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ON TWO-BRIDGED KNOT POLYNOMIALS

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Abstract

The extended diagram of a two-bridged knot is introduced, and it is shown how the coefficients of the Alexander polynomial of the knot may be read straight from this diagram. Using this result, it is shown by diagram manipulation that a conjecture of Fox about the coefficients of the Alexander polynomial of an alternating knot is true at least for two-bridged knots (which are all alternating).

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To draw a two-bridged knot or link of type (p,q) (where we assume q is odd, p and q are coprime and $0<q<2p$, see Schubert (1956)), one starts with two overarcs (= overcrossing arcs) placed end to end and oriented towards the centre. Points numbered 0 to p are marked off along each overarc starting from the middle. The (oriented) underarcs are then drawn, each spiralling outwards clockwise from the centre through the points numbered $q, 2q, \ldots$ until they reach the outside, and then spiralling inwards. Eventually after $q-1$ changes of direction (clockwise–anticlockwise) the tail end of an overarc is reached. One obtains a symmetric diagram if one changes direction when one is within a distance $q/2$ of each end.

More precisely, the underarc is divided by the crossing points into p segments which can be numbered consecutively. *The kth segment spirals clockwise or anti-clockwise according as* $[(2kq-q)/2p]$ *is even or odd*. See Fox (1957) or Schubert (1956) for more details, also see Figure 1 below. For a less cluttered diagram one can draw a number (however many are necessary) of overarcs parallel and, so to speak, 'unwind' the diagram. Thus one gets the *extended diagram*, denoted $(p,q)_\infty$. An underarc instead of spiralling clockwise proceeds from left to right, and instead of spiralling anticlockwise proceeds from right to left. Figure 1 shows the knot projection and the extended diagram of the knot (9, 5), with only one

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underarc shown in each case. One imagines the extended diagram to consist of an infinite number of parallel overarcs extending to infinity in both directions, and an infinite number of underarcs. Given one underarc, the others are obtained by translation the appropriate number of steps to the right or left. The diagram $(p,q)_\infty$ will be defined again in more generality in the proof of Theorem 2.

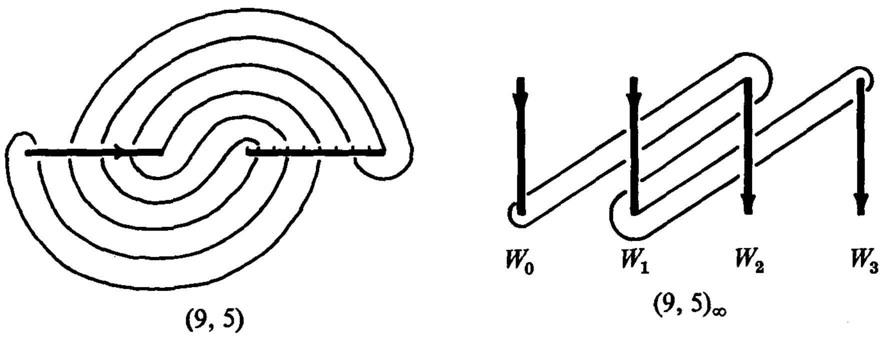

FIGURE 1.

On the extended diagram number all the overarcs, W_i, consecutively from $-\infty$ to ∞ from left to right. A single underarc is divided by the crossing points into p segments each of which joins two adjacent overarcs. Let the number of such segments which join W_i to W_{i+1} be denoted α_i. Then, surprisingly, the α_i are the absolute values of the coefficients of the Alexander polynomial $\Delta(t)$ of the knot (p,q) or the reduced Alexander polynomial in the case where (p,q) is a link. For example, knot $(9,5)$ shown in Figure 1 has Alexander polynomial $2-5t+2t^2$.

In this paper, the word knot will include the concept of a link, and the Alexander polynomial will be taken to mean the reduced Alexander polynomial in this case.

THEOREM 1. $\Delta(t)=\sum_{i=-\infty}^{\infty}(-1)^i\alpha_i t^i$.

PROOF. The right-hand side of the above expression depends of course on which underarc is considered; however, choice of a different underarc results only in multiplication by a factor $\pm t^j$. The Alexander polynomial is also defined only up to multiplication by such a factor. We consider the underarc which has its initial point at the overarc W_0.

The knot group has two generators, A and B and a single relator $A^{\varepsilon_1}B^{\varepsilon_2}\dots B^{\varepsilon_{2p}}$ where $\varepsilon_i=(-1)^{[iq/p]}$. This is the usual 'over presentation'. One sees that this is conjugate to the relator $A^{\delta_1}B^{\delta_2}\dots B^{\delta_{2p}}$, or to $B^{\delta_1}A^{\delta_2}\dots A^{\delta_{2p}}$ where $\delta_i=(-1)^{[(iq+\lambda)/p]}$ for any λ. We choose in particular $\lambda=-q/2$ and call the resulting relator R. Then, since $\delta_i=-\delta_{2p-i+1}$, one sees that R has the form $V.V^{*-1}$ where V^* is the word obtained from some word V of length p by interchanging A and B. Let

θ and ψ be homomorphisms from the knot group into Zt (the ring of integral L-polynomials) where A and B are sent to t by θ and to $-t$ by ψ. Define U_i to be the initial segment of V of length i when $\delta_i = -1$ and of length $i-1$ when $\delta_i = +1$. Now calculate $\Delta(-t)$.

$$\Delta(-t) = (\partial R/\partial A)^{\psi} = (\partial V/\partial A)^{\psi} - (\partial V^*/\partial A)^{\psi} = (\partial V/\partial A)^{\psi} - (\partial V/\partial B)^{\psi}$$
$$= \pm \sum_{i=1}^{p} \delta_i.(-1)^i(U_i\psi) = \pm \sum_{i=1}^{p} U_i\theta.$$

Now, to relate $\Delta(-t)$ to the extended diagram, compare the definition of δ_i with the rule for the direction of the ith underarc segment given above (in italic script). For $1 \leqslant i \leqslant p$, then, δ_i is positive if the ith underarc segment is directed from left to right, and negative if the segment is directed from right to left. It follows that the ith underarc segment lies between W_j and W_{j+1} exactly when $U_i\theta = t^j$. Thus, t^j occurs in the sum $\sum_{i=1}^{p} U_i\theta$ just as many times as there are underarc segments between W_j and W_{j+1}, that is, α_j times. So

$$\Delta(-t) = \pm \sum_{j=-\infty}^{\infty} \alpha_j t^j$$

which gives the result.

The picturesque result of Theorem 1 is useful for the rapid calculation of the polynomials of two-bridged knots. One reads the polynomial straight from the extended diagram. However it also has theoretical applications, and we will use it to prove the following theorem.

THEOREM 2. *The coefficients, α_i of the Alexander polynomial of a two-bridged knot satisfy, for some integer s:*

$$\alpha_0 < \alpha_1 < \dots < \alpha_s = \alpha_{s+1} = \dots = \alpha_{l-s-1} > \alpha_{l-s} > \dots > \alpha_{l-1}$$

where $l-1$ is the degree of the polynomial.

REMARK. K. Murasugi proved (see Murasugi (1958)) that one could write the Alexander polynomial of any alternating knot (and so, of any two-bridged knot) in the form $\sum_{i=0}^{l-1}(-1)^i a_i t^i$ in which $a_i > 0$. This also follows for two-bridged knots from our Theorem 1. Fox (1962) conjectured that for alternating knots, the coefficients satisfy the property expressed in Theorem 2. Thus, our theorem is a partial solution of that conjecture.

A pair of integers (p,q) with $p,q > 0$ and g.c.d.$(p,q) = 1$, and q odd will be called *admissible*. Consider the transformations on pairs of integers

$$T_1 : (p,q) \to (p+q,q); \quad T_2 : (p,q) \to (p,2p+q); \quad \text{and} \quad T_3 : (p,q) \to (p,2p-q),$$

the last defined only when $p > q$. Then starting with the pair $(1,1)$ one obtains any admissible pair after a sequence of such transformations. Proof of this last

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statement is by induction on $p+q$: Suppose $p+q>1$. When $p>q$, then (p,q) is obtained from $(p-q,q)$ by T_1. When $q<2p<2q$, then (p,q) is obtained from $(p,2p-q)$ by T_3. When $2p<q$, then (p,q) is obtained from $(p,q-2p)$ by T_2. These transformations give us a basis for induction on two-bridged knots by which we will prove Theorem 2. Unfortunately, the induction is complicated, and requires us to define carefully an extended diagram corresponding to any admissible pair. This will correspond to the already defined extended diagram of a knot in the case where $q<2p$.

Draw an infinite number of equally spaced infinite parallel lines (called grid lines), Y_i, in the plane, numbered from $-\infty$ to ∞ from left to right. On each mark off $p+q$ points consecutively numbered from $-(q-1)/2$ to $p+(q-1)/2$. The segment of each grid line, Y_i, between points marked 0 and p is the *overarc* W_i, the rest of the grid line acts simply as a guide line. The point numbered j on the grid line Y_i is called x_{ij}. Now, in the region between Y_i and Y_{i+1} join the following pairs of points by simple arcs, pairwise disjoint:

For $-(q-1)/2 \leqslant j \leqslant p-(q+1)/2$ join x_{ij} to $x_{i+1,j+q}$.

For $1 \leqslant j \leqslant (q-1)/2$ join $x_{i,p-j}$ to $x_{i,p+j}$ (call these *top loops*)

and also join $x_{i+1,j}$ to $x_{i+1,-j}$ (call these *bottom loops*).

See Figure 2. The union of these lines, for all i makes up the underarcs.

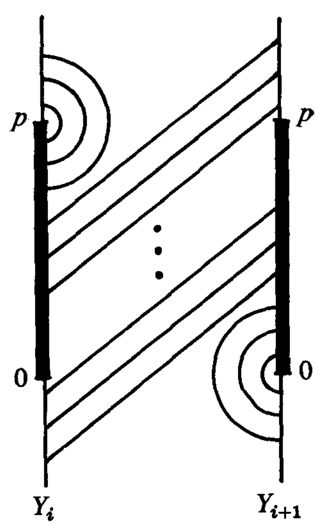

FIGURE 2.

We now consider the single underarc which meets the overarcs $W_0, W_1, \ldots, W_l$ for some l, and no other overarc. Call this the *principal underarc* and l the *length* of the diagram. The principal underarc in general contains bottom and top loops.

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We define the *bottom sequence* $\{b_i\}_{-\infty<i<\infty}$ of the diagram $(p,q)_\infty$ such that b_i is twice the number of bottom loops of the principal underarc at the overarc W_i (one bottom loop is counted each time the underarc loops around the bottom of W_i) plus one if the principal underarc starts at W_i. Thus, for example, the bottom sequence of $(9,5)_\infty$ shown in Figure 1 is: $b_0=2$, $b_1=3$, $b_i=0$ otherwise. The analogously defined *top sequence* defined in terms of top loops of the principal underarc satisfies, because of the obvious symmetry of the diagram,

(A1) $$t_i = b_{l-i}.$$

Now, if the principal overarc neither begins nor ends at W_i, then the number of times Y_i is approached from left and right are equal. This is expressed by $\alpha_{i-1}+b_i=\alpha_i+t_i$, which because of our definition of b_i and t_i remains true at the end points of the underarc also. Using (A1) we have

(A2) $$\alpha_i-\alpha_{i-1}=b_i-b_{l-i}.$$

Also useful in identifying a diagram are the formulae

(A3) $$q=\sum_{i=0}^{l} b_i \quad \text{and} \quad p=\sum_{i=0}^{l-1} \alpha_i$$

but since they are not essential to our argument we leave them unproven.

Given a diagram $(p,q)_\infty$ with bottom sequence $\{b_i\}$ and a nonnegative integer h, define the *h-alternate sequence* of the diagram to be the finite sequence $S_0,\dots,S_h$ defined by $S_{2j}=b_{h-j}$ and $S_{2j+1}=b_j$. The following induction hypothesis will be called IH(p,q): *If l is the length of the diagram $(p,q)_\infty$, then there exists a non-negative integer h; $1\leqslant h\leqslant l$ and an integer $r\leqslant h$ such that $b_i=0$ for $i>h$ and the h-alternate sequence satisfies*

$$0\leqslant S_0<S_1<\dots<S_r=S_{r+1}=\dots=S_h.$$

Our induction argument will show that IH(p,q) implies IH(p',q') where $(p',q')=T_i(p,q)$ for $i=1,2,3$. Since IH$(1,1)$ is obvious, this demonstrates IH(p,q) for all admissible pairs (p,q). As consequences of IH(p,q) we need the following properties of the diagram (p,q).

(H1) *If $h^*\geqslant h$ and $2j\leqslant h^*$, then $b_j\geqslant b_{h^*-j}$.*

(H2) *If $0\leqslant i<j$ and $b_i=b_j$, then $b_i=b_k=b_j$ for all k such that $i\leqslant k\leqslant j$.*

Proof of H1. By our definition, when $h/2\leqslant j\leqslant h$, then $b_j=S_{2h-2j}$. Thus, if $h/2\leqslant i\leqslant j$, then $b_i\geqslant b_j$. Now, if $j<h/2$, then $h^*-j\geqslant h-j>h/2$ and so

$$b_{h^*-j}\leqslant b_{h-j}=S_j\leqslant S_{2j+1}=b_j.$$

On the other hand, if $j\geqslant h/2$, then $h/2\leqslant j\leqslant h^*-j$ and so $b_j\geqslant b_{h^*-j}$.

Proof of H2. If $j > h$, then $b_i = b_j = 0$, which means that $i \geqslant h$. Thus, if $k > i$, then $b_k = 0 = b_i$. So, we can assume that $j \leqslant h$. Let $b_i = S_{i'}$, $b_k = S_{k'}$, $b_j = S_{j'}$. Now, if $i < k < j$, then $k' \geqslant \min(i',j')$. Thus, if $S_{i'} = S_{j'}$, then from $\mathrm{IH}(p,q)$ follows that $S_{i'} = S_{j'} = S_{k'}$, and hence that $b_i = b_k = b_j$.

Now we investigate the effect of the various transformations, T_i, on the extended diagram. In each case, we can think of T_i as a change of the grid lines, Y_i, whereby the underarcs remain unchanged. This change in the grid lines, shown in Figure 3, is then to be followed by an isotopic deformation of the plane to bring the new grid lines into the proper position (see definition above). Thereby, of course, the underarcs are altered.

Effect of T_1. If one replaces the lines Y_i in the extended diagram by new lines in the way shown in Figure 3, then one obtains a new extended diagram for which

$$l' = l+1 \quad \text{and} \quad b'_i = b_i \tag{A4}$$

where primed quantities refer to the new diagram. Using (A3), or directly, one sees that the new diagram is $(p+q,q)$. The bottom sequences of (p,q) and $(p+q,q)$ are the same. Therefore, choosing $h' = h$, we see that $\mathrm{IH}(p+q)$ implies $\mathrm{IH}(p+q,q)$.

Effect of T_2. In this transformation, the overarc portion of the grid line remains initially unchanged. When the grid lines are straightened by an isotopy of the plane, the overarcs are rotated clockwise through 180°. See Figure 3. For the new diagram

$$b'_i = 2\alpha_i + t_i = 2\alpha_i + b_{l-i} \quad \text{and} \quad l = l'. \tag{A5}$$

Either directly, or using (A3), one sees that the new diagram obtained is $(p, q+2p)$.

Let $h' = l' = l$, and let S' be the h'-alternate sequence of (p',q'). Then for $2j+1 \leqslant h' = l$, holds:

$$\begin{aligned} S'_{2j+1} - S'_{2j} = b'_j - b'_{l-j} &= b_{l-j} + 2\alpha_j - b_j - 2\alpha_{l-j} = b_{l-j} + 2(\alpha_j - \alpha_{j-1}) - b_j \\ &= b_{l-j} + 2(b_j - b_{l-j}) - b_j = b_j - b_{l-j} \geqslant 0, \quad \text{by H1.} \end{aligned}$$

And similarly, for $2j \leqslant l$,

$$S'_{2j} - S'_{2j-1} = b'_{l-j} - b'_{j-1} = b_j - b_{l-j+1} \geqslant 0, \quad \text{by H1.}$$

Now suppose $S'_{2j+1} = S'_{2j}$ and $2j+1 < l$. Then $b_j = b_{l-j}$. By H2, $b_j = b_{l-j-1}$ which means $S'_{2j+2} = S'_{2j+1}$. Similarly, $S'_{2j} = S'_{2j-1}$ implies $S'_{2j} = S'_{2j+1}$. This completes the induction step.

Effect of T_3. In this case, the grid lines are rotated anticlockwise when the grid lines are straightened. As a final step in the transformation, the whole

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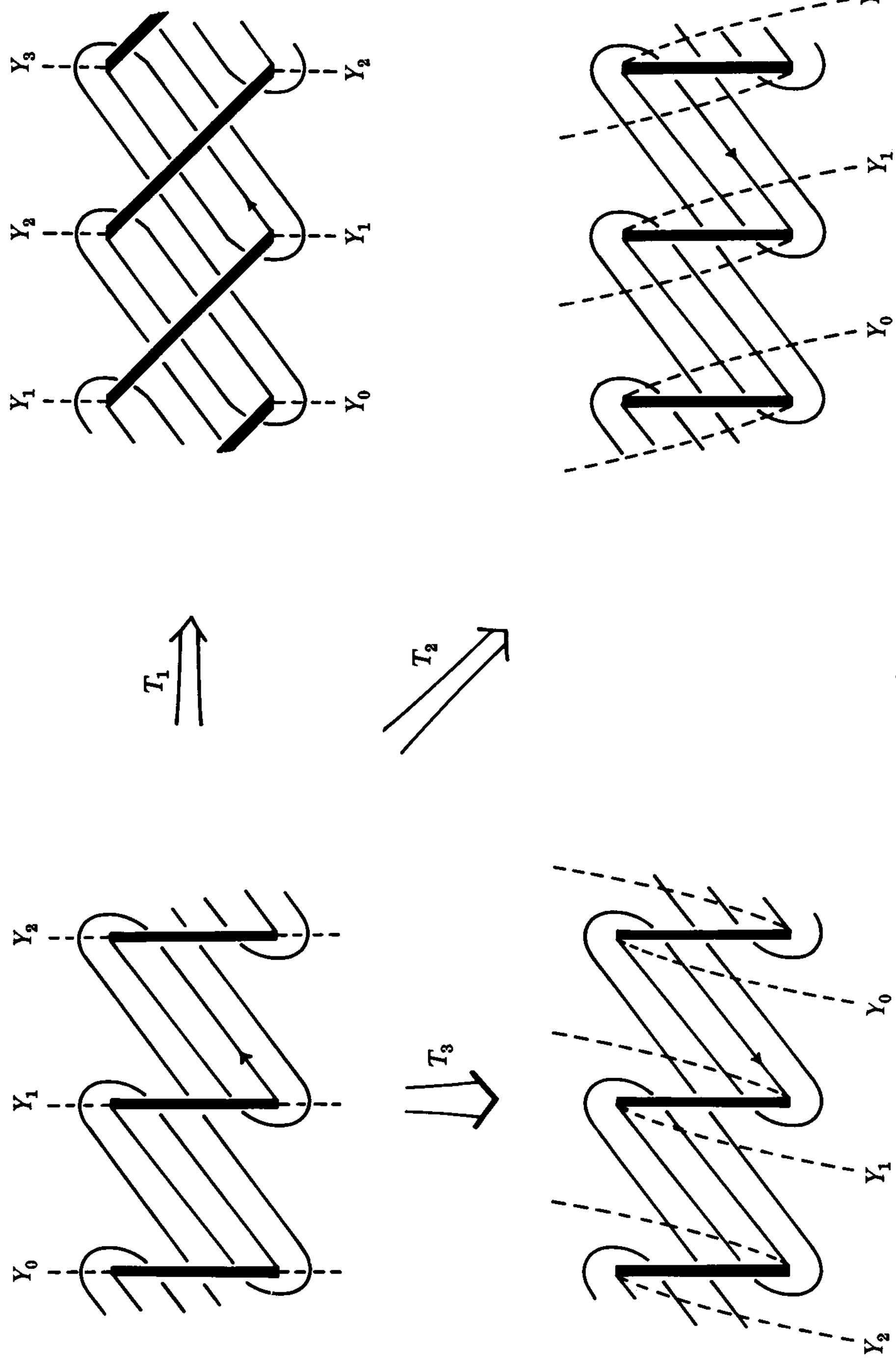

FIGURE 3.

The transformations T_i

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diagram must be reflected from right to left as is suggested by the backwards numbering of the grid lines in Figure 3.

For the new diagram:

$$\text{(A6)} \qquad b_i' = 2\alpha_i - b_i \quad \text{and} \quad l' = l.$$

Directly, or using (A3) one sees that the new diagram is $(p, 2p-q)_\infty$.

Put $h' = l$. Since T_3 is applied only when $p > q$, an application of T_3 is immediately preceded by T_1. Thus $h < h' = l$. Now in a similar fashion to that above, one sees that $\mathrm{IH}(p,q)$ implies $\mathrm{IH}(p, 2p-q)$. This completes the proof of $\mathrm{IH}(p,q)$ for all admissible (p,q).

We now complete the proof of Theorem 2. Suppose $2k < l$. Then (H1) yields $b_k - b_{l-k} \geqslant 0$. Therefore according to (A2), $\alpha_k \geqslant \alpha_{k-1}$. Furthermore, if $\alpha_k = \alpha_{k-1}$, then $b_k = b_{l-k}$. According to (H2), then, $b_{k+1} = b_{l-k-1}$ which gives $\alpha_k = \alpha_{k+1}$ and completes the proof.

Given a knot (p,q), one can determine the sequence of transformations, T_i, which transform $(1,1)$ to (p,q). Now using the formulae (A2), (A4), (A5) and (A6), one can calculate simultaneously the bottom sequence and Alexander polynomial of the knot (p,q). This gives a rapid algorithmic method of determining the Alexander polynomial, which for large p and q is far quicker than any standard method.

K. Funke (see Funke (1978)) has given an algorithmic method of calculating the genus of a two-bridged knot, and also a criterion for determining whether it is a fibred (Neuwirth) knot. His algorithmic method is a little different from the method discussed in this paper. As a corollary of our method, we also have:

Proposition 3. *Suppose $q < 2p$ and consider the sequence of transformations T_i which transform the pair $(1,1)$ to (p,q). If T_1 occurs m times in this sequence and n is the multiplicity of the knot (p,q) (that is 2 if p is even and 1 otherwise), then the genus of (p,q) is equal to $(m-n+1)/2$. The knot is fibred if and only if T_2 does not occur in the sequence.*

Proof. Since only T_1 changes the length of the diagram, m is the degree of $\Delta(t)$. Since the knot is alternating, the formula for the genus is a well-known property of alternating knots (Crowell (1959)). The knot is fibred if and only if $\Delta(0) = 1$. A quick glance at equations (A4), (A5) and (A6) shows that if T_2 occurs, the first term of the sequence b_i, and hence $\Delta(0)$, becomes greater than 1, whereas otherwise it remains equal to 1.

We remark that E. J. Mayland, Jr. has also considered similar methods in his paper: 'Inductive arguments on rational (two-bridged) knots' (unpublished). Prof.

K. Murasugi has also recently shown me some unpublished notes in which he obtains a proof of theorem 2 by quite different methods.

References

Richard H. Crowell (1959), 'Genus of alternating link types', *Ann. of Math.* (2) **69,** 258–275.

R. H. Fox (1957), 'Review of Schubert (1956)', *Math. Reviews* **18,** no. 6, p. 498.

R. H. Fox (1962), 'Problems in knot theory', in *Topology of three-manifolds and related topics,* Proceedings of the University of Georgia Institute, 1961, edited by M. K. Fort, Jr. (Prentice Hall Inc., Englewood Cliffs, N.J.).

Klaus Funke (1978), 'Geschlecht von Knoten und die Faserbarkeit ihrer Aussenräume', *Math. Z.* **159,** 3–24.

Kunio Murasugi (1958), 'On the Alexander polynomial of the alternating knot', *Osaka Math. J.* **10,** 181–189.

Horst Schubert (1956), 'Knoten mit 2 Brücken', *Math. Z.* **65,** 133–170.

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