K(i,j,A))$. For any pointed simplicial sheaf $F_\bullet$ on $(Sm/k)_{\mathrm{Nis}}$, one can take the following definition of reduced motivic cohomology:

$$\tilde{H}^{i,j}(F_\bullet, A) = \mathrm{Hom}_{\mathcal{H}o_{\mathbb{A}^1}}(F_\bullet, K(i,j,A)). \tag{4.2}$$

It is shown in [Del09, §§ 2.2–2.4] that there exists a weak equivalence between $K_n(A)$ and $K(2n,n,A)$, so the two constructions of Eilenberg–Mac Lane spaces agree. This extends the definition of motivic cohomology groups to the whole category of spaces.

We shall also need the notion of a cohomological operation.

Definition 4.1. A collection $\{\varphi\}_{p,q}$ of natural transformations of functors on **Spc**

$$\varphi_{p,q} : \tilde{H}^{p,q}(-,A) \to \tilde{H}^{p+i,q+j}(-,B),$$

where A and B are abelian groups and the index (p,q) runs through $\mathbb{Z}\times\mathbb{Z}$, is called an (unstable) cohomological A–B-operation of degree i and weight j.

Let us recall that in the category **Spc** there are two circles and hence two different suspension functors. Among all the cohomological operations there are special ones that commute with both suspension isomorphisms. These operations are called *bistable*, and Voevodsky showed, using a simple trick [Voe03, Proposition 2.6], that there exists a natural bijection between bistable operations and operations that *a priori* commute only with the T-suspension. (Recall that T is the Tate object.) We will call operations of the latter type *stable*.

Notation. We denote the set of all stable cohomological A–B-operations of degree i and weight j by $\mathcal{OP}^{i,j}(A,B)$. We always implicitly assume that all considered operations have non-negative degree and weight. Since, by [Voe03, Corollary 2.10], stable operations are additive, this set has a natural structure of an abelian group, induced by addition in cohomology.

If A (*respectively* B) has a ring structure, the set $\mathcal{OP}^{*,*}(A,B)$ also has a natural structure of a bigraded left (*respectively* right) $H^{*,*}$-module.

It is reasonable to expect that natural transformations of motivic cohomology functors can be classified by cohomology groups of motivic Eilenberg–Mac Lane spaces.

For every motivic Eilenberg–Mac Lane space $K_n(A)$, one can choose a universal element

$$\iota_n \in \tilde{H}^{2n,n}(K_n(A)),$$

corresponding to the identity morphism of the space $K(2n,n,A)$. Applying the T-suspension isomorphism map $\Sigma_T : \tilde{H}^{*,*}(-) \to \tilde{H}^{*+2,*+1}(T \wedge -)$ to the element ι_n, one obtains the element $\Sigma_T \iota_n \in \tilde{H}^{2n+2,n+1}(T \wedge K_n(A))$, corresponding to some homotopy class $\alpha_n \in [T \wedge K_n(A), K_{n+1}(A)]$. This class coincides with the homotopy class of the nth structure morphism of the motivic Eilenberg–Mac Lane spectrum $\mathbf{H}(A)$.

Finally, using the collection of classes $\{\alpha_\bullet\}$, one can construct an inverse system of the groups $\tilde{H}^{i+2n,j+n}(K_n(A),B)$ as shown in the diagram below.

$$\begin{array}{ccc} \vdots & & \\ \tilde{H}^{i+2n+2,j+n+1}(K_{n+1}(A),B) & \xrightarrow{\alpha_n^*} & \tilde{H}^{i+2n+2,j+n+1}(T\wedge K_n(A),B) \\ \downarrow & \nearrow_{\Sigma_T}^{\cong} & \\ \tilde{H}^{i+2n,j+n}(K_n(A),B) & & \\ \vdots & & \end{array} \tag{4.3}$$

A natural modification of [Voe03, Proposition 2.7] shows that

$$\mathcal{OP}^{i,j}(A,B) = \varprojlim_n \tilde{H}^{i+2n,j+n}(K_n(A),B). \tag{4.4}$$

We will see that the module $\mathcal{OP}^{*,*}(\mathbb{Z}/p,\mathbb{Z}/p)$ is naturally isomorphic to the motivic Steenrod algebra by Voevodsky (see the discussion on p. 2120).

Proposition 4.2. *Consider the motivic cohomology spectral sequence $(E_*^{*,*},d_*)$. Let us fix an integer $n>1$ and assume that for every $1<i<n$ and any variety $X\in Sm/k$, the differentials $d_i: H^{*,*}(X)\to H^{*+2i-1,*+i-1}(X)$ are trivial. Then the differential d_n can be identified with a stable cohomological operation of bidegree $(2n-1,n-1)$ up to multiplication by ± 1.*

Proof. Since all the previous differentials vanish, the differential d_n actually acts on the E_2-page of the spectral sequence. To prove the stability, one has to check the commutativity of the following diagram.

$$\begin{array}{ccc} H^{i,j}(X) & \xrightarrow{d_n} & H^{i+2n-1,j+n-1}(X) \\ \Sigma_T\downarrow & \pm 1 & \downarrow\Sigma_T \\ \tilde{H}^{i+2,j+1}(T\wedge X_+) & \xrightarrow{d_n} & \tilde{H}^{i+2n+1,j+n}(T\wedge X_+) \end{array} \tag{4.5}$$

Though the space $T\wedge X_+$ does not belong to Sm/k, its cohomology is a direct summand of cohomology of the scheme $\mathbb{P}^1\times X$. Namely, the retraction morphism $\mathbf{pt}\overset{\infty}{\to}\mathbb{P}^1\to\mathbf{pt}$ delivers the direct sum decomposition $H^{*,*}(\mathbb{P}^1)\cong\tilde{H}^{*,*}(\mathbb{P}^1,\infty)\oplus H^{*,*}$ and the pointed variety $(\mathbb{P}^1,\infty)$ is canonically weakly equivalent to T.

The motivic cohomology groups of $T\wedge X_+$ are the $(2,1)$-shifted cohomology groups of X and the isomorphism Σ_T is delivered by multiplication with the image of the Tate element σ_T. The MCSS is functorial and has a canonical multiplicative structure that is compatible with multiplication in motivic cohomology (see [FS02, § 14]). Hence, its differentials satisfy the Leibnitz rule and one has $d_n(\sigma_T\wedge x)=d_n(\sigma_T)\wedge x\pm\sigma_T\wedge d_n(x)$. Now, to prove the commutativity of (4.5) up to the sign, it suffices to verify that $d_n(\sigma_T)=0$. This element should lie in the cohomology group of the variety $\mathbb{P}^1$ of bidegree $(2n+1,n)$ that vanishes, since $2n+1>2n$ (see Statement 1.1(i)). So, the commutativity result follows for dimension reasons.

In order to complete the proof of the proposition, we only need to extend the differential to the whole category of spaces. It can be done using Levine's [Lev08] identification between the MCSS and the spectral sequence built by the slice filtration. Due to the functoriality of the spectral sequence construction, the differential d_n becomes a motivic cohomological operation of bidegree $(2n-1,n-1)$. It is not hard to show that the arguments above are also applicable to the category **Spc** and prove the stability of the operation. □

5. Some calculations in Steenrod modules

In this section we are going to perform some computations with cohomology of motivic Eilenberg–Mac Lane spaces and spectra, and we need some preliminary results and notation. We denote by $\mathfrak{K}_n : \mathcal{A}b \to \mathbf{Spc}$ (respectively $\mathfrak{K} : \mathcal{A}b \to \mathbf{Sp}$) the functor sending an abelian group A to the motivic Eilenberg–Mac Lane space $K_n(A)$ (respectively motivic Eilenberg–Mac Lane spectrum $\mathbf{H}(A)$).

Proposition 5.1. *For every $n > 0$, the functor $\mathfrak{K}_n$ preserves:*

(i) *limits;*
(ii) *filtered colimits.*

Proof. The functor $\mathfrak{K}_n$ can be considered as the following chain of functors:

$$\mathcal{A}b \to (Presheaves\ of\ \mathcal{A}b) \to (Presheaves\ of\ \mathbf{Sets}) \to (Nisnevich\ sheaves).$$

Since the groups $\mathbb{Z}_{\mathrm{tr}}(X)(U)$ are free abelian groups, one can easily check that the first functor preserves limits and filtered colimits.

Limits and colimits of presheaves are computed objectwise. The forgetful functor $\mathcal{A}b \to \mathbf{Sets}$ preserves limits, because it has a left adjoint functor sending every set X to the free abelian group $\mathbb{Z}[X]$ and also preserves filtered colimits (see, for example, [Art62, § 1.1]).

Finally, it is well known that the sheafification functor preserves arbitrary limits and colimits. □

For a field k, we call an abelian group *k-admissible* if it has a $\mathbb{Z}[\frac{1}{l}]$-module structure for $l = \mathrm{Char}\, k$.

Proposition 5.2. *Let k be a perfect field. Then the functor $\mathfrak{K}$ sends every short exact sequence of k-admissible groups to a distinguished triangle in the category* $\mathbf{Sp}$.

Proof. The following result was established by Röndigs and Østvær [RØ08, Theorem 1] for fields of characteristic zero and by Hoyois *et al.* [HKØ13, Theorem 5.8] for perfect fields of positive characteristic. Let k be a perfect field and R a ring such that $\mathrm{Char}\, k$ is invertible in R. Then Voevodsky's big category of motives $\mathrm{DM}(Sm/k, R)$ is equivalent to the homotopy category $\mathbf{H}(R){-}\mathbf{mod}$ of modules over the Eilenberg–Mac Lane spectrum $\mathbf{H}(R)$. The equivalence preserves the monoidal and triangulated structures.

Now it is not hard to check that the short exact sequence of k-admissible abelian groups leads to a distinguished triangle of motives in $\mathrm{DM}(Sm/k, R)$. Since the category of $\mathbf{H}(R)$-modules is a triangulated subcategory of $\mathbf{Sp}$, this proves the proposition. □

Remark 5.3. For the short exact sequence $0 \to A \to B \to C \to 0$ of abelian groups, the morphism $\mathfrak{K}(C) \to \mathfrak{K}(A)[1]$ in the corresponding distinguished triangle of T-spectra induces the Bockstein map $[A, B, C]$ in motivic cohomology. In particular, this implies the functoriality of Bockstein maps with respect to morphisms of short exact sequences.

All the relations below involving Bockstein maps are obvious consequences of this remark.

Statement 5.4. Let k be a perfect field of characteristic exponent mutually prime to p. Then the groups $H^{*,*}(\mathbf{H}(\mathbb{Z}/p), \mathbb{Z}/p)$ and $\mathcal{OP}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ are naturally isomorphic.

Proof. It is exactly the statement of [HKØ13, Theorem 3.2] and [Voe10, Corollary 2.71] that the $\lim^1$ groups in the short exact sequences

$$0 \to \varprojlim_n{}^1 \tilde{H}^{i+2n-1,j+n}(K_n(\mathbb{Z}/p),\mathbb{Z}/p) \to H^{i,j}(\mathbf{H}(\mathbb{Z}/p),\mathbb{Z}/p) \to \varprojlim_n \tilde{H}^{i+2n,j+n}(K_n(\mathbb{Z}/p),\mathbb{Z}/p) \to 0 \tag{5.1}$$

vanish (cf. also [HKØ13, Corollary 3.3]). We can identify the right-hand term with the group $\mathcal{OP}^{i,j}(\mathbb{Z}/p,\mathbb{Z}/p)$ using construction (4.4). $\square$

Our current aim is to compute the module of stable operations from cohomology with $\mathbb{Z}/p^\infty$-coefficients. We start with Voevodsky's computation of the motivic Steenrod algebra.

The module $\mathcal{OP}^{*,*}(\mathbb{Z}/p,\mathbb{Z}/p)$ has a natural bigraded algebra structure given by composites of operations. Consider its bigraded subalgebra, generated by Steenrod power operations P^i (of bidegrees $(2i(p-1),i(p-1))$) for $i>0$, the Bockstein homomorphism $\beta=[\mathbb{Z}/p,\mathbb{Z}/p^2,\mathbb{Z}/p]$ (of bidegree $(1,0)$) and operations of the form $x\mapsto ax$ for $a\in H^{*,*}$. This subalgebra is called the *motivic Steenrod algebra* $\mathcal{A}^{*,*}(k,\mathbb{Z}/p)$ in [Voe03, § 11, Lemma 9.5].

Let us also consider sequences $I=(\varepsilon_0,s_1,\varepsilon_1,s_2,\ldots,s_k,\varepsilon_k)$ of non-negative integers and such that one has $\varepsilon_i\in\{0,1\}$ and $s_i\geqslant ps_{i+1}+\varepsilon_i$ for every index i. These sequences are called *admissible*. To every admissible sequence I, one associates the operation $P^I=\beta^{\varepsilon_0}P^{s_1}\beta^{\varepsilon_1}\ldots P^{s_k}\beta^{\varepsilon_k}$. (Here we assume that $\beta^0=P^0=\mathrm{id}$.) These operations are called *admissible monomials*. There is a natural graded module map from the free graded left $H^{*,*}$-module generated by all admissible monomials to $\mathcal{A}^{*,*}(k,\mathbb{Z}/p)$.

It is proven in [Voe03, Lemma 11.1] that the latter homomorphism of $H^{*,*}$-modules is an epimorphism and in [Voe03, Corollary 11.5] that the admissible monomials are linearly independent with respect to the left $H^{*,*}$-module structure.

Moreover, Voevodsky showed [Voe10, Theorem 3.49] that over a field k of characteristic 0 there is a natural isomorphism of graded left $H^{*,*}$-modules between $\mathcal{OP}^{*,*}(\mathbb{Z}/p,\mathbb{Z}/p)$ and $\mathcal{A}^{*,*}(k,\mathbb{Z}/p)$.

In the sequel we are mostly dealing with the operations of weight $p-1$ and degree $>p$, so we will often omit the second (weight) index in the notation for operation and cohomology groups and implicitly assume that the first (degree) index is greater than p.

Up to the end of this section we will omit, for brevity, mentioning $\mathbb{Z}/p$-coefficients and write $\mathcal{OP}^*(-)$ for $\mathcal{OP}^{*,p-1}(-,\mathbb{Z}/p)$. We will also write $H^*(A,B)$ for $H^{*,p-1}(\mathbf{H}(A),B)$.

The arguments above immediately imply that $\mathcal{OP}^*(\mathbb{Z}/p)$ (we assume that $*>p$) is a free $\mathbb{Z}/p$-module with the set of generators $\{P^1,\beta P^1,P^1\beta,\beta P^1\beta\}$.

Remark 5.5. Voevodsky's theorem [Voe10, Theorem 3.49] mentioned in the previous discussion was originally proven only for base fields of characteristic 0. However, recently Hoyois *et al.* [HKØ13] could eliminate this annoying restriction and extend the result to the case of a perfect field k such that $(\mathrm{Char}\,k,p)=1$.

Remark 5.6. Using Voevodsky's computation of the motivic Steenrod algebra and the methods of this section, it is possible to compute modules of operations of weight $\leqslant p^2-p$. Leaving all the details to the reader, we just mention that the case $p=3$ is slightly more delicate.

Now we will explicitly compute weight $p-1$ cohomology groups of the T-spectra $\mathbf{H}(\mathbb{Z}/p^m)$ with integral and finite coefficients.

PROPOSITION 5.7. *For $m>0$, there are natural isomorphisms $H^*(\mathbb{Z}/p^m)\cong\mathcal{OP}^*(\mathbb{Z}/p^m)$ and $H^*(\mathbb{Z}/p^m,\mathbb{Z})\cong\mathcal{OP}^*(\mathbb{Z}/p^m,\mathbb{Z})$. The groups $H^*(\mathbb{Z}/p^m)$ and $H^*(\mathbb{Z}/p^m,\mathbb{Z})$ are the free graded $\mathbb{Z}/p$-modules with generators given by images of the following operations in the corresponding degrees:*

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	$2p-2$	$2p-1$	$2p$
$H^*(\mathbb{Z}/p^m)$	$P^1 r_m$	$P^1\beta_m,\ \beta_1 P^1 r_m$	$\beta_1 P^1 \beta_m$
$H^*(\mathbb{Z}/p^m, \mathbb{Z})$	$\emptyset$	$\beta_{\mathbb{Z}} P^1 r_m$	$\beta_{\mathbb{Z}} P^1 \beta_m$

Here r_m is induced by the coefficient reduction $\mathbb{Z}/p^m \to \mathbb{Z}/p$, $\beta_m = [\mathbb{Z}/p, \mathbb{Z}/p^{m+1}, \mathbb{Z}/p^m]$ and $\beta_{\mathbb{Z}} = [\mathbb{Z}, \mathbb{Z}, \mathbb{Z}/p]$.

Proof. We start with the case of $\mathbb{Z}/p$-coefficients. Setting $m = 1$, we get just Voevodsky's result cited above. Since in this case the higher inverse limits vanish, cohomology groups of spectra coincide with groups of operations. We now assume that $p > 3$. The case $p = 3$, which is similar, but requires a bit more calculations, is left to the reader. Consider the short exact sequence

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^{m+1} \xrightarrow{r} \mathbb{Z}/p^m \to 0$$

and assume that the groups $H^*(\mathbb{Z}/p^m)$ satisfy the theorem conclusions. By Theorem 5.2, one has a distinguished triangle of spectra:

$$\mathbf{H}(\mathbb{Z}/p) \to \mathbf{H}(\mathbb{Z}/p^{m+1}) \to \mathbf{H}(\mathbb{Z}/p^m). \tag{5.2}$$

Consider the following fragment of the corresponding cohomology long exact sequence.

$$\begin{array}{l}
0 \longrightarrow H^{2p-2}(\mathbb{Z}/p^m) \xrightarrow{r^*} H^{2p-2}(\mathbb{Z}/p^{m+1}) \longrightarrow H^{2p-2}(\mathbb{Z}/p) \xrightarrow{\beta_m^*} \\
H^{2p-1}(\mathbb{Z}/p^m) \xrightarrow{r^*} H^{2p-1}(\mathbb{Z}/p^{m+1}) \longrightarrow H^{2p-1}(\mathbb{Z}/p) \longrightarrow \\
H^{2p}(\mathbb{Z}/p^m) \longrightarrow H^{2p}(\mathbb{Z}/p^{m+1}) \longrightarrow H^{2p}(\mathbb{Z}/p) \longrightarrow 0
\end{array}$$

Since the map β_m^* delivers an isomorphism between the group $H^{2p-2}(\mathbb{Z}/p)$ and the direct summand of $H^{2p-1}(\mathbb{Z}/p^m)$ generated by the operation $P^1\beta_m$, one gets the isomorphism $H^{2p-2}(\mathbb{Z}/p^m) \cong H^{2p-2}(\mathbb{Z}/p^{m+1})$, which sends the generator $P^1 r_m$ to $r^*(P^1 r_m) = P^1 r_m r = P^1 r_{m+1}$.

In the same way, one can see that the direct summand of $H^{2p-1}(\mathbb{Z}/p^m)$ generated by the operation $\beta_1 P^1 r_m$ maps onto the direct summand of $H^{2p-1}(\mathbb{Z}/p^{m+1})$ with the generator $r^*(\beta_1 P^1 r_m) = \beta_1 P^1 r_m r = \beta_1 P^1 r_{m+1}$. The map $\bar{\beta}_{m+1} = [\mathbb{Z}/p^{m+1}, \mathbb{Z}/p^{m+2}, \mathbb{Z}/p]$ sends the group generated by $\beta_1 P^1 r_{m+1}$ to the group $H^{2p}(\mathbb{Z}/p)$ in such a way that the composite $\bar{\beta}_{m+1}^* r^* = \bar{\beta}_m^*$ makes an isomorphism between the direct summand of $H^{2p-1}(\mathbb{Z}/p^m)$ generated by the operation $\beta_1 P^1 r_m$ and the group $H^{2p}(\mathbb{Z}/p)$. Hence, the group $H^{2p-1}(\mathbb{Z}/p^{m+1})$ splits into two direct $\mathbb{Z}/p$-summands.

The rest of the exact sequence can be treated in a similar way. One can also immediately check that $H^i(\mathbb{Z}/p^{m+1}) = 0$ for $i > 2p$ and $p < i < 2p-2$.

It is also easy to show that the natural epimorphism $H^*(\mathbb{Z}/p^{m+1}) \twoheadrightarrow \mathcal{OP}^*(\mathbb{Z}/p^{m+1})$ is a monomorphism. For example, the image of $P^1 r_{m+1}$ is a non-zero element in the group $\mathcal{OP}^{2p-2}(\mathbb{Z}/p^{m+1})$, since we know that the operation $(P^1 r_{m+1})\bar{\beta}_{m+1} = P^1(r_{m+1}\bar{\beta}_{m+1}) = P^1\beta_1$ is non-trivial in $\mathcal{OP}^{2p-1}(\mathbb{Z}/p)$.

As a result, we conclude that $H^*(\mathbb{Z}/p^{m+1}) \cong \mathcal{OP}^*(\mathbb{Z}/p^{m+1})$ and

$$\varprojlim_n{}^1 \tilde{H}^{*+2n-1,*+n}(K_n(\mathbb{Z}/p^{m+1}), \mathbb{Z}/p) = 0.$$

The case of finite coefficients now follows by induction.

In order to proceed with the case of integral operations, we need the following simple lemma.

Lemma 5.8. *Let*

$$A \xrightarrow{\varphi} B \xrightarrow{\chi} B \xrightarrow{\psi} C$$

be an exact sequence of groups and the composite $\psi\varphi : A \to C$ be an isomorphism. Then $B \cong A \oplus Q$ for a group Q such that the restricted map $\bar{\chi} : Q \to Q$ is an automorphism.

Proof. The map $(\psi\varphi)^{-1}\psi$ (respectively $\varphi(\psi\varphi)^{-1}$) splits the exact sequence on the left (respectively right). Therefore, the group A is a direct summand of B. Denoting $B/\varphi A$ by Q, one can easily see that the four-term exact sequence splits into isomorphisms $A \xrightarrow{\varphi} B/Q$, $Q \xrightarrow{\bar{\chi}} Q$ and $B/Q \xrightarrow{\bar{\psi}} C$. □

End of the proof of Proposition 5.7. Now, using the theorem conclusion for the groups $H^*(\mathbb{Z}/p^m)$, we derive the integral case. Consider the fragment of the coefficient long exact sequence:

$$H^{2p-2}(\mathbb{Z}/p^m) \xrightarrow{\beta_{\mathbb{Z}}} H^{2p-1}(\mathbb{Z}/p^m, \mathbb{Z}) \xrightarrow{p} H^{2p-1}(\mathbb{Z}/p^m, \mathbb{Z}) \xrightarrow{r} C,$$

where $\beta_{\mathbb{Z}} = [\mathbb{Z}, \mathbb{Z}, \mathbb{Z}/p]$ and $C = \operatorname{Ker} \beta_{\mathbb{Z}}$ is the direct summand of $H^{2p-1}(\mathbb{Z}/p^m)$ generated by the element $\beta_1 P^1 r_m$. As one can easily verify, the map $\beta_1 = r\beta_{\mathbb{Z}}$ provides an isomorphism between $H^{2p-2}(\mathbb{Z}/p^m)$ and C. Therefore, the conditions of the above lemma are satisfied. Let us also mention that all the groups $H^*(\mathbb{Z}/p^m, \mathbb{Z})$ are p-groups. Together with the lemma above, this gives us an isomorphism $H^{2p-2}(\mathbb{Z}/p^m) \overset{\beta_{\mathbb{Z}}}{\cong} H^{2p-1}(\mathbb{Z}/p^m, \mathbb{Z})$. The case of degree $2p$ can be verified in the same way. Similarly, we can also check that $H^*(\mathbb{Z}/p^m, \mathbb{Z}) = 0$ for $p < * < 2p-1$ and $* > 2p$. □

Thus, we have re-proved a classical result of Cartan [Car54] in the motivic context.

The group inclusions $i_m : \mathbb{Z}/p^m \hookrightarrow \mathbb{Z}/p^{m+1}$ induce morphisms of spectra: $\mathbf{H}(\mathbb{Z}/p^m) \to \mathbf{H}(\mathbb{Z}/p^{m+1})$. Passing to cohomology, one obtains the inverse system of groups (with arbitrary coefficients)

$$H^*(\mathbb{Z}/p) \overset{i_1^*}{\leftarrow} H^*(\mathbb{Z}/p^2) \overset{i_2^*}{\leftarrow} \cdots .$$

Corollary 5.9.

$$\varprojlim_m H^l(\mathbb{Z}/p^m, \mathbb{Z}/p) = \begin{cases} \mathbb{Z}/p & \text{for } l = 2p-1, 2p, \\ 0 & \text{otherwise} \end{cases}$$

and $\varprojlim_m H^l(\mathbb{Z}/p^m, \mathbb{Z}) = \mathbb{Z}/p$ for $l = 2p$ and 0 otherwise.

Proof. We consider the case of the $\mathbb{Z}/p$-coefficients. The integral case is similar and left to the reader. Applying the map $i_m^* : H^*(\mathbb{Z}/p^{m+1}) \to H^*(\mathbb{Z}/p^m)$ to the generators, one has

$$i_m^*(P^1 r_{m+1}) = 0, \quad i_m^*(P^1\beta_{m+1}) = P^1\beta_m, \quad i_m^*(\beta_1 P^1 r_{m+1}) = 0 \quad \text{and} \quad i_m^*(\beta_1 P^1 \beta_{m+1}) = \beta_1 P^1 \beta_m.$$

Therefore, $\operatorname{Im}(i_m^*) \subseteq H^*(\mathbb{Z}/p^m)\beta_m$. Hence, only the elements of the form

$$\{X\beta_1 \leftarrow X\beta_2 \leftarrow \cdots\}$$

'survive' in the projective limit. The corollary follows immediately. □

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To complete the computation of p-cyclotomic operations, we need a lemma.

Lemma 5.10. *Let $X_1 \overset{\varphi_1}{\subseteq} X_2 \overset{\varphi_2}{\subseteq} \cdots$ be a sequence of abelian groups. Then, for an abelian group W, one has*

$$\varprojlim_i \mathcal{OP}^{*,*}(X_i, W) \cong \mathcal{OP}^{*,*}(\varinjlim_i X_i, W).$$

Proof. The system $\{X_i, \varphi_i\}$ induces the projective system of groups:

$$\mathcal{OP}^{*,*}(X_1, W) \overset{\varphi_1^\sharp}{\leftarrow} \mathcal{OP}^{*,*}(X_2, W) \overset{\varphi_2^\sharp}{\leftarrow} \cdots .$$

Let $\alpha \in \varprojlim \mathcal{OP}^{*,*}(X_i, W)$. In other words, one has a system of operations $\{\alpha_i \in \mathcal{OP}^{*,*}(X_i, W)\}$ such that $\alpha_i = \varphi_i^\sharp(\alpha_{i+1})$.

Let us also consider an element $y \in \tilde{H}^{*,*}(-, \varinjlim X_i)$. Since the homology functor on the category of complexes of abelian groups commutes with direct limits, it implies that

$$\tilde{H}^{*,*}(-, \varinjlim X_i) \cong \varinjlim \tilde{H}^{*,*}(-, X_i).$$

Hence, the element y determines a set of elements $\{y_j \in \tilde{H}^{*,*}(-, X_j)\}_{j \gg 0}$ such that $\varphi_*^j(y_j) = y_{j+1}$. We construct $\breve{\alpha} \in \mathcal{OP}^{*,*}(\varinjlim X_i, W)$, setting $\breve{\alpha}(y) := \alpha_N(y_N)$ for $N \gg 0$. Since

$$\alpha_{N+1}(y_{N+1}) = \alpha_{N+1}(\varphi_*^N(y_N)) = (\varphi_N^\sharp \alpha_{N+1})(y_N) = \alpha_N(y_N),$$

the operation $\breve{\alpha}$ is well defined.

In order to construct the map in the opposite direction, let us start with an operation $\gamma \in \mathcal{OP}^{*,*}(\varinjlim X_j, W)$ and construct for every index j the operation $\hat{\gamma}_j \in \mathcal{OP}^{*,*}(X_j, W)$ given by the through map

$$\tilde{H}^{*,*}(-, X_j) \to \tilde{H}^{*,*}(-, \varinjlim X_j) \overset{\gamma}{\to} \tilde{H}^{*,*}(-, W),$$

where the first arrow is canonical and the second is given by the operation γ. These operations fit together to make an element of the projective system and, therefore, the operation $\hat{\gamma} \in \varprojlim_j \mathcal{OP}^{*,*}(X_j, W)$. One can easily verify that the given constructions are mutually inverse. □

Corollary 5.11. *The natural map $H^*(\mathbb{Z}/p^\infty, G) \to \mathcal{OP}^*(\mathbb{Z}/p^\infty, G)$ is an isomorphism for $G = \mathbb{Z}/p$ or $\mathbb{Z}$.*

Proof. We have already seen above that $H^*(\mathbb{Z}/p^m, G) \cong \mathcal{OP}^*(\mathbb{Z}/p^m, G)$. These groups and their maps were explicitly computed in Proposition 5.7 and Corollary 5.9. The computation also implies that $\varprojlim_m{}^1 H^{*,p-1}(\mathbb{Z}/p^m, G) = 0$. The desired result now follows from the short exact sequence

$$0 \to \varprojlim_m{}^1 H^{*-1,p-1}(\mathbb{Z}/p^m, G) \to H^{*,p-1}(\mathbb{Z}/p^\infty, G) \to \varprojlim_m H^{*,p-1}(\mathbb{Z}/p^m, G) \to 0. \qquad □$$

Corollary 5.12. *If $G = \mathbb{Z}/p$, the $\mathbb{Z}/p$-module $\mathcal{OP}^*(\mathbb{Z}/p^\infty, G)$ has two generators $P^1\beta_\infty$, $\beta_1 P^1 \beta_\infty$, lying in degrees $2p-1$, $2p$, correspondingly. If $G = \mathbb{Z}$, it is generated by the element $\beta_{\mathbb{Z}} P^1 \beta_\infty$. Here $\beta_\infty = [\mathbb{Z}/p, \mathbb{Z}/p^\infty, \mathbb{Z}/p^\infty]$.*

Proof. Corollaries 5.9 and 5.11 give us an explicit description of the generators of the module $\mathcal{OP}^*(\mathbb{Z}/p^\infty, G)$. Since it follows from Proposition 5.1(ii) that there is a natural identification $\beta_\infty = \lim \beta_m$, this completes the proof. □

Let us now return from p-cyclotomic coefficients to p-local. Further, we will also need some auxiliary results about rational operations, which are presented in the appendix.

Proposition 5.13. *The Bockstein homomorphism $B = [\mathbb{Z}_{(p)}, \mathbb{Q}, \mathbb{Z}/p^\infty]$ induces an isomorphism of $\mathbb{Z}/p$-modules: $\mathcal{OP}^*(\mathbb{Z}_{(p)}) \cong \mathcal{OP}^{*+1}(\mathbb{Z}/p^\infty)$, so that the group $\mathcal{OP}^l(\mathbb{Z}_{(p)})$ is $\mathbb{Z}/p$ in degrees $l = 2p-2, 2p-1$ and trivial otherwise. One can take operations $P^1 r, \beta_1 P^1 r \in \mathcal{OP}^*(\mathbb{Z}_{(p)})$ as generators in the corresponding degrees. Here $r : \mathbb{Z}_{(p)} \to \mathbb{Z}/p$ is the coefficient reduction map.*

Proof. Let us recall that by (4.4), one has $\mathcal{OP}^{i,j}(A, B) = \varprojlim_n \tilde{H}^{i+2n,j+n}(K_n(A), B)$. Consider the commutative square

$$\begin{array}{ccc} H^{*+1,p-1}(\mathbb{Z}/p^\infty) & \xrightarrow{\cong} & \mathcal{OP}^{*+1}(\mathbb{Z}/p^\infty) \\ \uparrow{\scriptstyle B^*} & & \uparrow \\ H^{*,p-1}(\mathbb{Z}_{(p)}) & \twoheadrightarrow & \varprojlim_m \tilde{H}^{*+2m,p-1+m}(K_m(\mathbb{Z}_{(p)})) \end{array}$$

where the vertical arrows are induced by the Bockstein homomorphism B and both horizontal arrows are epimorphisms from (5.1). The top arrow is an isomorphism by Lemma 5.11.

Taking the short exact sequence of abelian groups $0 \to \mathbb{Z}_{(p)} \to \mathbb{Q} \to \mathbb{Z}/p^\infty \to 0$ and applying Proposition 5.2, one gets a distinguished triangle

$$\mathbf{H}(\mathbb{Z}_{(p)}) \to \mathbf{H}(\mathbb{Q}) \to \mathbf{H}(\mathbb{Z}/p^\infty) \tag{5.3}$$

of spectra. Using the triangle and Lemma A.1, one shows that the map B^* in the diagram is also an isomorphism. Hence, all maps in the diagram are isomorphisms. This proves the isomorphism $\mathcal{OP}^*(\mathbb{Z}_{(p)}) \overset{B}{\cong} \mathcal{OP}^{*+1}(\mathbb{Z}/p^\infty)$.

Finally, the equality $rB = \beta_\infty = [\mathbb{Z}/p, \mathbb{Z}/p^\infty, \mathbb{Z}/p^\infty]$, together with the description of the groups $\mathcal{OP}^*(\mathbb{Z}/p^\infty, \mathbb{Z}/p)$ given above, supplies us with the desired set of generators. This proves the proposition. □

Our current purpose is to compute the group $\mathcal{OP}^*(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$.

Proposition 5.14.

$$H^m(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) = \begin{cases} \mathbb{Z}/p & \text{for } m = 2p-1, \\ 0 & \text{for } m \geqslant 2p. \end{cases}$$

Proof. We show, first, that $H^m(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$ is a p-group for $m \geqslant 2p-1$. Using the distinguished triangle (5.3) and the universal coefficient formula, one can write the exact sequence

$$H^m(\mathbb{Q}, \mathbb{Z}_{(p)}) \longrightarrow H^m(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \longrightarrow H^{m+1}(\mathbb{Z}/p^\infty, \mathbb{Z}) \otimes \mathbb{Z}_{(p)}.$$

By Corollary 5.12, we already know that $H^{m+1}(\mathbb{Z}/p^\infty, \mathbb{Z})$ is either $\mathbb{Z}/p$ for $m = 2p-1$, or 0. So, it suffices to show that $H^m(\mathbb{Q}, \mathbb{Z}_{(p)})$ is a p-group. By A.2, one has $0 = H^m(\mathbb{Q}, \mathbb{Q}) = H^m(\mathbb{Q}, \mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$ and this group is the p-localization of $H^m(\mathbb{Q}, \mathbb{Z}_{(p)})$. So, the statement follows.

Now consider the fragment of the coefficient long exact sequence

$$H^{2p-2}(\mathbb{Z}_{(p)}) \xrightarrow{\mathfrak{B}} H^{2p-1}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \xrightarrow{p} H^{2p-1}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \xrightarrow{r} H^{2p-1}(\mathbb{Z}_{(p)}),$$

where $\mathfrak{B} = [\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}, \mathbb{Z}/p]$. As we already know from the computation above (Proposition 5.13), both the groups with finite coefficients are isomorphic to $\mathbb{Z}/p$ and the isomorphism between them can be performed by the map $\beta_1 : H^{2p-2}(\mathbb{Z}_{(p)}) \to H^{2p-1}(\mathbb{Z}_{(p)})$. One can easily verify the relation $\beta_1 = \mathfrak{B}r$. From Lemma 5.8 and the fact that $H^{2p-1}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$ is a p-group, we have $H^{2p-2}(\mathbb{Z}_{(p)}) \overset{\mathfrak{B}}{\cong} H^{2p-1}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$.

The same arguments can be used to show that $H^m(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) = 0$ for $m \geqslant 2p$. One just should mention, in addition, that the map $r : H^{2p-1}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \to H^{2p-1}(\mathbb{Z}_{(p)})$ in the sequence above is an isomorphism. □

Corollary 5.15. *The Bockstein homomorphism $\mathfrak{B}$ induces the isomorphism*

$$\mathcal{OP}^{2p-2}(\mathbb{Z}_{(p)}) \overset{\mathfrak{B}}{\cong} \mathcal{OP}^{2p-1}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}).$$

Proof. In the commutative diagram

$$\begin{array}{ccc} H^{2p-2}(\mathbb{Z}_{(p)}) & \longrightarrow & H^{2p-1}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \\ \downarrow & & \downarrow \\ \mathcal{OP}^{2p-2}(\mathbb{Z}_{(p)}) & \xrightarrow{\mathfrak{B}} & \mathcal{OP}^{2p-1}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \end{array}$$

the top arrow is an isomorphism by Proposition 5.14. The right vertical arrow is an epimorphism by the standard argument (cf. (5.1)). So, the map $\mathfrak{B}$ is an epimorphism. Since $\mathcal{OP}^{2p-2}(\mathbb{Z}_{(p)}) \cong \mathbb{Z}/p$ and the group $\mathcal{OP}^{2p-1}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$ is non-trivial by results in the next section, the statement follows. □

Summarizing the results of 5.13–5.15, we obtain the following.

Theorem 5.16. *There are no non-trivial stable cohomological $\mathbb{Z}_{(p)}$–$\mathbb{Z}_{(p)}$-operations of weight $p-1$ and degree greater than $2p-1$. Every non-trivial stable operation of the form*

$$H^{*,*}(-, \mathbb{Z}_{(p)}) \to H^{*+2p-1, *+p-1}(-, \mathbb{Z}_{(p)})$$

in motivic cohomology coincides, after multiplication by a unit of $\mathbb{Z}/p$, with the operation $\mathfrak{B}P^1 r$, where P^1 denotes the first $\mathbb{Z}/p$ motivic Steenrod power, $\mathfrak{B} = [\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}, \mathbb{Z}/p]$ and r is the corresponding coefficient reduction operation $\mathbb{Z}_{(p)} \to \mathbb{Z}/p$.

6. $d_p \neq 0$

The purpose of the current section is to construct for every prime p a smooth variety having the property that the pth differential d_p is non-zero. Although in the previous discussion we systematically avoided the case of $p = 2$, in this section we decided to give slightly more general statements for completeness. So, let p just be a prime number. All coefficient rings are by default $\mathbb{Z}_{(p)}$. Abusing the notation, we omit mentioning coefficients unless it is absolutely necessary.

Below we give two examples demonstrating non-triviality of the differential d_2 for $p = 2$ and d_p for odd primes.

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Example 6.1. Consider the motivic cohomology spectral sequence for the variety $\operatorname{Spec}\mathbb{Q}$. One can check that the Milnor symbol $\{-1,-1,-1,-1\} \in K_4^M(\mathbb{Q})$ is non-trivial of order 2. The group $K_4^M(\mathbb{Q}) = \mathbb{Z}/2$ is canonically isomorphic to $E_2^{0,-4}$. On the other hand, the spectral sequence converges in the degree $i+j=-4$ to $K_4(\mathbb{Q})$ and the map $K_4^M(\mathbb{Q}) \to K_4(\mathbb{Q})$ should pass through the stable homotopy group of the sphere spectrum π_S^4. The latter group is trivial; therefore, one gets from the short exact sequence $E_2^{-2,-3} \xrightarrow{d_2} E_2^{0,-4} \to E_\infty^{0,-4}$ that the differential $d_2 : H^{1,3}(\operatorname{Spec}\mathbb{Q}) \to E_2^{0,-4} = K_4^M(\mathbb{Q})$ is non-zero. This is, certainly, true with $\mathbb{Z}_{(2)}$ coefficients as well.

A more detailed explanation can be found in [Wei13, III.7.2, VI.4.3, Exercise IV.1.12].

PROPOSITION 6.2. *Let us assume that for an odd prime number p and a variety $G \in Sm/k$ the following conditions are satisfied:*

(i) $K_0(G, \mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)} \cdot \mathbf{1}$, *where the class* $\mathbf{1}$ *lies in codimension* 0*;*
(ii) $CH^{p+1}(G, \mathbb{Z}_{(p)}) \neq 0$.

Then the differential $d_p : E_p^{1,-2} \to E_p^{p+1,-p-1}$ in the motivic spectral sequence

$$E_2^{i,j} = H^{i-j,-j}(G, \mathbb{Z}_{(p)}) \Rightarrow K_{-i-j}(G, \mathbb{Z}_{(p)})$$

is non-trivial.

Proof. Since motivic cohomology groups coincide with higher Chow groups, the term

$$E_2^{p+1,-p-1} = CH^{p+1}(G, 0) = CH^{p+1}(G) \neq 0$$

by (ii). By Proposition 3.1, one has $E_2 = E_p$. On the other hand, $E_\infty^{p+1,-p-1} = 0$, since, by (i), the whole group $K_0(G)$ is concentrated in the term $E_\infty^{0,0} = \mathbb{Z}_{(p)}$. Again, by Proposition 3.1 and the triviality of groups $E_2^{i,j} = H^{i-j,-j}(X)$ for $j > 0$ or $i + j > 0$ (see Statement 1.1(i, iii)), one also has $E_{p+1}^{p+1,-p-1} = 0$. (For the case $p = 3$, we also need to use triviality of the group $H^{-1,0}(X)$ (see Statement 1.1(iv)).) Hence, there should be a non-trivial differential that 'kills' the term $E_p^{p+1,-p-1}$ and the only possibility is that $0 \neq d_p : E_p^{1,-2} \to E_p^{p+1,-p-1}$. □

Example 6.3. Consider a non-split central simple algebra $\mathcal{D}$ of degree p over k. Set $G = \mathrm{SL}_{1,\mathcal{D}}$ to be the norm variety, the subvariety of $\mathcal{D}$, given by the equation $\operatorname{Nrd} x = 1$, where Nrd denotes the reduced norm (see [GS06, § 2.6]). We will show below that this gives us an example of a variety with $d_p \neq 0$.

Now we are almost ready to complete the proof of Theorem 2.1.

Let us fix an odd prime number p and an integer l such that either $l = 0$ or $(l, p) = 1$. We also introduce a field F of characteristic l, setting

$$F = \begin{cases} \mathbb{Q} & \text{for } l = 0, \\ \mathbf{F}_l(t) & \text{for } l > 0. \end{cases}$$

Here $\mathbf{F}_l(t)$ is the field of rational functions over the prime finite field $\mathbf{F}_l$. Global class field theory tells us that in all the cases the Brauer group $\operatorname{Br}(F)$ has many non-trivial p-torsion elements and there are non-split central simple algebras of degree p over F. Hence, for any characteristic l, we obtain examples of fields and varieties over them with non-trivial differentials d_p.

Let us consider now the case $k = \mathbf{F}_l$ for an odd prime number l. Choose, as before, a non-split central simple algebra $\mathcal{D}$ over $F = \mathbf{F}_l(t)$. By Example 6.3 (cf. Proposition 6.5 and discussion above it), its associated norm variety $\mathrm{SL}_{1,\mathcal{D}}$ satisfies the conditions of Proposition 6.2. Using standard limit arguments, one can find a finitely generated k-algebra $A \subset F$ and a variety $M \in Sm/k$ over $\operatorname{Spec} A$ that also satisfies the conditions of Proposition 6.2 and such that $M \times_{\operatorname{Spec} A} \operatorname{Spec} F = \mathrm{SL}_{1,\mathcal{D}}$. This implies that the coefficient α in the relation $d_p = \mathfrak{B}\alpha P^1 r$ of Theorem 2.1 is non-zero over k.

This argument shows that $\alpha \neq 0$ for every prime field, because we already know that the conclusion is true for the case of characteristic 0. Using the functoriality of the MCSS and cohomological operations, one can show that the same statement holds for an arbitrary field. This completes the proof of Theorem 2.1. □

Remark 6.4. Alternatively, in the previous proof, one could consider the variety $\mathrm{SL}_{1,\mathcal{D}}$ as a motivic space over k. It shows that the corresponding slice spectral sequence has non-trivial differential d_p given by a non-zero cohomological operation on the category **Spc**. By [Lev08] (cf. also p. 2118), this implies that the differential in the MCSS is also given by this non-zero operation.

It is left to show that the variety G from Example 6.3 satisfies the assumptions of Proposition 6.2. The first one is checked in [Sus91, Theorem 6.1]. The rest of the paper is devoted to proving the second one.

Below we denote by $X = \mathrm{SB}(\mathcal{D})$ the Severi–Brauer variety, corresponding to the algebra $\mathcal{D}$ (see [GS06, ch. 5]). This is a twisted form of the projective space $\mathbb{P}^{p-1}$. So, one has $\dim X = p-1$. Let us also mention that since G is a twisted form of SL_p, one has $\dim G = p^2 - 1$.

Proposition 6.5. *For the variety $G = \mathrm{SL}_{1,\mathcal{D}}$ introduced above, one has $CH^{p+1}(G) \neq 0$.*

Proof. Setting, as above, $X = \mathrm{SB}(\mathcal{D})$, for the projection map $G \times X \to G$ consider a filtration of the base by codimension of points and write down the corresponding spectral sequence (see Rost [Ros96, § 8]):

$$E_1^{s,t}(n) = \coprod_{g \in G^{(s)}} H^t(X_{F(g)}, \mathcal{K}_{n-s}) \;\Rightarrow\; H^{s+t}(G \times X, \mathcal{K}_n), \tag{6.1}$$

where $X_{F(g)} = X \times \operatorname{Spec} F(g)$ is a fiber over the generic point g of codimension s. This spectral sequence is a natural generalization of the Brown–Gersten–Quillen (BGQ) spectral sequence (the cohomology groups here are the $\mathcal{K}$-cohomology, i.e. the Zariski cohomology groups with coefficients in the sheaf $\mathcal{K}$ associated to Quillen's K-theory).

For convenience, we have included a diagram below of the case $n = p+1$, which is the most important case for us. For brevity, we have used the following notation:

$$\coprod_{g \in G^{(s)}} H^t(X_{F(g)}, \mathcal{K}_u) =: R^s_{t,u}.$$

The non-zero part of the E_1-page is concentrated in the strip given by the inequalities $s \geqslant 0$, $0 \leqslant t \leqslant p-1$ and $s + t \leqslant n$. Let us denote the spectral sequence $E_*^{*,*}(p+1)$ by $E_*^{*,*}$, so that $E_1^{s,t} = R^s_{t,p+1-s}$.

$$
\begin{array}{cccccc}
R^0_{p-1,p+1} & R^1_{p-1,p} & R^2_{p-1,p-1} & 0 & 0 & \\
\vdots & \vdots & & \ddots & 0 & 0 \\
R^0_{1,p+1} & R^1_{1,p} & \cdots & R^{p-1}_{1,2} & R^p_{1,1} & 0 \\
R^0_{0,p+1} & R^1_{0,p} & \cdots & & R^p_{0,1} & R^{p+1}_{0,0}
\end{array}
\qquad d_p : R^1_{p-1,p} \to R^{p+1}_{0,0}
$$

Let the following statements hold:

(i) $E_2^{p+1,0} = CH^{p+1}(G)$;

(ii) the boundary map $H^p(G \times X, \mathcal{K}_{p+1}) \xrightarrow{\varphi} E_p^{1,p-1}$ is not an epimorphism.

Then the proposition follows easily. Actually, just consider a fragment of the boundary short exact sequence

$$H^p(G \times X, \mathcal{K}_{p+1}) \xrightarrow{\varphi} E_p^{1,p-1} \xrightarrow{d_p} E_p^{p+1,0}.$$

By (ii), φ is not an epimorphism, and so one has $E_p^{p+1,0} \neq 0$. But by (i) and for dimension reasons there exists an epimorphism $CH^{p+1}(G) = E_2^{p+1,0} \twoheadrightarrow E_p^{p+1,0}$ that proves the desired result.

The rest of the paper is devoted to the proof of the auxiliary statements: (i) is established in Lemma 6.6 right below, (ii) is proven in Proposition 6.11. $\square$

LEMMA 6.6. *In the spectral sequence considered in Proposition 6.5 above, one has* $E_2^{p+1,0} = CH^{p+1}(G)$.

Proof. One has $E_2^{p+1,0} = R^{p+1}_{0,0}/R^p_{0,1}$. Decoding the notation, we get

$$E_2^{p+1,0} = \operatorname{Coker}\left(\coprod_{g \in G^{(p)}} F(g)^* \to \coprod_{g \in G^{(p+1)}} \mathbb{Z} \right) = CH^{p+1}(G) \tag{6.2}$$

that completes the proof. The same is, certainly, true with $\mathbb{Z}_{(p)}$ coefficients. $\square$

In order to check statement (ii), we should perform some computation with the term $E_p^{1,p-1}$. The following lemma simplifies our life showing that we actually work with the term $E_2^{1,p-1}$.

LEMMA 6.7. *In the spectral sequence in Proposition 6.5, one has* $E_2^{1,p-1} = E_p^{1,p-1}$.

Proof. By the next lemma, one has $E_2^{p+1-t,t} = 0$ for $1 \leqslant t \leqslant p-1$. So, for dimension reasons, the only non-trivial differential with domain $E_*^{1,p-1}$ is d_p. $\square$

LEMMA 6.8. *The differential maps* $d_1^t : R^{p-t}_{t,t+1} \to R^{p-t+1}_{t,t}$ *are epimorphisms, provided that* $1 \leqslant t \leqslant p-1$. *In other words, in these cases* $E_2^{p+1-t,t} = 0$.

Proof. We have to prove that the maps

$$\coprod_{g\in G^{(p-t)}} H^t(X_{F(g)},\mathcal{K}_{t+1}) \to \coprod_{g\in G^{(p+1-t)}} H^t(X_{F(g)},\mathcal{K}_t)$$

are epimorphisms. The inner groups $H^t(X_{F(g)},\mathcal{K}_{t+m})$ can be computed using the BGQ spectral sequence. Writing down Gersten resolutions for different values of t, one gets natural maps of the resolutions, induced by embeddings of points of different codimensions. This implies natural maps of BGQ spectral sequences and, finally, natural maps of $\mathcal{K}$-cohomology groups

$$\cdots \to H^t(X_{F(g)},\mathcal{K}_{t+m}) \to H^{t+1}(X_{F(g)},\mathcal{K}_{t+1+m}) \to \cdots .$$

By Statement 6.9 below, these maps are isomorphisms for $m=0,1$ and $1 \leqslant t \leqslant p-1$. By functoriality of the construction, this implies that

$$E_2^{p+1-t,t} = R_{t,t}^{p+1-t}/R_{t,t+1}^{p-t} \cong R_{p-1,p-1}^{p+1-t}/R_{p-1,p}^{p-t} = E_2^{p+1-t,p-1}(2p-t).$$

The rest follows from Lemma 6.10 below, setting there $n=2p-t$. □

In the proof of the previous proposition we used a result of Merkurjev and Suslin, which we reproduce here.

Statement 6.9 [MS82, Corollary 8.7.2]. Let $\bar{k}$ be the algebraic closure of k. For a Severi–Brauer variety X of dimension $p-1$, set $\bar{X} = X \times \operatorname{Spec} \bar{k}$. Then

$$H^i(X,\mathcal{K}_i) = CH^i(X) = p\mathbb{Z}_{(p)} \subset \mathbb{Z}_{(p)} = CH^i(\bar{X}) \tag{6.3}$$

and

$$H^i(X,\mathcal{K}_{i+1}) = \operatorname{Nrd} \mathcal{D}^* \subset \bar{k}^* = H^i(\bar{X},\mathcal{K}_{i+1}), \tag{6.4}$$

provided that $1 \leqslant i \leqslant p-1$. (Here Nrd denotes the group of the reduced norms.)

LEMMA 6.10. *For $n > p$, one has $E_2^{n-p+1,p-1}(n) = 0$.*

Proof. Consider now $G \times X$ as a group variety over X. By Suslin's computations [Sus91, Theorem 4.2], $H^*(G\times X,\mathcal{K}_*)$ becomes a module over $H^*(X,\mathcal{K}_*)$ generated by Chern classes c_j for $j \geqslant 1$, where $c_j \in H^j(G\times X,\mathcal{K}_{j+1})$. In particular, this implies that $CH^i(G\times X) = 0$ for $i > p-1$. Therefore, the spectral sequence converges to zero in the nth diagonal. In particular, $E_\infty^{n-p+1,p-1}(n) = 0$. For dimension reasons, there are no differentials affecting the term $E_2^{n-p+1,p-1}(n)$. So, one has $E_2^{n-p+1,p-1}(n) = E_\infty^{n-p+1,p-1}(n) = 0$. □

PROPOSITION 6.11. *The map $\varphi : H^p(G\times X,\mathcal{K}_{p+1}) \to E_p^{1,p-1}$ has non-trivial cokernel.*

Proof. Let us mention, first, that by the previous lemma, one has $E_p^{1,p-1} = E_2^{1,p-1}$. Consider the base-change commutative diagram corresponding to the morphism $\operatorname{Spec}\bar{k} \to \operatorname{Spec} k$, where $\bar{k}$ is the algebraic closure of k. Later we denote $E_2^{1,p-1}$ by V and the corresponding group $E_2^{1,p-1}$ over $\bar{k}$ by $\bar{V}$.

$$\begin{array}{ccc} H^p(G\times X,\mathcal{K}_{p+1}) & \xrightarrow{\varphi} & V \\ \chi\downarrow & & \downarrow\psi \\ H^p(\bar{G}\times \bar{X},\mathcal{K}_{p+1}) & \xrightarrow{\bar{\varphi}} & \bar{V} \end{array} \tag{6.5}$$

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The desired statement can be derived easily from the following three claims:

(i) $\operatorname{Im}\chi$ is divisible by p;
(ii) $\psi : V \to \bar{V}$ is an epimorphism;
(iii) $\bar{V} = \mathbb{Z}_{(p)}$.

Assume that φ is an epimorphism. Since ψ is also an epimorphism, we can choose an element $x \in H^p(G \times X, \mathcal{K}_{p+1})$ such that $\psi\varphi(x) = 1$. Then, by (i), $1 = \bar{\varphi}\chi(x)$ is p-divisible. This gives a contradiction. We prove (i) in Lemma 6.12 and (ii) in Proposition 6.14 below. Finally, (iii) appears in the proof of 6.14 as an indirect result. □

Lemma 6.12. *For the base-change morphism $\chi : H^p(G \times X, \mathcal{K}_{p+1}) \to H^p(\bar{G} \times \bar{X}, \mathcal{K}_{p+1})$, the image of χ is divisible by p.*

Proof. This follows from the above-mentioned (see the proof of Lemma 6.10) decomposition

$$H^p(G \times X, \mathcal{K}_{p+1}) = \coprod_{i>0} c_i CH^{p-i}(X) \tag{6.6}$$

and the fact that the map $CH^i(X) \to CH^i(\bar{X})$ is a multiplication by p due to Statement 6.9. □

Before we can prove the last proposition, we need to construct one map. To this end, let us reproduce here one important definition (see [Pan91, 3.1] for details).

Definition 6.13. For a quasi-compact locally Noetherian scheme Y, let A be a sheaf of algebras on Y locally isomorphic in the étale topology to the sheaf of split algebras $M_n(\mathcal{O}_Y)$. In other words, A is an Azumaya algebra on Y.

Consider the category of sheaves of left A-modules and denote by $\mathcal{P}(Y; A)$ its full subcategory, whose objects are locally free coherent $\mathcal{O}_Y$-modules. We set $K_*(Y; A) := K_*(\mathcal{P}(Y; A))$, where the functor on the right-hand side is obtained by application of Quillen's Q-construction [Qui73].

We will also write, for brevity, $H^*(G, \mathcal{K}_*; A)$ for $H^*(G, \mathcal{K}_*(-; A))$.

Currently, we are going to construct a natural epimorphism $\tilde{\rho} : V \to H^1(G, \mathcal{K}_2; \mathfrak{D})$, where $\mathfrak{D} := \mathcal{D}^{\otimes(p-1)}$ and $V = E_2^{1,p-1}$ (see Proposition 6.11).

First, consider the BGQ spectral sequence converging to the K-groups of the Severi–Brauer variety X. Since $(p-1)!$ is invertible in the coefficient ring, this spectral sequence has no non-trivial differentials affecting the two highest diagonals. Moreover, if the base field is algebraically closed, all the differentials in the spectral sequence vanish (see [MS82, 8.6.2]). Again, by the invertibility of $(p-1)!$, the topological filtration on the K-groups coincides with γ-filtration. The latter filtration is generated by the image of the corresponding γ-operation.

The E_∞-terms of the BGQ spectral sequence are the graded parts of $K(X)$. Taking into account the triviality of differentials, mentioned in the previous paragraph, there exist boundary maps

$$H^{p-1}(X, \mathcal{K}_{p-1+m}) \to K_m(X)^{(p-1)}, \tag{6.7}$$

where $m = 0, 1, 2$ and we have the smallest non-trivial filtration group on the right-hand side. These maps are isomorphisms for $m = 0, 1$. Provided that the base field is algebraically closed, they are isomorphisms also for $m = 2$.

By Quillen's computation of K-groups of Severi–Brauer varieties [Qui73], one has isomorphisms $K_m(X)^{(p-1)} \cong K_m(\mathfrak{D})$, so that we obtain the maps $H^{p-1}(X_g, \mathcal{K}_{p-1+m}) \xrightarrow{\rho_m} K_m(F(g); \mathfrak{D})$

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for $m = 0, 1, 2$, which are isomorphisms for $m = 0, 1$ and, provided that the base field is algebraically closed, are isomorphisms for $m = 2$. As a result, one gets the map of complexes ρ_*:

$$\begin{array}{ccccc} R^0_{p-1,p+1} & \longrightarrow & R^1_{p-1,p} & \longrightarrow & R^2_{p-1,p-1} \\ \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_0} \\ K_2(F(G);\mathfrak{D}) & \longrightarrow & \coprod_{g\in G^{(1)}} K_1(F(g);\mathfrak{D}) & \longrightarrow & \coprod_{g\in G^{(2)}} K_0(F(g);\mathfrak{D}) \end{array} \tag{6.8}$$

inducing the epimorphism map $\tilde{\rho}$ on the middle-term homology groups. The latter map becomes an isomorphism after passing to the algebraic closure. The middle-term homology groups in the upper and bottom lines can be identified with V and $H^1(G, \mathcal{K}_2; \mathfrak{D})$, correspondingly, which gives us the desired epimorphism $\tilde{\rho}$.

PROPOSITION 6.14. *Let V and $\bar{V}$ be as before. Then the map $\psi : V \to \bar{V}$ is an epimorphism.*

Proof. Let us consider the base-change diagram corresponding to the morphism $\operatorname{Spec} \bar{k} \to \operatorname{Spec} k$.

$$\begin{array}{ccc} V & \xrightarrow{\psi} & \bar{V} \\ {\scriptstyle \tilde{\rho}}\downarrow & & \| \\ H^1(G, \mathcal{K}_2; \mathfrak{D}) & \xrightarrow{\omega} & H^1(\bar{G}, \mathcal{K}_2; \bar{\mathfrak{D}}) \end{array} \tag{6.9}$$

Observe now that $\bar{G} = \mathrm{SL}_n(\bar{k})$ and $H^1(\bar{G}, \mathcal{K}_2; \bar{\mathfrak{D}}) = H^1(\mathrm{SL}_n, \mathcal{K}_2) = \mathbb{Z}_{(p)}$ with a natural choice of a generator, given by the first Chern class (see [Sus91, Theorem 2.7]). This gives us the following commutative diagram.

$$\begin{array}{ccc} V & \xrightarrow{\psi} & \bar{V} \\ {\scriptstyle \tilde{\rho}}\downarrow & & \| \\ H^1(G, \mathcal{K}_2; \mathfrak{D}) & \xrightarrow{\omega} & H^1(\mathrm{SL}_n, \mathcal{K}_2) \\ \uparrow{\scriptstyle c_1} & & {\scriptstyle \bar{c}_1}\uparrow \\ K_1(G; \mathfrak{D}) & \xrightarrow{f} & K_1(\mathrm{SL}_n) \end{array} \tag{6.10}$$

Consider the universal element $\alpha \in K_1(G; \mathfrak{D})$ determined as in [Sus91, § 4]. It is constructed in such a way that its image $f(\alpha)$ in $K_1(\mathrm{SL}_n)$ is the universal matrix element. Then, due to [Sus91, Theorem 2.7], $\bar{c}_1 f(\alpha) = 1$. Hence, the map ω is an epimorphism and so is ψ. □

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Appendix. Something about the groups of rational operations

In this short appendix we give two statements concerning cohomology groups of the spectrum $\mathbf{H}(\mathbb{Q})$, which we need in the paper.

Statement A.1. All motivic cohomology groups of the spectrum $\mathbf{H}(\mathbb{Q})$ with $\mathbb{Z}/p$-coefficients vanish.

Proof. This is left to the reader. □

PROPOSITION A.2. *For integers $n, \varepsilon > 0$, one has $H^{2n+\epsilon,n}(\mathbf{H}(\mathbb{Q}), \mathbb{Q}) = 0$.*

Proof. We want to compute the group $H^{2n+\varepsilon,n}(\mathbf{H}(\mathbb{Q}), \mathbb{Q}) = [\mathbf{H}(\mathbb{Q}), \mathbf{H}(\mathbb{Q})[2n+\varepsilon](n)]$. Since the spectrum $\mathbf{H}(\mathbb{Q})$ is a direct summand of the spectrum $\mathbf{BGL}_{\mathbb{Q}}$, it suffices to show that $[\mathbf{BGL}_{\mathbb{Q}}, \mathbf{BGL}_{\mathbb{Q}}[2n+\varepsilon](n)] = 0$. Using the Bott periodicity and [Rio10, Corollary 5.3.1], we have

$$[\mathbf{BGL}_{\mathbb{Q}}, \mathbf{BGL}_{\mathbb{Q}}[2n+\varepsilon](n)] = [\mathbf{BGL}_{\mathbb{Q}}, \mathbf{BGL}_{\mathbb{Q}}[\varepsilon]] = \varprojlim (K_{-\varepsilon}(k))^{\Omega}.$$

(Here we use the notation of [Rio10].) Since the group $K_{-\varepsilon}(k)$ is the algebraic K-group of the base field k and, obviously, vanishes for $\varepsilon > 0$, then so does $\varprojlim (K_{-\varepsilon}(k))^{\Omega}$. □

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