Mathematica and Maple on hand, it takes no longer to press a key to evaluate a hypergeometric function as it does to find a cosine function. So how about (7) and (9) as solutions of the cubic equation for the irreducible case?

References
1. H. W. Turnbull, Theory of equations, Oliver and Boyd (1952) Chapter IX.

I. J. ZUCKER
16 Highview Gardens, London N3 3EX

92.35 Real roots of cubics: explicit formula for quasi-solutions

Introduction
The Cardano/Tartaglia formulae are of no practical use for cubics with three real roots. They depend, for example, on the evaluation of inverse trigonometric functions such as arc cosx or arc tanx, which require iterative methods for their evaluation, as emphasised in [1] and [2]. This paper develops a procedure to determine the real roots of a cubic, based on the explicit calculation of the intermediary root of a canonical form, obtained by a transformation of the classic depressed form $x^3 + px + q$.

Canonical form
We are interested in the monic cubic

$$g(y) = y^3 + ay^2 + by + c$$  \hspace{1cm} (1)

with real coefficients, but already transformed, by the translation $y = x - \frac{b}{3}$, to

$$h(x) = x^3 + px + q$$ \hspace{1cm} (Depressed form)  \hspace{1cm} (2)

where $p = b - 3 \left[\frac{c}{3}\right]^2$ and $q = c - \frac{b}{3} \left[ p + \left(\frac{c}{3}\right)^2 \right]$.

It happens that (2) has three real roots if, and only if,

$$p \leq 0 \hspace{0.5cm} \text{and} \hspace{0.5cm} |q| < q_M,$$

where $q_M = 2 \left[ \text{sign}(p) \frac{p}{3} \right]^\frac{3}{2}$.  \hspace{1cm} (3)

If also $p < 0$, the real roots are distinct.

We shall recall some known facts while establishing an appropriate nomenclature.

In order to have three real roots, it is necessary and sufficient that $h(x)$ has two local extrema, say, $x_M$ (local maximum) and $x_m$ (local minimum), at which $h(x_M) h(x_m) \leq 0$, with at least one of these two values non-zero.
Evidently, with the leading coefficient of the cubic being 1, \( x_m = \sqrt{-\frac{1}{3}p} \) and \( x_M = -x_m \); furthermore we always have \( x_M < 0 < x_m \).

In the case of three roots, \( x_1 \leq x_2 \leq x_3 \), the following ordering prevails: \( x_1 \leq x_M \leq x_2 \leq x_m \leq x_3 \). We now introduce another change of variable:

\[
w = \frac{x}{x_m} \quad \text{with} \quad x_m = \left[ \frac{\text{sign}(p)\frac{p}{3}}{3} \right]^{1/3}
\]

so that (2) becomes

\[
f(w) = w^3 + \text{sign}(p)3w + 2\alpha \quad \text{canonical form}
\]

where

\[
f(w) = \frac{h(w)}{x_m^3} = \frac{2h(w)}{q_M} \quad \text{and} \quad \alpha = \frac{q}{q_M} = \frac{q}{2} \left[ \text{sign}(p)\frac{p}{3} \right]^{1/3}.
\]

If \( r_1, r_2, r_3 \) are roots of \( f(w) \), Vieta's relations yield

\[
r_1 + r_2 + r_3 = 0; \quad r_1r_2 + r_1r_3 + r_2r_3 = \text{sign}(p)3 \quad \text{and} \quad r_1r_2r_3 = -2\alpha.
\]

**Definition:** A real root \( r_k \) is called intermediary if, there being other roots, \( r_j \leq r_k \leq r_l \). The other two roots are the extreme roots.

Whenever we have three real roots, they will always be ordered monotonically: \( r_1 \leq r_2 \leq r_3 \).

**Geometric interpretation of the canonical form**

The transformation defined in (4) has the effect (in the case of three real roots) of 'shaping' the cubic so that its graph assumes a local maximum at the point \( w_m = -1 \) and a local minimum at the point \( w_M = 1 \). Consequently, \( f(w_m) = 2(\alpha + 1) \) and \( f(w_M) = 2(\alpha - 1) \). Thus \(-4 < f(w_m) < 0 < f(w_M) < 4\). Furthermore, the difference between these values is independent of the cubic, for we always have \( f(w_M) - f(w_m) = 4 \).

We also notice that the lowest valued root is never smaller than \(-2\) whereas the largest root does not exceed \(2\). As a matter of fact, Taylor's expansion about \( x \) yields

\[
f(x + d) = f(x) + f'(x)d + f''(x)\frac{d^2}{2} + f'''(x)\frac{d^3}{6}.
\]

Taking \( x = -2 \) gives

\[
f(-2 + d) = 2(\alpha - 1) + d(d^2 - 6d + 9) = 2(\alpha - 1) + d(d - 3)^2
\]

and, since \( |\alpha| < 1 \), we verify that \( f(-2 + d) \) is negative, for any \( d < 0 \).

We are in a position to consolidate these results as follows.

**Theorem 1:** Separation and proximity of roots

A cubic has three real roots if, and only if, in its canonical form, \( |\alpha| < 1 \) and \( p < 0 \). In that case, the following root separation prevails: \(-2 < r_1 < -1 < r_2 < 1 < r_3 < 2\), and the part of the graph including these roots is contained in a square with sides of length 4.
The following properties hold, where $E$ denotes the interval $[-1, 1]$

- Viewed as a function of $\alpha$, $r_2(\alpha)$ is a bijection on $E$: the intermediary root $r_2$ assumes all values on $E$, as $\alpha$ goes through $E$, and vice versa.
- $r_2(\alpha)$ is an odd function: $r_2(\alpha) = -r_2(-\alpha)$, and the root $r_2(\alpha)$ shares the same sign with $\alpha$.
- The extreme roots satisfy $-3 < r_1 r_3 < -2$ (since $r_1 r_3 = r_2 - 3$) and $3 \leq r_3 - r_1 \leq 2\sqrt{3}$.
- The root with highest absolute value is the only one whose sign is contrary to that of $\alpha$.

By direct substitution we verify that the condition $\alpha^2 < 1$ ($|\alpha| < 1$) is equivalent to the irreducible case (casus irreducibilis) in the Cardano/Tartaglia method: $\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^2 < 0$.

In [7], Nickalls presented several basic properties related to the shape and symmetry of the cubic, with $p < 0$. These results were of great help to us. On the other hand, following a different path from ours, his proposed numerical solution procedure for the irreducible case, although original, also depends on the calculation of inverse trigonometric functions. That means iterations are required.

For illustrative purposes, we present the roots of the canonical form (5), for some particular values of $\alpha \in E$. The accuracy of the results indicated in Table 1 below is of the order of $10^{-12}$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$r_1(\alpha)$</th>
<th>$r_2(\alpha)$</th>
<th>$r_3(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>-2.000000000000</td>
<td>1.000000000000</td>
<td>1.000000000000</td>
</tr>
<tr>
<td>0.90</td>
<td>-1.977439743482</td>
<td>0.729299275657</td>
<td>1.248140467825</td>
</tr>
<tr>
<td>0.75</td>
<td>-1.942241850970</td>
<td>0.557874698332</td>
<td>1.384367152638</td>
</tr>
<tr>
<td>0.50</td>
<td>-1.879385241572</td>
<td>0.347296355334</td>
<td>1.532088886238</td>
</tr>
<tr>
<td>0.25</td>
<td>-1.810037929234</td>
<td>0.168254401781</td>
<td>1.641783527453</td>
</tr>
<tr>
<td>0.20</td>
<td>-1.795219749245</td>
<td>0.134137845705</td>
<td>1.661081903540</td>
</tr>
<tr>
<td>0</td>
<td>$-\sqrt{3}$</td>
<td>0</td>
<td>$\sqrt{3}$</td>
</tr>
<tr>
<td>-0.20</td>
<td>-1.661081903540</td>
<td>-0.134137845705</td>
<td>1.795219749245</td>
</tr>
<tr>
<td>-0.25</td>
<td>-1.641783527453</td>
<td>-0.168254401781</td>
<td>1.810037929234</td>
</tr>
<tr>
<td>-0.50</td>
<td>-1.532088886238</td>
<td>-0.347296355334</td>
<td>1.879385241572</td>
</tr>
<tr>
<td>-0.75</td>
<td>-1.384367152638</td>
<td>-0.557874698332</td>
<td>1.942241850970</td>
</tr>
<tr>
<td>-0.90</td>
<td>-1.248140467825</td>
<td>-0.729299275214</td>
<td>1.977439743482</td>
</tr>
<tr>
<td>-1.00</td>
<td>-1.000000000000</td>
<td>-1.000000000000</td>
<td>2.000000000000</td>
</tr>
</tbody>
</table>

TABLE 1: Roots of the canonical form, for particular values of $\alpha$
Estimation of the intermediary root: inverse function interpolation [3]

We note that it suffices to work with $\alpha > 0$. If $\alpha < 0$, we change its sign and, at the end, the sign of the roots. We want to obtain an estimate for the intermediary root $r_2(\alpha)$ of (5), for the three real roots case. This will be done by inverse function interpolation which is based on the series expansion of the inverse function $\varphi(y)$ of $y = f(x)$. Although unknown, the existence of the function $\varphi$ is guaranteed. The clue is then to solve $x = \varphi(0)$.

The function $\varphi(y)$ is developed by the Taylor series around point $x_0$ which is 'close' to the root $\xi: f(\xi) = 0 \to \xi = \varphi(0)$:

$$x = \varphi(y) = x_0 + \sum_{k=1}^{\infty} \frac{(y - y_0)^k}{k!} \varphi^{(k)}(y_0)$$

where $x_0 = \varphi(y_0)$. When $y = 0$, $x = \xi$ so that

$$\xi - x_0 = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} (y_0)^k \varphi^{(k)}(y_0).$$

This expression depends on the derivatives of $\varphi$ calculated at the points

$$y_0' = f'(x_0), \quad y_0'' = f''(x_0), \quad y_0''' = f'''(x_0); \ldots .$$

The various derivatives are obtained by repeated differentiation:

$$\frac{dx}{dy} = \varphi' = \frac{1}{f'} = \frac{1}{y};$$

$$\varphi'' = \frac{d}{dx} (\varphi'(y)) \frac{dx}{dy} = \frac{d}{dx} \left( \frac{1}{f'(x)} \right) \frac{dx}{dy} = \frac{-y''}{(y')^2} \frac{1}{y} = \frac{-y''}{(y')^3}; \ldots .$$

Recursively, this amounts to performing the following recurrence [3, p. 18], for $k = 1, 2, \ldots .$

$$\varphi^{(k)} = \frac{X_k}{(y')^{2k-1}}$$

$$X_{n+1} = (X_n)'y' - (2n - 1)X_n y''; \quad \text{with} \quad X_1 = 1, \quad X_2 = -y''.$$

So if $f(x)$ is nearly zero ($x$ is almost a root) whereas in an interval containing $x$, the derivative $f'(x)$ is not so 'small' ($|f'(x)| \geq m$), the existence of the inverse function is guaranteed and, in our case where $f(x)$ is a polynomial, also its higher derivatives. Thus, the following series expansion holds true ([3], pages 15-25):

$$h = k - \frac{f''}{2f'}k^2 + \frac{3(f''')^2 - 2f'f'''}{6(f')^2}k^3 + \frac{10f'f''f''' - (f')^2f'''}{24(f')^3}k^4$$

$$+ \frac{10(f''')^2(f')^2 - 105f'''f''(f')^2 + 105(f')^2}{120(f')^4}k^5$$

$$+ \frac{-280(f''')^2f'(f')^2 + 1260f'''(f')^3f' - 945(f')^5}{720(f')^5}k^6 + \ldots ,$$

\(9a\)
where \( f \) and its derivatives are calculated at point \( x \), with

\[
h = r - x \quad \text{(} r \text{ is a quasi-root)}; \quad k = -\frac{f}{f'}.
\] (9b)

For the function \( f(x) = x^3 - 3x + 2\alpha \), with \(|\alpha| \leq 1\), we shall determine expressions for the corrections \( h \) near the two points \( x = 0 \) and \( x = -2 \); we designate them by \( h_0 \) and \( h_{-2} \):

\[
h_0 = k\left[1 + \frac{1}{3}k^2 + \frac{1}{9}k^4 + \frac{4}{81}k^6 + \frac{55}{81}k^8\right] + \text{negl. terms}; \quad k = \frac{2}{3}
\] (10a)

\[
h_{-2} = k\left[1 + \frac{2}{3}k + \frac{7}{9}k^2 + \frac{10}{9}k^3 + \frac{143}{81}k^4 + \frac{728}{243}k^5\right] + \text{negl. terms}; \quad k = \frac{2}{9}[1 - \alpha]
\] (10b)

We notice that for \( h_0 \), the coefficients of the even powers of \( k \) turn out to be identically equal to zero and, in (9a), the resulting coefficients of \( k^7 \) and \( k^9 \) (used in (10a)), after cancellation and use of \( f''(x) = 0 \), result in:

\[
\frac{-280}{5040}(f''')^3k^7 = \frac{-5}{18}(f'')^3k^7 \quad \text{and} \quad \frac{154000}{9!(f')^4}k^9 = \frac{55}{1296}(f''')^4k^9.
\]

Equation (10a), as expected, is an odd function of \( \alpha \) and yields an explicit estimate of the intermediary root; on the other hand, equation (10b) gives an explicit estimate of the smaller root.

The reasoning leading to the choice goes as follows: When \( \alpha \) is near to 1, the intermediary root also approaches 1, but in this situation the derivative approaches 0. It is precisely to overcome this situation that we switch from the neighbourhood of 0 to the neighbourhood of \(-2\), as follows:

Practical choice criterion:

For \( 0 < \alpha \leq 0.35 \) we use equation (10a);

for \( \alpha > 0.35 \) we use equation (10b).

The estimate of \( r_2 \) will be denoted by \( \psi(\alpha) \) and will be made equal to \( h_0 \); the estimate of the smallest root \( r_1 \) will be denoted by \( \xi(\alpha) \) and will be made equal to \( h_{-2} - 2 \).

**Results of the preliminary estimates: quasi-solutions**

Extensive numerical calculations revealed the results obtained to be of extraordinary quality, with percentage errors (in absolute value) not exceeding 0.0003 %.

\[
\psi(\alpha) = h_0 \quad \rightarrow \quad |r_2 - \psi(\alpha)|/r_2 < 3 \times 10^{-6}, \quad \text{whenever} \quad 0 < \alpha \leq 0.35
\]

\[
\xi(\alpha) = h_{-2} - 2 \quad \rightarrow \quad |r_1 - \xi(\alpha)|/r_1 < 3 \times 10^{-6}, \quad \text{whenever} \quad 0.35 < \alpha < 1.
\]

We therefore call them *quasi-solutions*.

Having the estimate \( \xi(\alpha) \) for the smallest root, we can directly determine the estimate \( \psi(\alpha) \) of the intermediary root \( r_2 \), in accordance with
the formula given by equation (13b):

$$\psi = \frac{-\xi}{2} - \frac{\sqrt{12 - 3\xi^2}}{2}.$$ 

Adjusting the initial estimate: Ostrowsky [3]

Despite the excellent quality of the estimates given by $\psi(\alpha)$ (or $\xi(\alpha)$), higher precision may be securely achieved, by the well-known procedure:

$$u_0 = \psi(\alpha) \quad \text{(or } u_0 = \xi(\alpha))$$

$$u_1 = \theta(u_0) = u_0 - K(u_0) \quad (11)$$

where

$$K(w) = \frac{f(w)}{f'(w)} \left[ 1 - \frac{f(w)f''(w)}{f'(w)^2} \right]^{-1} = \frac{w^3 - 3w + 2\alpha}{3(w^2 - 1)} \left[ 1 - \frac{2w(w^3 - 3w + 2\alpha)}{3(w^2 - 1)} \right]^{-1}. \quad (12)$$

Then, $r_2 = u_1$ (or $r_1 = u_1$) will be the desired root.

This is precisely the extended Newton's method, which in the case of polynomials has a cubic rate of convergence [3], whenever we start sufficiently close to a solution.

If $\alpha$ is equal to 0 or 1, so is the intermediary root and nothing more remains to be done. For $0 < \alpha < 1$, the initial estimate satisfies the hypothesis of theorem 15.1 in [3], thus guaranteeing the convergence to the intermediary root located between 0 and 1. The cubic rate of convergence is guaranteed by theorem 15.2, also in [3].

Note 1: As mentioned above, the initial estimates given by $\psi(\alpha)$ produce errors less than 0.0003%. Applying the correction $\theta(u)$ (one iteration) we attained precision of at least $10^{-15}$ for all the roots calculated, with $\alpha$ assuming values between 0, step 0.05, up to 1.

Determination of the extreme roots

If a real root is known, the remaining two (real or imaginary) can be explicitly obtained. Indeed, by (7), if $r = r_2$ is a root, $f(r_2) = 0$, and the cubic in (5) gives

$$f(r_2 + d) = d \left[ 3(r_2^2 + \text{sign}(p)) + 3r_2d + d^2 \right]. \quad (13)$$

Evidently, $(r_2 + d)$ will be a non-trivial root of (11), for the two values $d_+$ and $d_-:

$$d_+ = \frac{-3r_2 + \sqrt{-\text{sign}(p)12 - 3r_2^2}}{2}, \quad (14)$$

$$d_- = \frac{-3r_2 - \sqrt{-\text{sign}(p)12 - 3r_2^2}}{2}. \quad (15)$$
Therefore,
\[ r_3 = r_2 + d_+ \quad \text{and} \quad r_1 = r_2 + d_- \quad (16) \]
and, for the three real roots case \((p < 0)\), there follows:

**Lemma 1:** If \( r_2 \) is an intermediary root of \( f(w) \), the extreme roots will be:
\[ r_3 = \frac{-r_2}{2} + \frac{\sqrt{12 - 3r_2^2}}{2} \quad \text{and} \quad r_1 = \frac{-r_2}{2} - \frac{\sqrt{12 - 3r_2^2}}{2}. \]

**Corollary**
When \( r_2 = 0 \), \[ r_1 = -\sqrt{3} \quad \text{and} \quad r_3 = \sqrt{3} \quad (\alpha = 0). \]
When \( r_2 = 1 \), \[ r_1 = -2 \quad \text{and} \quad r_3 = 1 \quad (\alpha = 1). \]
When \( r_2 = -1 \), \[ r_1 = -1 \quad \text{and} \quad r_3 = 2 \quad (\alpha = -1). \]

This can be verified by direct substitution in (5).

**Obtaining the roots of the original cubic**
By back substitution we get
\[ x_1 = r_1 x_m; \quad x_2 = r_2 x_m; \quad x_3 = r_3 x_m \quad (17) \]
\[ y_1 = x_1 - \frac{a}{3}; \quad y_2 = x_2 - \frac{a}{3}; \quad y_3 = x_3 - \frac{a}{3}; \quad (18) \]

**Note 2:** This procedure has shown precision of order \(10^{-15}\), for any cubic with \(|\alpha| < 1\). If a still higher precision is wanted, a second step of applying \(\theta\), in step c, \( r_2 = \theta(\theta(u_0)) \), will most surely suffice.

**Single real root case**
If \( p > 0 \) or \(\alpha^2 > 1\), the canonical form of (5)
\[ f(w) = w^3 + 3 \text{ sign}(p) w + 2\alpha \]
has one real root, that may be directly obtained by the Cardano/Tartaglia formulae [2] and [3], which in our case reduce to the following function of \(\alpha\):
\[ r_2 = \sqrt[3]{-\alpha + \sqrt{\alpha^2 + \text{sign}(p)}} + \sqrt[3]{-\alpha - \sqrt{\alpha^2 + \text{sign}(p)}} \quad (19) \]

**Examples**
1. Solve \( g(y) = y^3 - [3 \times 10^{-2}] y^2 + [2.4 \times 10^{-6}] = 0 \).
   With \( y = x - \frac{a}{3} \) and \( a = -3 \times 10^{-2} \), we obtain the Depressed Form
   \[ h(x) = x^3 - [3 \times 10^{-4}] x + [4 \times 10^{-7}] = 0. \]
   Then with \( a = \frac{4}{5} \left[ \frac{2}{3} a \right]^{3/2} = \frac{4 \times 10^{-7}}{[2 \times 10^{-6}]^{3/2}} = 0.2 \), we get the canonical form
   \[ f(w) = w^3 - 3w + 0.4 = 0 \quad (\alpha = 2). \]
Intermediary root estimate: \( \psi(0.2) = 0.134137170472 \); error 
\( r_2 - u_0 = 0.000000675233 \).

Test: \( f(\psi(0.2)) < 10^{-12} \). But, for a higher precision we make one correction: 
\( r_2 = \theta(u_0) = 0.134137845705 \).

By the formulae of Lemma 1: \( r_1 = -1.795219749245 \) and \( r_3 = 1.661081903541 \).

Back-substituting gives

\[
\begin{align*}
x_1 &= -0.01795219749245 \text{ so that } y_1 = -0.00795219749245 \\
x_2 &= 0.00134137845705 \text{ so that } y_2 = 0.01134137845705 \\
x_3 &= 0.01661081903541 \text{ so that } y_3 = 0.02661081903541.
\end{align*}
\]

2. Solve \( h(x) = x^3 - [3 \times 10^{-2}]x + [3 \times 10^{-5}] = 0 \).

We are already in the depressed form which leads to the canonical form: 
\( f(w) = w^3 - 3w + 0.03, \alpha = 0.015 \) whence \( u_0 = \psi(\alpha) = 0.010000333367 \).

Test: \( f(u_0) = 0.000000000000027 \).

If we accept \( u_0 \) as being the value of the intermediary root, the deviation from the exact value is 
\( r_2 - u_0 = 9 \times 10^{-15} \).

Thus, we assume \( r_2 = u_0 = 0.010000333367 \).

Since \( x_m = 0.1 \), the intermediary root of the original equation is 
\( x_2 = 0.0010000333367 \).

**Complementary notes:**

Note 3: In the case of three real roots, even disregarding \( \psi(\alpha) \), any initial estimate of \( r_2 \) satisfying \( 0 < r_2 < 1 \) for the case of \( \alpha > 0 \), or any initial estimate satisfying \( -1 < r_2 < 0 \) for the case \( \alpha < 0 \), will be sufficient to guarantee convergence of Newton's extended method, always to the intermediary root. However a few applications of \( \theta \) will be required to attain the same accuracy.

Note 4: For most practical purposes we may consider the abridged expressions of the roots

\[
\begin{align*}
r_2 &= \frac{2}{3} \alpha + \frac{8}{81} \alpha^3; \quad \text{if } \alpha \leq 0.3 \\
r_1 &= \frac{2}{9} [1 - \alpha] + \frac{8}{243} [1 - \alpha]^2 - 2; \quad \text{if } \alpha \geq 0.7.
\end{align*}
\]

Illustration:

\((x - 1)(x - 100)(x + 101) = 0\) i.e. \(x^3 - [10101]x + 10100 = 0\).

Canonical form: \(w^3 - 3w + 2[0.025847986] = 0\) gives \( \alpha = 0.025847986 \).
\[ x_m = \left[ \frac{10101}{3} \right]^{\frac{1}{3}} = 58.02585631 \]

\[ r_2 \approx \frac{2}{3} \alpha = 0.01723199 \Rightarrow x_2 = x_m r_2 = 0.999900976 \]
\[ \Rightarrow \text{error} = 1 - 0.999900976 \approx 0.000099. \]

References

EDGAR RECHTSCHAFFEN
Department of Computer Science, UNIFESO, Teresopolis, RJ, Brazil
e-mail: edgarxrecht@terra.com.br

92.36 A singular integral

In the March 2006 *Gazette*, Harry V. Smith [1] gives a numerical evaluation of

\[ J = \int_{-1}^{1} \frac{\sinh^2 \frac{x}{2}}{(1 - x^{0.005})(1 + x^{0.995})} \, dx \]

using rather heavy machinery in order to deal with the singularities of the integrand at the endpoints. Integration by parts reduces the problem to two simpler integrals without singularities. This makes the example accessible to school students.

More generally, let us consider a definite integral of the form

\[ I = \int_{-1}^{1} \frac{f(x)}{(1 - x)^\alpha (1 + x)^\beta} \, dx \]

where \( f(x) \) is a smooth function on \([-1, 1]\), \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \).