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ABOUT A LOCAL APPROXIMATION THEOREM AND AN INVERSE FUNCTION THEOREM

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Abstract

This paper gives a theorem by which it is possible to derive in an easy way a local approximation theorem and an inverse function theorem. The latter theorems are not new. The main advantage of our paper is in giving a relatively short route to these results.

1. Introduction

Suppose we are given two Banach spaces X and Z and a mapping g: $X \to Z$ which is continuously Fréchet differentiable at a given point $x_0 = 0 \in X$. Denote this derivative by $g'(x_0)$ and assume that $g'(x_0)(X) = Z$, and that $g'(\,.\,)$ is continuous at x_0. As is well known an *inverse function theorem* in this situation takes the following form, see [3]:

> There is a neighbourhood V of the origin in Z such that, for any $v \in V$, $g(x) = v$ has a solution $x(v)$ with $|x(v) - x_0| \leqslant K|v|$, for some K, only depending on V.

Assume now that, for some $h \in X$, one has $g'(x_0)(h) = 0$; then a *local approximation theorem*, see [1, 3 or 4] says:

> There is a mapping $\eta(\lambda) = o(\lambda)$ such that $g(x_0 + \lambda h + \eta(\lambda)) = 0$.

Both results are applied in mathematical programming when deriving Lagrange multiplier theorems: the inverse function theorem, for instance, in [4] and the local approximation theorem, for instance, in [1].

Extensions of both results can be found in [7]. There, the inequality system $-g(x) \in Z_+$ is studied, where Z_+ is a closed convex cone, with apex at the origin, in Z. The proofs of these generalizations are, however, rather long.

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The aim of this paper is to provide a clearer understanding of Robinson's proofs in [7]. In doing so we are less general than Robinson is; the route we take, however, is considerably shorter.

The main ingredient in our approach, as well as in that of Robinson, is the notion of convex process, a concept originated by Rockafellar [8] and developed further by Robinson [5]. We will also use a general *contraction theorem* due to Robinson [6].

1. Preliminaries

In the sequel, X, Y and Z will be real Banach spaces.

DEFINTION 1. ([5]). *A mapping* $T: X \to 2^Z$ *is called convex process from* X *to* Z *if*

(*a*) $Tx + Tx' \subset T(x+x')$ *for all* $x, x' \in X$ *such that* Tx *and* Tx' *are non empty subsets of* Z,

(b) $T(\lambda x) = \lambda Tx$ *for all* $x \in X$ *such that* $Tx \neq \varnothing$ *and for all* $\lambda > 0$,

and

(c) $0 \in T0$.

When a convex process T from X to Z assigns to every $x \in X$ only one point of Z, then T can be identified with a linear operator from X to Z. If T is a convex process from X to Z then T^{-1} defined by $x \in T^{-1}z$ if and only if $z \in Tx$ too is a convex process but now from Z to X. A non-trivial example is the following:

EXAMPLE. Let A be a linear operator from X to Z and Z_+ a convex cone in Z with apex at the origin. Then $T: X \to 2^Z$ defined by $Tx = Ax + Z_+$ is a convex process from X to Z.

It is easy to see that $T: X \to 2^Z$ is a convex process from X to Z if and only if *graph* $T = \{(x,z) \mid z \in Tx\}$ is a convex cone in (X, Z) with apex at the origin.

The next definition extends the notion of the "norm of a linear mapping".

DEFINITION 2. ([5].) *If* T *is a convex process from* X *to* Z *then*
$|T| = \sup \{\inf \{|z| \mid z \in Tx\} \mid |x| \leqslant 1, Tx \neq \varnothing\}$.

Following Robinson [5] we speak of a *normed* convex process T when $|T|$ is finite. One can, for instance, show that the sum of two normed convex processes is again normed, see [5].

In order to clarify the latter definition we make the following remark. Let A be a linear continuous mapping from X to Z with null-space N. Then A^{-1} can be considered a mapping from Z to X/N, the quotient space of X mod N. Let us denote an element of X/N by $[x]$, $x \in X$. Then $[X]$ can be considered a translate of N over x in X. When we define $|[x]| = \inf\{|x'| \mid x' \in [x]\}$ then, under this norm, X/N is a Banach space (see [4]). It is now easy to see that the norm of A^{-1} regarded as a point to point mapping from Z to X/N equals the norm of A^{-1} considered a convex process from Z to X.

DEFINITION 3. *A convex process T: $X \to 2^Z$ is called closed if graph T is a closed set in (X, Z).*

Obviously, if T is closed, so is T^{-1}. The next results are devices to prove our theorem.

LEMMA 1. ([5].) *If T is a closed convex process from X to Z with $X = \{x \in X \mid Tx \neq \varnothing\}$ then $|T|$ is finite.*

The next result is concerned with the Hausdorff distance.

DEFINITION 4. *Let A and B be subsets of Y, and y a point of Y; then*
$A - B = \{y \in Y \mid y = a - b$ for some $a \in A$ and some $b \in B\}$,
$d(y, A) = \inf\{|y - a| \mid a \in A\}$,
$d(A, B) = \sup\{d(a, B) \mid a \in A\}$,
$\rho(A, B) = \max\{d(A, B), d(B, A)\}$ and $\rho(A, B)$ is the Hausdorff distance between A and B.

LEMMA 2. ([5].) *Let P and Q be non-empty subsets of Z. Let T be a convex process from Z to X such that $|T|$ is finite and such that $T(P)$ and $T(Q)$ are both non-empty. If, further, $(Q - P) \cup (P - Q) \subset \{z \mid Tz \neq \varnothing\}$ then $\rho(T(P), T(Q)) \leqslant |T| \rho(P, Q)$.*

It is easily seen that this lemma generalizes the fact that $|Az| \leqslant |A||z|$ where A is a continuous linear mapping from Z to X.

LEMMA 3. ([6].) *Let T be a mapping from X to 2^Z. Suppose there are non-negative numbers α and r with $0 < \alpha < 1$ and point $\hat{\omega}_0 \in X$ such that:*

(1) *For some $\varepsilon > 0$ and all $\omega_1, \omega_2 \in \bar{B}(\hat{\omega}_0, r + \varepsilon)$, which is the closed ball around $\hat{\omega}_0$ with radius $r + \varepsilon$, $T\omega_1$ and $T\omega_2$ are non-empty closed sets with $\rho(T\omega_1, T\omega_2) \leqslant \alpha\rho(\omega_1, \omega_2)$, and*
(2) *$d(\hat{\omega}_0, T\hat{\omega}_0) \leqslant (1 - \alpha) r$.*
Then there is a point $\omega_\infty \in \bar{B}(\hat{\omega}_0, r + \varepsilon)$ with $d(\hat{\omega}_0, \omega_\infty) \leqslant (1 - \alpha)^{-1} d(\hat{\omega}_0, T\hat{\omega}_0) + \varepsilon$ such that $\omega_\infty \in T\omega_\infty$.

This lemma can be considered a contraction result. For completeness, we state the following generalized mean value theorem.

LEMMA 4. ([4].) *Let $f\colon X \to Z$ be Fréchet differentiable on an open set D. Let $x \in D$ and suppose $x+\alpha h \in D$, $0 \leqslant \alpha \leqslant 1$. Then $|f(x+h)-f(x)| \leqslant |h| \sup_{0<\alpha<1} |f'(x+\alpha h)|$.*

3. Main results

THEOREM. *Let X and Z be Banach spaces and X_+ and Z_+ closed convex cones, with apex at the origin, in X and Z respectively. Let g be a Fréchet differentiable mapping from U to Z, where U is an open convex neighbourhood of the origin in X. Further let the Fréchet derivative $g'(.)$ be continuous at $x_0 = 0$, $-g(x_0) \in Z_+$ and $g'(x_0)(X_+)+Z_+ = Z$. Then there is a function $\sigma(\lambda, h) = o(\lambda h)$ and a $\lambda^* > 0$ such that $h \in \bar{B}(0,1)$, $\lambda \in [0, \lambda^*]$, $v \in Z$ and $-g(x_0)+v-g'(x_0)(h) \in Z_+$ imply the existence of an $\omega_0 \in \bar{B}(0, \sigma(\lambda, h))$ such that $-g(x_0+\lambda h+\omega_\infty)+\lambda v \in Z_+$ and $\omega_\infty \in X_+$.*

PROOF. Defining $G'(x_0)\colon X \to 2^Z$ (the power set of Z) by

$$G'(x_0)(x) = g'(x_0)(x)+Z_+ \quad \text{for } x \in X_+,$$
$$= \varnothing \quad \text{for } x \notin X_+,$$

we have that $G'(x_0)$ is a closed convex process from X to Z. Together with $G'(x_0)(X) = Z$, this implies by Lemma 1 that $|G'(x_0)^{-1}|$ is finite. Now choose $\lambda^* \in (0, 1]$ and $\delta > 0$ such that

(a) $x_0+\lambda h+\omega \in U$ for all $\lambda \in [0, \lambda^*]$, for all $h \in \bar{B}(0,1)$ and for all $\omega \in \bar{B}(0, \delta)$,

(b) $\sigma(\lambda, h) = 2\{|G'(x_0)^{-1}||r(\lambda, h)|+\lambda^2|h|^2\} \leqslant \delta$ for all $h \in \bar{B}(0,1)$ and for all $\lambda \in [0, \lambda^*]$, where $r(\lambda, h) = g(x_0)+g'(x_0)(\lambda h)-g(x_0+\lambda h)$,

and

(c) $|G'(x_0)^{-1}||g'(x_0+\lambda h+\omega)-g'(x_0)| \leqslant \frac{1}{2}$ for all $\lambda \in [0, \lambda^*]$, for all $h \in \bar{B}(0,1)$ and for all $\omega \in \bar{B}(0, \delta)$.

This is possible because of the continuity of $g'(.)$ at x_0. Notice that $\sigma(\lambda, h) = o(\lambda h)$. Now take $h \in B(0,1)$, $h \neq 0$, $\lambda \in (0, \lambda^*]$ and $v \in Z$ such that $-g(x_0)+v-g'(x_0)(h) \in Z_+$, and define

$$q_\lambda(\omega) = g(x_0+\lambda h+\omega)-g'(x_0)(\omega)-\lambda v \quad \text{for all } \omega \in \bar{B}(0,\delta) \text{ and}$$
$$T_\lambda(\omega) = G'(x_0)^{-1}(-q_\lambda(\omega)) \quad \text{for } \omega \in \bar{B}(0, \delta),$$
$$= \{0\} \quad \text{elsewhere.}$$

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Now we will prove that $\omega_1, \omega_2 \in \bar{B}(0, \sigma(\lambda, h))$ imply

(1) $T_\lambda(\omega_1)$ and $T_\lambda(\omega_2)$ are non-empty closed sets,

(2) $\rho(T_\lambda(\omega_1), T_\lambda(\omega_2)) \leqslant \frac{1}{2}|\omega_1 - \omega_2|$,

and

(3) $d(0, T_\lambda(0)) \leqslant \frac{1}{2}\sigma(\lambda, h)$.

Taking this for granted for the moment, Lemma 2 applies with $T = T_\lambda$, $\hat{\omega}_0 = 0$, $r = \varepsilon = \frac{1}{2}\sigma(\lambda, h)$ and $\alpha = \frac{1}{2}$ and therefore there is a $\omega_\infty \in T_\lambda(\omega_\infty)$ with $\omega_\infty \in \bar{B}(0, \sigma(\lambda, h))$.

But $\omega_\infty \in T_\lambda(\omega_\infty)$ is equivalent to

$$-g(x_0 + \lambda h + \omega_\infty) + g'(x_0)(\omega_\infty) + \lambda v \in g'(x_0)(\omega_\infty) + Z_+ \quad \text{for } \omega_\infty \in X_+,$$

and the proof would be complete, except for the case $h = 0$ or $\lambda = 0$.

Let us now return to the implications (1), (2) and (3). As the proof of (1) is trivial, we omit it. Now we prove (2). Applying Lemma 2 with $T = G'(x_0)^{-1}$, $P = \{-q_\lambda(\omega_1)\}$ and $Q = \{-q_\lambda(\omega_2)\}$, it follows that

$$\begin{aligned}\rho(T_\lambda(\omega_1), T_\lambda(\omega_2)) &\leqslant |G'(x_0)^{-1}||q_\lambda(\omega_1) - q_\lambda(\omega_2)| \\ &\leqslant |G'(x_0)^{-1}||\omega_1 - \omega_2| \sup_{0<\alpha<1} |q'_\lambda(\alpha\omega_1 + (1-\alpha)\omega_2)| \\ &\leqslant \tfrac{1}{2}|\omega_1 - \omega_2|, \quad \text{at least when } |G'(x_0)^{-1}| \neq 0.\end{aligned}$$

Here the latter inequality follows from (c). In case $|G'(x_0)^{-1}| = 0$ we even have that $\rho(T_\lambda(\omega_1), T_\lambda(\omega_2)) = 0$.

To end, we prove (3). By definition,

$$\begin{aligned}d(0, T_\lambda(0)) &= \inf\{|x| \mid x \in T_\lambda(0)\} \\ &= \inf\{|x| \mid -g(x_0 + \lambda h) + \lambda v \in g'(x_0)(x) + Z_+\}.\end{aligned}$$

Now, by assumption, we have $-g(x_0) - g'(x_0)(h) + v \in Z_+$ and $-g(x_0) \in Z_+$. As Z_+ is a convex cone it follows that $-g(x_0) + \lambda v - g'(x_0)(\lambda h) \in Z_+$. Hence

$$\begin{aligned}d(0, T_\lambda(0)) &\leqslant \inf\{|x| \mid -g(x_0 + \lambda h) + \lambda v \in g'(x_0)(x) - g(x_0) + \lambda v - g'(x_0)(\lambda h) + Z_+\} \\ &= \inf\{|x| \mid r(\lambda, h) \in g'(x_0)(x) + Z_+\} \\ &\leqslant |G'(x_0)^{-1}||r(\lambda, h)|,\end{aligned}$$

because of the definition of the norm of a convex process. Applying (b), it immediately follows that $d(0, T_\lambda(0)) \leqslant \frac{1}{2}\sigma(\lambda, h)$.

In the case of $h = 0$ or $\lambda = 0$ we may take $\omega_\infty = 0$ and we are done with the proof.

The proof of the foregoing result is very similar to Robinson's proof of an inverse function theorem in [6], the main difference being another choice of $q_\lambda(\omega)$ and $T_\lambda(\omega)$. Notice further that the main argument in the proof is the choice of these two mappings and proving the existence of a fixed point of $T_\lambda(.)$. The rest are technicalities.

In the case of $v = 0$ we can considerably strengthen the above result. This will be done below.

We will study the system $-f(y) \in Z_+, y \in C \subset Y$, where C is a closed convex set of a Banach space Y and where f is a Fréchet differentiable mapping from Y to Z. It is assumed that $f'(.)$ is continuous at $y_0 = 0 \in C$.

We define $L_0 = \{(y, \lambda) \mid \lambda > 0, \lambda^{-1} y \in C\}$, $L = \text{cl}\, L_0$ (note that L is a closed convex cone) and make $X = Y \times R$ a Banach space by introducing a norm on it as follows: $|(y, \lambda)| = \max\{|y|, |\lambda|\}$.

COROLLARY 1. *Let $\bar{h} \in C$ be such that $-f(y_0) - f'(y_0)(\bar{h}) \in Z_+$. If $-f(y_0) \in Z_+$ and if every $z \in Z$ can be written as*

$$z = f'(y_0)(\lambda y) + \omega f(y_0) + z_+$$

$$\text{for some } \lambda \geqslant 0, \quad \omega \geqslant 0, \quad y \in C \quad \text{and} \quad z_+ \in Z_+,$$

then there is a mapping $\eta(\lambda) = o(\lambda)$ such that

$$-f(y_0 + \lambda \bar{h} + \eta(\lambda)) \in Z_+ \quad \text{for } \lambda \bar{h} + \eta(\lambda) \in C.$$

PROOF. The whole trick is to define $g(y, r) = (1+r) f(y_0 + (1+r)^{-1} y)$ for (y, r) in a neighbourhood of $(0, 0)$ and then to apply the theorem to this mapping. Now $g'(y, r) = (f'(y_0 + (1+r)^{-1} y), \quad f(y_0 + (1+r)^{-1} y) - (1+r)^{-1} f'(y_0 + (1+r)^{-1} y)(y))$; hence $g'(0, 0) = (f'(y_0), f(y_0))$ and $g'(.,.)$ is continuous at $(0, 0)$.

We will show that we are allowed to apply the theorem when defining $X = Y \times R$, $X_+ = L$, $v = 0$, $h = (\bar{h}, 0)$ and $x_0 = (y_0, 0) = (0, 0)$. One trivially has that $-g(y_0, 0) \in Z_+$ and $-g(y_0, 0) - g'(y_0, 0)(\bar{h}, 0) \in Z_+$. By assumption, every $z \in Z$ can be written as follows:

$$\begin{aligned} z &= f'(y_0)(\lambda y) + \omega f(y_0) + z_+ \\ &= f'(y_0)(\lambda y) + (\omega + r) f(y_0) - r f(y_0) + z_+ \quad \text{for all } r \in R. \end{aligned}$$

But $-r f(y_0) \in Z_+$ and $\lambda y \in \lambda C \subseteq (\omega + r) C$ for r large enough, and this implies that $g'(0,0)(L) + Z_+ = Z$. Without loss of generality, we may take $h \in \bar{B}(0, 1)$ and the theorem applies, leading to the existence of a mapping $\eta(\lambda) = o(\lambda) \in L$ such that $-g(x_0 + \lambda h + \eta(\lambda)) \in Z_+$. Translated back to $Y \times R$ this means for $\eta(\lambda) = (\eta_1(\lambda), \eta_2(\lambda))$, $-g(\lambda \bar{h} + \eta_1(\lambda), \eta_2(\lambda)) \in Z_+$. Hence $-f(y_0 + (1 + \eta_2(\lambda))^{-1} (\lambda \bar{h} + \eta_1(\lambda))) \in Z_+$ for λ small enough. Now it is easy to see that $(1 + \eta_2(\lambda))^{-1} (\lambda \bar{h} + \eta_1(\lambda)) = \lambda \bar{h} + o(\lambda)$. Take a fixed $\lambda \in (0, 1)$; then there is a sequence $(v_i, \delta_i) \to (0, 0)$ such that $\eta_1(\lambda) + v_i \in (\eta_2(\lambda) + \delta_i) C \subseteq (1 + \eta_2(\lambda) - \lambda) C$. The latter inclusion does hold because C is convex and $0 \in C$, whereas the first inclusion is a consequence of the fact that

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$(\eta_1(\lambda), \eta_2(\lambda)) \in L = \mathrm{cl}\, L_0$. Hence, for λ small enough,

$$(\lambda\bar{h} + \eta_1(\lambda) + v_i)(1 + \eta_2(\lambda))^{-1} = \lambda(1 + \eta_2(\lambda))^{-1}\bar{h} + (1 - \lambda(1 + \eta_2(\lambda))^{-1})$$
$$\times (\eta_1(\lambda) + v_i)(1 + \eta_2(\lambda) - \lambda)^{-1} \in C,$$

because C is convex; hence, since C is closed, $(\lambda\bar{h} + \eta_1(\lambda))(1 + \eta_2(\lambda))^{-1} \in C$, and we are done with the proof.

Corollary 1 is part of Corollary 2 to Theorem 1 of [7]. The special case of Theorem 1 where $X_+ = X$ is a corollary of the local solvability theorem of [2, page 150]; however, Corollary 1 proves a similar conculsion under a somewhat weaker hypothesis. The advantage of our proof is that it is much shorter than Robinson's. We must admit that our approach by using the function $g(y, r)$, in fact the only trick in Corollary 1, was suggested to us, when reading the proofs of Robinson's results in [7].

Now we proceed with proving an inverse function theorem by applying the Theorem. An advantage is the shortness of proof; a disadvantage is that it is less general than Theorem 1 of [7].

COROLLARY 2. *Under the same assumptions as in the Theorem, there is a neighbourhood V of the origin in Z such that for every $v \in V$ there is a $x(v) \in V$ such that $|x_0 - x(v)| \leqslant K|v|$, for some K, depending on V only, and that $-g(x(v)) \in -v + Z_+$.*

PROOF. Take an $\varepsilon > 0$; define $\delta = |G'(x_0)^{-1}| + \varepsilon$ and let $\tilde{V} = \{\tilde{v} \in Z \,|\, |\tilde{v}| \leqslant \delta^{-1}\}$. Further, let $\hat{\lambda} > 0$ be such that $0 \leqslant \lambda \leqslant \hat{\lambda}$ and $|h| \leqslant 1$ imply that $|\sigma(\lambda, h)| \leqslant \lambda |h|$, where $\sigma(\lambda, h)$ is as in the Theorem. Take an arbitrary $\tilde{v} \in \tilde{V}$; then

$$\begin{aligned} &\inf\{|h| \,|\, -g(x_0) + \tilde{v} - g'(x_0)h \in Z_+\} \\ &\leqslant \inf\{|h| \,|\, -g(x_0) + \tilde{v} - g'(x_0)(h) \in -g(x_0) + Z_+\} \\ &= \inf\{|h| \,|\, h \in G'(x_0)^{-1}(\tilde{v})\} \leqslant |G'(x_0)^{-1}|\,|\tilde{v}|, \end{aligned}$$

where the first inequality follows from the fact that $-g(x_0) \in Z_+$. Hence there is an $\bar{h} \in X_+$ with $|\bar{h}| \leqslant \delta|\tilde{v}| \leqslant 1$ such that $-g(x_0) + \tilde{v} - g'(x_0)(\bar{h}) \in Z_+$. Take a $\lambda \in (0, \min\{\lambda^*, \hat{\lambda}\}]$, where λ^* is as in the Theorem; then, by the same Theorem, there is a $\omega_\infty \in \bar{B}(0, \sigma(\lambda, \bar{h}))$ such that $-g(x_0 + \lambda\bar{h} + \omega_\infty) + \lambda\tilde{v} \in Z_+$. Defining $\bar{\lambda} = \min\{\hat{\lambda}, \lambda^*\}$ and $V = \bar{\lambda}\tilde{V}$, we have that every $v \in V$ can be written as $v = \lambda\tilde{v}$ with $\tilde{v} \in \tilde{V}$ and $\lambda \in (0, \bar{\lambda}]$. Putting $x(v) = x_0 + \lambda\bar{h} + \omega_\infty$, we therefore have that $-f(x(v)) \in -v + Z_+$ and $|x_0 - x(v)| = |\lambda\bar{h} + \omega_\infty| \leqslant 2\delta|\lambda\tilde{v}| = 2\delta|v|$ and the proof is complete.

Notice that again we use the finiteness of $|G'(x_0)^{-1}|$. Notice further that, in the case when we take $Z_+ = \{0\}$ and $X_+ = X$, we have Liusternik's results [3].

Finally, we remark that Tuy derives in [9] local approximation theorems like Corollary 1. In the case of $Z = R^K$, he derives far more general results because he allows for so-called "convex derivatives" instead of Fréchet derivatives. In proving his results he relies on fixed point theorems due to Kakutani and Nadler.

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