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ORE EXTENSIONS AND POISSON ALGEBRAS

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Abstract. For a derivation δ of a commutative Noetherian $\mathbb{C}$-algebra A, a homeomorphism is established between the prime spectrum of the Ore extension $A[z;\delta]$ and the Poisson prime spectrum of the polynomial algebra $A[z]$ endowed with the Poisson bracket such that $\{A, A\} = 0$ and $\{z, a\} = \delta(a)$ for all $a \in A$.

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1. Introduction. The best known example of a simple Poisson algebra is the coordinate ring of the symplectic plane, that is, the polynomial algebra $\mathbb{C}[z, x]$ with $\{z, x\} = 1$. This corresponds to the best known example of a simple Ore extension $A[z;\delta]$, namely the Weyl algebra $A_1(\mathbb{C})$, generated by x and z subject to the relation $zx - xz = 1$. Here $A = \mathbb{C}[x]$ and $\delta = d/dx$. The first known example of a Poisson bracket on $\mathbb{C}[x, y, z]$ for which $\mathbb{C}[x, y, z]$ is a simple Poisson algebra, due to Farkas [**6**, Example following Lemma 15], is such that $\{x, y\} = 0$ and the Hamiltonian $\{z, -\}$ acts on $\mathbb{C}[x, y]$ as the derivation $\delta = \partial_x + (1 - xy)\partial_y$, where ∂_x and ∂_y are the partial derivatives. In the first known example, due to Bergman, see [**3**], of a derivation δ for which the Ore extension $\mathbb{C}[x, y][z;\delta]$ is simple, the derivation δ is $\partial_x + (1 + xy)\partial_y$. The proofs of simplicity in both [**6**] and [**3**] remain valid for the common generalization where $\delta = \partial_x + (1 + \lambda xy)\partial_y$ for some $\lambda \in \mathbb{C}^*$, giving rise to corresponding families of simple Poisson algebras and simple Ore extensions. Unlike the case of the symplectic plane and the Weyl algebra, this correspondence does not appear to have been noted. These examples of simple Poisson algebras with corresponding simple Ore extensions are special cases of a general situation. Given any non-zero derivation δ of a commutative $\mathbb{C}$-algebra A, there is a Poisson bracket on the polynomial algebra $A[z]$ such that $\{A, A\} = 0$ and $\{z, a\} = \delta(a)$ for all $a \in A$. We shall show that if A is Noetherian then the Poisson prime spectrum of $A[z]$ is homeomorphic to the prime spectrum of $A[z;\delta]$. This fits into the philosophy of [**9**], in that $A[z]$ is the commutative fibre version of the semi-classical limit of the family of non-commutative algebras $R_\alpha := A[h][z; h\delta]/(h - \alpha)A[h][z; h\delta]$, where $\alpha \in \mathbb{C}^*$ and the derivation δ is extended to the polynomial algebra $A[h]$ by setting $\delta(h) = 0$. Note that $R_\alpha \simeq A[z; \alpha\delta]$.

In addition to Bergman's example, there are many known examples of simple derivations of $\mathbb{C}[x, y]$, for example see [**2**, **3**, **4**, **12**, **16**, **21**, **22**]. All such examples give rise to Poisson brackets for which $\mathbb{C}[x, y, z]$ is a simple Poisson algebra. In [**10**], Goodearl and Warfield illustrated their study of Krull dimension in Ore extensions with some non-simple Ore extensions of $\mathbb{C}[x, y]$ with interesting prime spectra. In the final section we shall transfer these and some other known examples to the Poisson setting and also answer a question from [**10**] on Ore extensions by constructing an accessible example of a derivation of $\mathbb{C}[x, y]$ giving rise to a Poisson bracket on $B := \mathbb{C}[x, y, z]$ for which the height two prime ideal $yB + zB$ is Poisson but no height one prime ideal is Poisson.

2. Background on Poisson algebras. Our base field will always be $\mathbb{C}$, although the results are valid over any field of characteristic 0. In Remark 3.7 algebraic closure is pertinent. We denote the prime spectrum of a not-necessarily commutative ring by Spec R.

DEFINITION 2.1. A *Poisson algebra* is $\mathbb{C}$-algebra A with a Poisson bracket, that is, a bilinear product $\{-, -\} : A \times A \to A$ such that A is a Lie algebra under $\{-, -\}$ and, for all $a \in A$, the *Hamiltonian* $\mathrm{ham}(a) := \{a, -\}$ is a $\mathbb{C}$-derivation of A.

The following definitions and claims made for them are well known. One comprehensive reference is [**8**, Lemma 1.1 and thereabouts].

DEFINITIONS 2.2. Let Δ be a set of derivations of a commutative $\mathbb{C}$-algebra A. The Δ-*centre*, $Z_\Delta(A)$, of A is $\{a \in A : \delta(a) = 0 \text{ for all } \delta \in \Delta\}$.

An ideal I of A is a Δ-*ideal* if $\delta(I) \subseteq I$ for all $\delta \in \Delta$ and a proper Δ-ideal P of A is Δ-*prime* if, for all Δ-ideals I and J of A, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. If $\Delta = \{\delta\}$ is a singleton then, in these and subsequent definitions, we replace Δ by δ rather than by $\{\delta\}$.

To say that A is Δ-*simple* means that 0 is the only proper Δ-ideal I of A. A derivation δ of A is said to be *simple* if A is δ-simple.

The Δ-*core* of an ideal I of A, denoted $(I : \Delta)$, is the largest Δ-ideal of A contained in I. If P is a prime ideal of A then $(P : \Delta)$ is prime, see [**8**, Lemma 1.1(a)].

If I is a Δ-ideal of A then each derivation $\delta \in \Delta$ induces a derivation $\overline{\delta}$ of A/I such that $\overline{\delta}(a + I) = \delta(a) + I$ for all $a \in A$. If I is a Δ-ideal and is also prime then $\overline{\delta}$ extends to the quotient field $Q(R/I)$ by the quotient rule $\overline{\delta}(as^{-1}) = (\overline{s}\overline{\delta}(\overline{a}) - \overline{a}\overline{\delta}(\overline{s}))\overline{s}^{-2}$.

A Δ-ideal P of A is Δ-*primitive* if $P = (M : \Delta)$ for some maximal ideal M of A. Every Δ-primitive ideal is Δ-prime.

If A is a Poisson algebra and $\Delta = \{\mathrm{ham}(b) : b \in A\}$ then we replace the prefix Δ- by the word Poisson. In particular, an ideal I of a Poisson algebra is a *Poisson ideal* if $\{i, a\} \in I$ for all $a \in A$ and $i \in I$ and A is a *simple Poisson algebra* if and only if the only Poisson ideals of A are 0 and A. The Poisson centre of A and the Poisson core of a Poisson ideal I of A will be denoted by $\mathrm{PZ}(A)$ and $\mathcal{P}(I)$ respectively.

LEMMA 2.3. *Let A be a commutative Noetherian $\mathbb{C}$-algebra, and let Δ be a set of derivations of A. If P is a Δ-prime ideal of A then P is prime.*

Proof. See [**8**, Lemma 1.1 (d)]. □

DEFINITIONS 2.4. Let Δ be a set of derivations of a commutative Noetherian $\mathbb{C}$-algebra A. The Δ-prime spectrum of A, denoted Δ-Spec(A), is the set of all Δ-prime ideals of A with the topology induced from the Zariski topology in Spec(A). The Poisson spectrum of A will be denoted by P. Spec(A). Thus, a closed set in P. Spec(A) has the form $V(I) := \{P \in \mathrm{P.\,Spec}(A) : P \supseteq I\}$ for some ideal I of A. As is observed in [**9**, Section 6.1], replacing I by the Poisson ideal that it generates, I can be assumed to be a Poisson ideal.

DEFINITION 2.5. Let A be a Poisson algebra and I be a Poisson ideal of A. If the induced Poisson bracket on A/I is zero, we say that I is *residually null*. This is

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equivalent to saying that I contains all elements of the form $\{a, b\}$ where $a, b \in A$, or that I contains all such elements where $a, b \in G$ for some generating set G for A. The set of residually null Poisson prime ideals of A is clearly closed in P. Spec(A).

DEFINITIONS 2.6. By a *Poisson maximal ideal* we mean a maximal ideal that is also Poisson whereas by a *maximal Poisson ideal* we mean a Poisson ideal that is maximal in the lattice of Poisson ideals. These notions are not equivalent. Any Poisson maximal ideal is maximal Poisson but the converse is false as can be seen by considering the ideal 0 in any simple Poisson algebra that is not simple as an associative algebra, such as $\mathbb{C}[y, z]$ with $\{y, z\} = 1$.

DEFINITIONS 2.7. A G-*domain* is a commutative integral domain R such that the intersection of the non-zero prime ideals is non-zero, in other words 0 is locally closed in Spec R. See [**20**, Theorems 19 and 20 and the intermediate text]. With A and Δ as in Definitions 2.2, let P be a Δ-prime ideal of A. We shall say that P is Δ-G if it is locally closed in Δ-Spec(A). To say that A is Δ-G means that 0 is a Δ-G ideal of A.

If P is a Δ-ideal and prime, in particular if A is Noetherian and P is Δ-prime, we say that P is Δ-*rational* if $Z_{\overline{\Delta}}(Q(A/P)) = \mathbb{C}$, where $\overline{\Delta}$ is the set of derivations of the quotient field $Q(A/P)$ induced, via R/P, by derivations belonging to Δ.

3. Semi-classical limits of Ore extensions. Let A denote a commutative $\mathbb{C}$-algebra that is also a domain and let D be the polynomial algebra $A[h]$. Let δ be a derivation of A and extend δ to D by setting $\delta(h) = 0$. Then $h\delta$ is a derivation of D and we can form the Ore extension (or skew polynomial ring or ring of formal differential operators) $T := D[z; h\delta]$ in which elements have the form $\sum_0^n d_i z^i$, $d_i \in D$ and $zd - dz = h\delta(d)$ for all $d \in D$. Note that $hz = zh$ and h is a central non-unit regular element of T such that T/hT is isomorphic to the commutative polynomial algebra $B := A[z]$ and $T/(h-1)T$ is isomorphic to the Ore extension $R := A[z; \delta]$. If $\alpha \in \mathbb{C}^*$, then $T/(h-\alpha)T \simeq A[z; \alpha\delta] \simeq A[z; \delta]$, where the final isomorphism maps z to $\alpha^{-1}z$. In this situation, there is a well-defined Poisson bracket on B such that

$$\{\overline{u}, \overline{v}\} = \overline{h^{-1}[u, v]}$$

for all $\overline{u} = u + hT$ and $\overline{v} = v + hT \in B$. With this bracket, B is the *semi-classical limit* of the family $A[z; \alpha\delta]$, $\alpha \in \mathbb{C}^*$, as in [**9**, 2.1], T is a *quantization* of the Poisson algebra B in the sense of [**1**, Chap. III.5] and R is a *deformation* of B in the sense of [**19**]. A familiar example is obtained by taking $A = \mathbb{C}[x]$ and $\delta = d/dx$. Here R is the Weyl algebra $A_1(\mathbb{C})$, with generators x and z subject to the relation $zx - xz = 1$, and the semi-classical limit B is $\mathbb{C}[x, z]$ with $\{z, x\} = 1$, that is the coordinate ring of the symplectic plane.

To emphasise the role of the single derivation δ, the Poisson bracket on B will sometimes be written $\{-, -\}_\delta$. Thus, $\{a, b\}_\delta = 0$ and $\{z, a\}_\delta = \delta(a)$ for all $a, b \in A$. In the terminology of [**23**], B is a Poisson polynomial ring over A for which the Poisson bracket on A and the derivation α are both zero.

LEMMA 3.1. *Let A be a commutative $\mathbb{C}$-algebra with a derivation δ and let $B = A[z]$ equipped with the Poisson bracket $\{-, -\}_\delta$.*

(i) *For all $a, b \in A$ and all $m, n \in \mathbb{N}$, $\{az^m, bz^n\} = (ma\delta(b) - nb\delta(a))z^{m+n-1}$.*

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(ii) *Let Q be a δ-ideal of A. Then QB is a Poisson ideal of B and there is an isomorphism of Poisson algebras, $\theta_Q : B/QB \to (A/Q)[z]$ given by*

$$\theta_Q\left(\left(\sum_{i=0}^{n} a_i z^i\right) + QB\right) = \sum_{i=0}^{n} (a_i + Q)z^i,$$

where the Poisson bracket on $A/Q[z]$ is $\{-,-\}_{\overline{\delta}}$.

Proof. (i) is routine using the fact that the Hamiltonians are derivations. The first statement in (ii) is immediate from (i) and the second statement is straightforward. □

LEMMA 3.2. *Let A be a commutative Noetherian $\mathbb{C}$-algebra that is also a domain, let δ be a non-zero derivation of A and let P be a non-zero Poisson prime ideal of $B := A[z]$ for the Poisson bracket $\{-,-\}_\delta$. Let $Q = P \cap A$.*

(i) *Q is a non-zero δ-prime ideal of A.*

(ii) *If $\delta(A) \not\subseteq Q$ then $P = QB$.*

Proof. (i) Let $p = \sum_{i=0}^{n} a_i z^i$, with each $a_i \in A$, be an element of minimal degree n in z among non-zero elements of P. Let $a \in A$ be such that $\delta(a) \neq 0$. Then $(\text{ham}\, a)(p) = -\sum_{i=0}^{n} i\delta(a)a_i z^{i-1} \in P$. This contradicts the minimality of n unless $n = 0$. Thus, $n = 0$ and $Q \neq 0$.

As P is a Poisson ideal of B, Q is a δ-ideal of A. Let I and J be δ-ideals of A such that $IJ \subseteq Q$. By Lemma 3.1(ii), IB and JB are Poisson ideals of B. Clearly, $IBJB \subseteq P$ so either $IB \subseteq P$, whence $I \subseteq P \cap A$, or $JB \subseteq P$, whence $J \subseteq P \cap A$. Thus, $P \cap A$ is δ-prime.

(ii) By Lemma 2.3, A/Q is a domain. Suppose $\delta(A) \not\subseteq Q$, so that the induced Poisson bracket on the domain A/Q is non-zero. Clearly, $QB \subseteq P$. If $QB \neq P$ then $\theta_Q(P/BQ)$ is a non-zero Poisson ideal of $(A/Q)[z]$ intersecting A/Q in 0. This is impossible by (i) applied to A/Q, so $QB = P$. □

The situation is similar for the prime spectrum of the Ore extension $R = A[z; \delta]$. Let A be a commutative $\mathbb{C}$-algebra with a derivation δ, and let Q be a δ-ideal of A. By [**7**, Section 1, final paragraph], QR is an ideal of R and there is an isomorphism $\psi_Q : R/QR \to A/Q[z; \overline{\delta}]$ given by

$$\psi_Q\left(\left(\sum_{i=0}^{n} a_i z^i\right) + QR\right) = \sum_{i=0}^{n} (a_i + Q)z^i.$$

LEMMA 3.3. *Let A be a commutative $\mathbb{C}$-algebra that is also a domain, let δ be a non-zero derivation of A and let P be a non-zero prime ideal of $R := A[z; \delta]$. Let $Q = P \cap A$.*

(i) *Q is a non-zero δ-prime ideal of A.*

(ii) *If $\delta(A) \not\subseteq Q$ then $P = QR$.*

Proof. (i) By [**14**, Lemma 1.3], Q is δ-prime and, by [**15**, Lemma 1], $Q \neq 0$.

(ii) By Lemma 2.3 with $\Delta = \{\delta\}$, A/Q is a domain. Suppose that $\delta(A) \not\subseteq Q$ so that the induced derivation $\overline{\delta}$ on the domain A/Q is non-zero. The ideal QR is prime by [**14**, Lemma 1.3] and $QR \subseteq P$. If $QB \neq P$ then $\psi_Q(P/RQ)$ is a non-zero ideal of $(A/Q)[z; \overline{\delta}]$ intersecting A/Q in 0. This is impossible by (i) applied to A/Q, so $QR = P$. □

COROLLARY 3.4. *Let A be a commutative $\mathbb{C}$-algebra that is a domain and let δ be a non-zero derivation of A. Let $R = A[z; \delta]$ and let B be the Poisson algebra $A[z]$ with the*

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Poisson bracket $\{-,-\}_\delta$. *Then* B *is Poisson simple if and only if* R *is simple if and only if* δ *is simple.*

Proof. It follows from Lemmas 3.2 and 3.3 that if δ is simple then B is Poisson simple and R is simple. On the other hand, if J is a non-zero δ-ideal of A then JR is a non-zero proper ideal of R, by [**14**, Lemma 1.3], and JB is a non-zero proper Poisson ideal of B by Lemma 3.1(ii). $\square$

We now aim to generalize Corollary 3.4 to establish a homeomorphism between Spec R and P. Spec B. On each side, we shall partition the spectrum into two types of prime ideal.

NOTATION 3.5. Let A be a commutative $\mathbb{C}$-algebra and domain with a non-zero derivation δ, let $R = A[z;\delta]$ and let $B = A[z]$ equipped with the Poisson bracket $\{-,-\}_\delta$. Let $J = \delta(A)A$, which is a δ-ideal of A, and let $S = (A/J)[z]$. Then

(i) $JB = \{B, B\}B$ is a residually null Poisson ideal of B and is contained in all residually null Poisson ideals of B.

(ii) $\theta_J : B/JB \to S$ is an isomorphism of $\mathbb{C}$-algebras. The Poisson brackets are both 0.

(iii) JR is an ideal of R such that R/JR is commutative and JR is contained in all ideals I of R such that R/I is commutative.

(iv) $\psi_J : R/JR \to S$ is an isomorphism of commutative $\mathbb{C}$-algebras. The induced derivation $\overline{\delta}$ on A/J is 0.

Let

$$\mathrm{P.Spec}_1(B) = \{P \in \mathrm{P.Spec}\, B : P \text{ is residually null}\} = \{P \in \mathrm{P.Spec}\, B : JB \subseteq P\}$$

and let $\mathrm{P.Spec}_2(B) = \mathrm{P.Spec}\, B \setminus \mathrm{P.Spec}_1(B)$. By analogy, let

$$\mathrm{Spec}_1(R) = \{P \in \mathrm{Spec}\, R : R/P \text{ is commutative}\} = \{P \in \mathrm{Spec}\, R : JR \subseteq P\}$$

and let $\mathrm{Spec}_2(R) = \mathrm{Spec}\, R \setminus \mathrm{Spec}_1(R)$. Note that $\mathrm{P.Spec}_1(B)$ and $\mathrm{Spec}_1(R)$ are closed in P. Spec B and Spec R respectively. Also, $\mathrm{P.Spec}_1(B)$ is homeomorphic to $\mathrm{Spec}(B/JB)$ and $\mathrm{Spec}_1(R)$ is homeomorphic to $\mathrm{Spec}(R/JR)$.

Let κ be the isomorphism $\psi_J^{-1}\theta_J : B/JB \to R/JR$. Thus,

$$\kappa\left(\left(\sum_{i=0}^{n} a_i z^i\right) + JB\right) = \left(\sum_{i=0}^{n} a_i z^i\right) + JR.$$

Then κ induces a homeomorphism between $\mathrm{Spec}(R/JR)$ and $\mathrm{Spec}(B/JB)$ and there is a homeomorphism $\Gamma_1 : \mathrm{P.Spec}_1(B) \to \mathrm{Spec}_1(R)$ such that $\Gamma_1(P)/JR = \kappa(P/JB)$ for all $P \in \mathrm{P.Spec}_1(B)$.

THEOREM 3.6. *Let* A *be a Noetherian* $\mathbb{C}$*-algebra that is a domain, and let* δ *be a non-zero derivation of* A. *Let* $R = A[z;\delta]$, *and let* B *be the Poisson algebra* $A[z]$ *with the Poisson bracket* $\{-,-\}_\delta$. *There is a homeomorphism* Γ *between* Spec R *and* P. Spec B.

Proof. We have seen in 3.5 that there is a homeomorphism $\Gamma_1 : \mathrm{P.Spec}_1(B) \to \mathrm{Spec}_1(R)$ such that $\Gamma_1(P)/JR = \kappa(P/JB)$ for all $P \in \mathrm{P.Spec}_1(B)$. We aim to extend this to a homeomorphism $\Gamma : \mathrm{P.Spec}(B) \to \mathrm{Spec}(R)$.

By Lemma 3.2, every element of $\mathrm{P.Spec}_2 B$ has the form QB for a δ-prime ideal Q of A such that $J \not\subseteq Q$ and, by Lemma 3.3, every element of $\mathrm{Spec}_2 R$ has the form QR

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for such an ideal Q. Define $\Gamma_2 : \mathrm{P.Spec}_2 B \to \mathrm{Spec}_2 R$ by setting $\Gamma_2(QB) = QR$. Then Γ_2 is bijective and Γ_2 and Γ_2^{-1} preserve inclusions. Combine Γ_1 and Γ_2 by defining a bijection $\Gamma : \mathrm{P.Spec}\, B \to \mathrm{Spec}\, R$ by

$$\Gamma(P) = \begin{cases} \Gamma_1(P) \text{ if } P \in \mathrm{P.Spec}_1(B) \\ \Gamma_2(P) \text{ if } P \in \mathrm{P.Spec}_2(B) \end{cases} .$$

Inclusions within $\mathrm{P.Spec}_1 B$ and $\mathrm{Spec}_1 R$ and within $\mathrm{P.Spec}_2 B$ and $\mathrm{Spec}_2 R$ are preserved by Γ and Γ^{-1}. There are no inclusions $P' \subseteq P$ with $P' \in \mathrm{P.Spec}_1 B$ and $P \in \mathrm{P.Spec}_2 B$ or with $P' \in \mathrm{Spec}_1 R$ and $P \in \mathrm{Spec}_2 R$. Let $P' = QB \in \mathrm{P.Spec}_2 B$ and $P \in \mathrm{P.Spec}_1 B$. Then

$$\begin{aligned} QB \subseteq P &\Leftrightarrow \frac{QB + JB}{JB} \subseteq \frac{P}{JB} \\ &\Leftrightarrow \kappa\left(\frac{QB + JB}{JB}\right) \subseteq \kappa\left(\frac{P}{JB}\right) = \frac{\Gamma(P)}{JR} \\ &\Leftrightarrow \frac{QR + JR}{JR} \subseteq \frac{\Gamma(P)}{JR} \\ &\Leftrightarrow \Gamma(QB) = QR \subseteq \Gamma(P). \end{aligned}$$

Thus, both Γ and Γ^{-1} preserve inclusions. By [**9**, Lemma 9.4], Γ is a homeomorphism. □

REMARK 3.7. For many affine algebras, particularly enveloping algebras and quantum algebras, there are prime ideals that are not completely prime and there is an established homeomorphism between the completely prime part of the spectrum of a deformation and the Poisson prime spectrum of a corresponding semi-classical limit. Some such algebras are discussed in [**17**], where a common theme is that the incompletely prime ideals are annihilators of finite-dimensional simple modules of dimension $d > 1$ and it is such a module, rather than its annihilator, that is reflected on the Poisson side through a d-dimensional simple Poisson module. In the context of this paper, this issue is not present on either side. On the Ore side, Sigurdsson [**24**] shows that all prime ideals of $A[z;\delta]$ are completely prime. On the Poisson side, by [**17**, Theorem 1], a d-dimensional simple Poisson module over an affine Poisson algebra corresponds to a d-dimensional simple Lie module for the Lie algebra M/M^2 for some maximal Poisson ideal M. In the context of the present paper, M/M^2 is always soluble and, by [**5**, Corollary 1.3.13], every finite-dimensional simple Lie module for M/M^2 has dimension one.

4. Primitivity. The purpose of this section is to show that, for a commutative affine domain A with derivation δ, the Ore extension $A[z;\delta]$ is primitive if and only if $A[z]$ is Poisson primitive, for the Poisson bracket $\{-,-\}_\delta$, and that, under the homeomorphism Γ of Theorem 3.6, Poisson primitive ideals of $A[z]$ correspond to primitive ideals of $A[z;\delta]$.

It follows from [**15**, Theorems 1,2], where A is not necessarily affine, that if $\delta \neq 0$ and A is either δ-primitive or δ-G then $A[z;\delta]$ is primitive. The converse in the Noetherian case was established in [**11**, Theorem 3.7]. The logical independence, in the general case, of the two conditions, δ-primitive and δ-G, was shown in [**15**] by means of the non-affine examples $A = \mathbb{C}[[y]]$ with $\delta = y\,d/dy$, which is δ-G but not δ-primitive,

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and $A = \mathbb{C}(t)[y]$ with $\delta = t\partial/\partial t + y\partial/\partial y$, which is δ-primitive but not δ-G. If A is affine and δ-G, then A is δ-primitive by [**8**, Proposition 1.2]. It would be interesting to know whether there is an affine δ-primitive $\mathbb{C}$-algebra A which is not δ-G. Such an example would have consequences for the Poisson Dixmier–Moeglin equivalence, as it would give rise to a Poisson bracket on $A[z]$ for which 0 is Poisson primitive, and hence Poisson rational, but not locally closed.

In the Poisson setting we have analogues of [**15**, Theorems 1,2].

THEOREM 4.1. *Let δ be a non-zero derivation of a commutative $\mathbb{C}$-algebra A. Then $A[z]$, with the Poisson bracket $\{-,-\}_\delta$, is Poisson primitive if A is δ-primitive or δ-G.*

Proof. Suppose that A is δ-G and let $I \neq 0$ be the intersection of the non-zero δ-prime ideals of A. As A is a domain, the Jacobson radical $\mathrm{Jac}(A[z]) = 0$, for example by [**13**, Theorem 4]. Suppose that $A[z]$ is not Poisson primitive and let M be a maximal ideal of $A[z]$. Then $\mathcal{P}(M) \neq 0$ and, by Lemma 3.2(i), $\mathcal{P}(M) \cap A$ is a non-zero δ-prime ideal of A. Therefore, $I \subseteq M$ for all maximal ideals M of $A[z]$, so $I \subseteq \mathrm{Jac}(A[z]) = 0$. This contradiction shows that $A[z]$ is Poisson primitive.

Suppose that A is δ-primitive and let M be a maximal ideal of A containing no non-zero δ-ideal of A. Let N be any maximal ideal of $A[z]$ containing M. Thus, $N \cap A = M$. Let $P = \mathcal{P}(N)$. Then $P = 0$ for otherwise, by Lemma 3.2, $P \cap A$ is a non-zero δ-ideal of A contained in M. Thus, $A[z]$ is Poisson primitive. $\square$

It would be interesting to know whether the converse is true in the Noetherian case. As the next result shows, it is true in the affine case.

THEOREM 4.2. *Let δ be a non-zero derivation of a commutative affine $\mathbb{C}$-algebra A. Then $A[z]$ is Poisson primitive if and only if A is δ-primitive.*

Proof. Suppose that $A[z]$ is Poisson primitive and let M be a maximal ideal of $A[z]$ containing no non-zero Poisson ideal of $A[z]$. Then $M \cap A$ contains no non-zero δ-ideal of A for if J is a non-zero δ-ideal of A contained in $M \cap A$, then by Lemma 3.1(ii) $JA[z]$ is a non-zero Poisson ideal of $A[z]$ contained in M. But, by [**20**, Theorem 27], $A/(M \cap A)$ is a G-domain. As A is affine, it is a Hilbert ring, by [**20**, Theorem 31], so $M \cap A$ is a maximal ideal of A. Thus, A is δ-primitive. The converse holds by Theorem 4.1. $\square$

COROLLARY 4.3. *Let δ be a non-zero derivation of a commutative affine $\mathbb{C}$-algebra A. Then $A[z]$, with the Poisson bracket $\{-,-\}_\delta$, is Poisson primitive if and only if $A[z;\delta]$ is primitive.*

Proof. As we observed above, [**8**, Proposition 1.2] tells us that, in the affine case, if A is δ-primitive then A is δ-G. The result is then immediate from Theorem 4.2 and [**11**, Theorem 3.7]. $\square$

COROLLARY 4.4. *Let δ be a non-zero derivation of a commutative affine $\mathbb{C}$-algebra A, let $B = A[z]$ with the Poisson bracket $\{-,-\}_\delta$ and let $R = A[z;\delta]$.*

(i) *In* $\mathrm{P.Spec}(B)$*, the Poisson primitive ideals are the maximal elements of* $\mathrm{P.Spec}_1\, B$*, that is, the Poisson ideals P of B such that $B/P \simeq \mathbb{C}$, and the ideals of the form QB, where Q is a δ-primitive ideal of A.*

(ii) *In* $\mathrm{Spec}(R)$*, the primitive ideals are the maximal elements of* $\mathrm{Spec}_1\, R$*, that is, the ideals P of R such that $R/P \simeq \mathbb{C}$, and the ideals of the form QR, where Q is a δ-primitive ideal of A.*

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(iii) *In the homeomorphism* Γ *between* $\mathrm{P.Spec}(B)$ *and* $\mathrm{Spec}(R)$ *in Theorem 3.6, the Poisson primitive ideals of* B *correspond to the primitive ideals of* R.

Proof. (i) Let P be a Poisson prime ideal of B. First suppose that $P \in \mathrm{P.Spec}_1 B$. Then the Poisson bracket on B/P is 0, so P is Poisson primitive if and only if P is maximal if and only if $B/P \simeq \mathbb{C}$. Now suppose that $P \in \mathrm{P.Spec}_2 B$. Then $P = QB$ for some δ-prime ideal Q of A with $\delta(A) \not\subseteq Q$ and, by Theorem 4.2 applied to A/Q, P is Poisson primitive if and only if Q is δ-primitive.

(ii) Let P be a prime ideal of R. First suppose that $P \in \mathrm{Spec}_1 R$. Then R/P is commutative, so P is primitive if and only if P is maximal if and only if $R/P \simeq \mathbb{C}$. Now suppose that $P \in \mathrm{Spec}_2 R$. Then $P = QR$ for some δ-prime ideal Q of A with $\delta(A) \not\subseteq Q$ and, by Theorem 4.2 and Corollary 4.3 applied to A/Q, P is primitive if and only if Q is δ-primitive.

(iii) This follows from (i) and (ii). $\square$

5. Examples in $\mathbb{C}[x, y, z]$**.** Here we look at some examples where $A = \mathbb{C}[x, y]$ so that $B = \mathbb{C}[x, y, z]$, the polynomial algebra in three indeterminates. For $w = x, y$ or z, we denote the derivation $\partial/\partial w$ of B by ∂_w and, for $a \in B$, we write a_w for $\partial_w(a)$ and $\mathrm{grad}\, a$ for the triple $(a_x, a_y, a_z) \in B^3$. Poisson brackets on $\mathbb{C}[x, y, z]$ are the subject of [**18**]. Any such bracket is determined by the triple $(f, g, h) \in B^3$ such that

$$\{y, z\} = f, \quad \{z, x\} = g \text{ and } \{x, y\} = h.$$

A triple $F = (f, g, h) \in B^3$ is a *Poisson triple* if it does determine a Poisson bracket in this way. By [**18**, Proposition 1.17(1)], a triple $F = (f, g, h) \in B^3$ is a Poisson triple if and only if $F.\mathrm{curl}\, F = 0$. Similar results are true for the rational function field $Q(B) = \mathbb{C}(x, y, z)$ and the completion $\widehat{B}$ of B at any maximal ideal.

For any $a, b \in B$, there is a Poisson bracket on B such that

$$\{y, z\} = ba_x, \quad \{z, x\} = ba_y \text{ and } \{x, y\} = ba_z.$$

We call such a bracket *exact* if $b = 1$ and *m-exact* in general. A Poisson bracket on B is *qm-exact*, respectively *cm-exact*, if there exist $a, b \in Q(B)$, respectively $a, b \in \widehat{B}$, for some maximal ideal of B, such that

$$\{y, z\} = ba_x \in B, \quad \{z, x\} = ba_y \in B \text{ and } \{x, y\} = ba_z \in B.$$

It is shown in [**18**] that every Poisson bracket on B is cm-exact and the Poisson spectrum is determined for a qm-exact bracket with $a = st^{-1}$ and $b = t^2$, s and t being co-prime elements of B. Taking $t = 1$, this includes the exact brackets.

In the remainder of this section, we consider non-exact Poisson brackets on $B = \mathbb{C}[x, y, z]$ that extend the zero Poisson bracket on $A = \mathbb{C}[x, y]$, that is, we consider Poisson brackets on B with $\{x, y\} = 0$.

LEMMA 5.1. *Let* $f, g \in B$, *and let* $F = (f, g, 0)$. *Then* F *is a Poisson triple if and only if there exist* $h \in B$ *and* $f_1, g_1 \in A$ *such that* $f = hf_1$ *and* $g = hg_1$.

Proof. Suppose that F is a Poisson triple. If $g = 0$ we can take $h = f$, $f_1 = 1$ and $g_1 = 0$ so we may assume that $g \neq 0$. As $\mathrm{curl}((f, g, 0)) = (-g_z, f_z, g_x - f_y)$, we have $fg_z = gf_z$. Hence, $\partial_z(f/g) = 0$ and $pf = qg$ for some $p, q \in A$. Let h be the highest common factor of f and g in B, and let $f_1, g_1 \in B$ be such that $f = hf_1$ and $g = hg_1$.

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Then $pf_1 = qg_1$. If $f_1 \notin A$ then f_1 has an irreducible factor u in $B\backslash A$ and, as $q \in A$, u must divide g_1, contradicting the choice of h. Thus, $f_1 \in A$ and similarly $g_1 \in A$.

Conversely, suppose that $F = (hf_1, hg_1, 0)$ where $h \in B$ and $f_1, g_1 \in A$. Then curl F has the form $(-g_1 h_z, f_1 h_z, \ell)$, where $\ell \in B$, so $F.\operatorname{curl} F = 0$, and hence F is a Poisson triple. □

The Poisson prime ideals of B for a Poisson triple $F = (hf_1, hg_1, 0)$ are the prime ideals containing h and the Poisson primes for the Poisson triple $(f_1, g_1, 0)$, so it suffices to consider the case where $f, g \in A$. Thus, $\operatorname{ham} x = -g\partial_z$, $\operatorname{ham} y = f\partial_z$ and $\operatorname{ham} z = g\partial_x - f\partial_y$. Also, $\operatorname{ham} z(A) \subseteq A$ so that $\operatorname{ham} z$ restricts to a derivation of A and the results of Section 3 apply with δ being the restriction to A of $g\partial_x - f\partial_y$.

If $a \in A$ then the corresponding exact bracket on B has $\{y, z\} = a_x$, $\{z, x\} = a_y$ and $\{x, y\} = 0$. The following theorem is a special case of [**18**, Theorem 3.8].

THEOREM 5.2. *Let $a \in A\backslash\{0\}$. The Poisson prime ideals of B under the exact bracket determined by a are* 0*, the residually null Poisson prime ideals and the height one prime ideals uA, where u is an irreducible factor of $a - \lambda$ for some $\lambda \in \mathbb{C}$ such that $a - \lambda$ is a non-zero non-unit.*

Combining this with Theorem 3.6 and its proof, we obtain the following corollary.

COROLLARY 5.3. *Let $a \in A\backslash\{0\}$, let δ be the derivation of A such that $\delta(x) = a_y$ and $\delta(y) = -a_x$, and let $R = A[x; \delta]$. Let $J = a_yA + a_xA$. Then*

(i) *JR is an ideal of R and $R/JR \simeq (A/J)[z]$.*

(ii) *The prime ideals of R under the exact bracket determined by a are* 0*, the height one prime ideals uR, where u is an irreducible factor of $a - \lambda$ for some $\lambda \in \mathbb{C}$ such that $a - \lambda$ is a non-zero non-unit and the prime ideals of the form $\pi^{-1}(Q)$, where Q is a prime ideal of $(A/J)[z]$ and π is the composition $R \twoheadrightarrow R/JR \simeq (A/J)[z]$.*

EXAMPLE 5.4. Let $a = x^2 + y^2$. Then $\{z, x\} = 2y$, $\{y, z\} = 2x$ and $\{x, y\} = 0$. The residually null Poisson prime ideals of B are $xB + yB$ and the maximal ideals that contain it. The other Poisson prime ideals of B are 0, the height one prime ideals $(x + iy)A$, $(x - iy)A$ and $(x^2 + y^2 - \lambda)A$, where $\lambda \in \mathbb{C}^*$. Note that those of the form $(x^2 + y^2 - \lambda)A$ are maximal Poisson ideals.

If $\delta = 2y\partial_x - 2x\partial_y$ so that $\delta(x) = 2y$ and $\delta(y) = -2x$ and $R = A[z; \delta]$ then the prime spectrum of R consists of 0, the height one prime ideals $(x + iy)R$, $(x - iy)R$ and $(x^2 + y^2 - \lambda)R$, where $\lambda \in \mathbb{C}^*$, $xR + yR$ and $xR + yR + (z - \mu)R$, where $\mu \in \mathbb{C}$. For each $\lambda \in \mathbb{C}^*$, the algebra $R/(x^2 + y^2 - \lambda)R$ is simple.

In the remainder of the paper we consider non-exact Poisson brackets on B, beginning with some for which B is Poisson simple. The following result of Shamsuddin, for which a proof may be found in [**2**, Proposition 3.2], is useful in identifying Poisson brackets for which B, or a localization of B, is Poisson simple.

PROPOSITION 5.5. *Let C be a commutative domain and let $g = at + b \in C[t]$, where $a, b \in C$. Suppose that there exists a derivation δ of $C[t]$ such that $\delta(C) \subseteq C$, C is $\delta|_C$-simple, $\delta(t) = g$ and, for all $r \in C$, $\delta(r) \neq ar + b$. Then $C[t]$ is δ-simple.*

EXAMPLES 5.6. In the case where $A = \mathbb{C}[x, y]$ and $B = \mathbb{C}[x, y, z]$, there are many known examples of simple derivations $\delta = g\partial_x - f\partial_y$ of A. For all of these, B is Poisson simple for the Poisson bracket determined by the triple $(f, g, 0)$. In many of these

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examples $g = 1$ so that

$$\{x, y\} = 0, \quad \{z, x\} = 1 \text{ and } \{y, z\} = f. \tag{5.1}$$

In the best known example which is due to, but not published by, Bergman and is documented in [**3**, Section 6], $f = -(1 + xy)$. The simplicity of δ follows easily from Proposition 5.5, with $C = \mathbb{C}[x]$ and $t = y$ and the same argument works for $f = -(1 + \lambda xy)$, $\lambda \in \mathbb{C}^*$. When $\lambda = -1$ and x, y and z are written $-x_1, x_3$ and x_2 respectively, this gives the first published example, due to Farkas [**6**, Example following Lemma 15], of a Poisson bracket on B for which B is Poisson simple.

Other examples of polynomials $f \in A$ for which B is Poisson simple under the Poisson bracket in (5.1) include:

(i) $f = p(x) - y^2$, where $p(x) \in \mathbb{C}[x]$ has odd degree. See [**21**, Theorem 6.2].

(ii) $f = -(y^m + ax^n)$, where $m, n \in \mathbb{N}$, $m \geq 2$ and $a \in \mathbb{C}\backslash\{0\}$. See [**12**, Theorem 1], which generalised an earlier result [**22**] for the case $n = 1$.

Example 5.7. In contrast to the examples in Examples 5.6, B is also Poisson simple for the Poisson bracket such that

$$\{x, y\} = 0, \quad \{z, x\} = y^3 \text{ and } \{y, z\} = xy - 1,$$

which has the property that, for all $b \in B$, $\{z, b\}$, $\{x, b\}$ and $\{y, b\}$ are not units. Clearly, $\{x, b\} = -y^3\partial_z(b)$ and $\{y, b\} = (xy - 1)\partial_z(b)$ are never units. For $\{z, (\sum a_i z^i)\} = \sum\{z, a_i\}z^i$ to be a unit, it is necessary that $\{z, a_0\}$ is a unit, and it is shown in [**16**] that if $a \in A$ then $\{z, a\} = \delta(a)$ is not a unit.

Remark 5.8. The examples in 5.7 and 5.6(i) have analogues in the polynomial algebra $\mathbb{C}[x_1, x_2, \ldots, x_n]$ when $n > 3$. In these $\{x_i, x_j\} = 0$ for $1 \leq i, j \leq n - 1$ and ham z is a simple derivation of $\mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$. See [**16**, Section 3] and [**21**, Section 9] for details from the point of view of Ore extensions.

Example 5.9. Coutinho [**3, 4**] has used the theory of foliations to make a substantial contribution to the understanding of the simple derivations of $\mathbb{C}[x, y]$. Let A_2 be the subspace of $\mathbb{C}[x, y]$ consisting of polynomials of total degree at most 2 and let $\mathcal{U}_2$ be the set of unimodular rows (a, b), where $a, b \in A_2 \times A_2$ are such that at least one of a and b has total degree 2. It is shown in [**4**] that the closure $\overline{\mathcal{U}_2}$ in $A_2 \times A_2$ has four irreducible components $\mathcal{P}_i$, $1 \leq i \leq 4$ and examples of simple derivations from each component are given. For the first two types, the class of examples is dense in $\mathcal{P}_i$. Below we give the details of examples of four corresponding types of Poisson bracket on B for which B is Poisson simple. Full details, presented from the point of view of derivations of $\mathbb{C}[x, y]$, can be found in [**4**].

Type 1, $\mathcal{P}_1$: Let $a, b, c \in \mathbb{Q}[i]\backslash 0$, with $a \neq 1$ be such that the quadratic polynomial $y^2 + bx^2 + cxy$ is irreducible over $\mathbb{Q}[i]$. Then, by [**4**, Proposition 4.1] and Corollary 3.4, $\mathbb{C}[x, y, z]$ is Poisson simple for the Poisson bracket such that

$$\{x, y\} = 0, \quad \{y, z\} = c(xy + a) + bx^2 \text{ and } \{z, x\} = xy + a.$$

Type 2, $\mathcal{P}_2$: Let $\beta \in \mathbb{Q}[i][x, y]$ be homogeneous of degree 2 and irreducible over $\mathbb{Q}[i]$. Then by [**4**, Proposition 5.4] and Corollary 3.4 $\mathbb{C}[x, y, z]$ is Poisson simple for the Poisson bracket such that

$$\{x, y\} = 0, \quad \{y, z\} = -\beta \text{ and } \{z, x\} = 1.$$

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Type 3, $\mathcal{P}_3$: By [**4**, Proposition 6.1] and Corollary 3.4, $\mathbb{C}[x, y, z]$ is Poisson simple for the Poisson bracket such that

$$\{x, y\} = 0, \quad \{y, z\} = -x \text{ and } \{z, x\} = xy + 1.$$

Type 4, $\mathcal{P}_4$: In Examples 5.6(i), take $p(x) = \rho x$, where $\rho \in \mathbb{C}\backslash\{0\}$.

For discussion of some classes of simple derivations $\delta = g\partial_x - f\partial_y$ of A where the degrees of f and g may be greater than 2, see [**3**, Corollary 4.3, Theorems 4.4 and 5.5 and Proposition 6.2].

EXAMPLE 5.10. Let $f = -1$ and $g = x$, so that $\delta(x) = x$ and $\delta(y) = 1$ and the Poisson bracket on A is such that $\{y, z\} = -1$, $\{z, x\} = x$ and $\{x, y\} = 0$. The Poisson triple here is the cm-exact triple $x\,\mathrm{grad}(y - \log x)$. It is clear that xB is a Poisson prime ideal and $x \notin \mathrm{PZ}(B)$. Applying Proposition 5.5 with $C = \mathbb{C}[x^{\pm 1}]$, $a = 0$, $b = 1$ and $\delta|_C = xd/dx$, it is easy to see that xA is the only non-zero δ-prime ideal of A. As $\delta(A) \nsubseteq xA$, it follows from Theorem 3.2 that $\mathrm{P.Spec}(A) = \{0, xA\}$. By Theorem 3.6, if $R = A[z; \delta]$ then $\mathrm{Spec}\, R = \{0, xR\}$.

EXAMPLE 5.11. Let $M = xB + yB$ and $N = xA + yA$ and suppose that $f, g \in A$ are such that if $\delta = g\partial_x - f\partial_y$ then N is the unique non-zero δ-prime ideal of A, in other words, there are no height one prime ideals invariant under δ and N is the only maximal ideal of A invariant under δ. Then $f = -\delta(y) \in N$, $g = \delta(x) \in N$ and $\delta(A) \subseteq N$. By Theorems 3.2 and 3.6,

$$\mathrm{P.Spec}\, B = \{0, xB + yB\} \cup \{xB + yB + (z - \alpha)B : \alpha \in \mathbb{C}\},$$

and if $R = A[z; \delta]$,

$$\mathrm{Spec}\, R = \{0, xR + yR\} \cup \{xR + yR + (z - \alpha)R : \alpha \in \mathbb{C}\}.$$

Note that $\mathrm{P.Spec}_2\, B = \{0\}$ and $\mathrm{Spec}_2\, R = \{0\}$. In other words, there is no proper Poisson prime homomorphic image of B with a non-zero Poisson bracket and every proper prime homomorphic image of R is commutative. However, if j is such that $f \notin N^j$ or $g \notin N^j$ then $B/(N^jB)$ is a proper Poisson homomorphic image of B with a non-zero Poisson bracket and $R/(N^jR)$ is a non-commutative proper homomorphic image of R. Such a j must exist as f and g must be non-zero and $\cap_{j\geq 1} N^j = 0$.

Goodearl and Warfield [**10**, p. 61] specify such an example with $f = -(x^2 + y^2)$ and $g = x + y$. Although the condition on the base field in [**10**] is satisfied by $\mathbb{R}$ but not by $\mathbb{C}$, the conclusion is also valid for $\mathbb{C}$. The details of this example were omitted from [**10**] as the proof was 'exceedingly tedious'. Interest was expressed in any similar example with a short proof. Here, subject to the reader's interpretation of the word 'short', we present such an example.

Let $f = -x(1 + xy)$ and $g = y$ so that $\delta(y) = x(1 + xy)$ and $\delta(x) = y$. Let $'$ denote differentiation with respect to x. Clearly, N is the unique maximal ideal of A invariant under δ. Let $Q \neq N$ be a non-zero δ-prime ideal of A. Then Q has height one and is principal, $Q = qA$, say, with $0 \neq q = \sum_{i=0}^{n} q_i(x)y^i$, each $q_i(x) \in \mathbb{C}[x]$, $q_n(x) \neq 0$, and as $\delta(q_0(x)) = yq_0'(x)$, $n > 0$. Let $h \in A$ be such that $\delta(q) = hq$. Note that for $0 \leq i \leq n$,

$$\delta(q_i(x)y^i) = q_i'(x)y^{i+1} + ix^2q_i(x)y^i + ixq_i(x)y^{i-1}.$$

Also note that $q_n'(x) \in q_n(x)\mathbb{C}[x]$, whence $q_n'(x) = 0$ and $q_n(x) \in \mathbb{C}^*$. Therefore, $\deg_y(\delta(q)) \leq n$ so $h = h(x) \in \mathbb{C}[x]$. Comparing coefficients of y^i, $0 \leq i \leq n$, in the

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equation $\delta(q(x)) = h(x)q(x)$, we obtain

$$(i+1)xq_{i+1}(x) = (h(x) - ix^2)q_i(x) - q'_{i-1}(x), \tag{5.2}$$

where $q_{-1}(x) = 0 = q_{n+1}(x)$. Note that $q_0(x) \neq 0$, otherwise $q_i(x) = 0$ for all i. For $i \geq 0$, let $d_i = \deg(q_i(x))$, let $d = d_0$ and let $e_i = \deg(h(x) - ix^2)$. By (5.2) with $i = 0$, $d_1 = e_0 + d_0 - 1$. It follows from (5.2) that

$$\text{if } d_i + e_i > d_{i-1} - 1 \text{ then } d_{i+1} = d_i + e_i - 1. \tag{5.3}$$

In the following five situations, (5.3) can be used to show, inductively, that the sequence $\{d_i\}$ is eventually strictly increasing. Hence, these cases can be excluded.

(i) If $h(x) = 0$ then $e_i = 2$, when $i > 0$, $q_1(x) = 0$ and $d_i = d - 4 + i$ whenever $i > 1$.

(ii) If $h(x)$ has degree $r = 0$ or 1 then $e_0 = r$, $e_i = 2$ when $i > 0$, $d_1 = d + r - 1$ and $d_i = d - r - 2 + i$ when $i > 1$.

(iii) If $h(x)$ has degree $r \geq 3$ or $h(x) = ax^2 + bx + c$ has degree $r = 2$ and $a \notin \mathbb{N}$ then $d_i = d + i(r-1)$ for $i > 0$.

(iv) If $h(x) = ax^2 + bx + c$ has degree 2, $a \in \mathbb{N}$ and $b \neq 0$ then $d_i = d + i$ for $0 \leq i \leq a$, $d_{a+1} = d + a$ and $d_{a+j} = d + a + j - 1$ for $j \geq 2$.

(v) If $h(x) = ax^2 + c$ has degree 2, $a \in \mathbb{N}$ and $c \neq 0$ then $d_i = d + i$ for $0 \leq i \leq a$, $d_{a+1} = d + a - 1$ and $d_{a+j} = d + a + j - 2$ for $j \geq 2$.

This leaves only the case $h(x) = ax^2$, $a \in \mathbb{N}$, in which we need to keep track of leading coefficients as well as degrees. Let α be the leading coefficient of $q_0(x)$. By repeated use of (5.2), the leading coefficient of $q_i(x)$ is $\binom{a}{i}\alpha$ for $0 \leq i \leq a$. In particular, the leading coefficients of $q_{a-1}(x)$ and $q_a(x)$ are $a\alpha$ and α respectively. By (5.3), $d_i = d + i$ for $0 \leq i \leq a$.

By (5.2), with $i = a$, $(a+1)xq_{a+1}(x) = -q'_{a-1}(x)$ so $d_{a+1} = d + a - 3$ and the leading coefficient in $q_{a+1}(x)$ is $-(d + a - 1)a\alpha/(a+1)$.

From (5.2), with $i = a + 1$, we see that $d_{a+2} \leq d + a - 2$ and the coefficient of x^{d+a-2} in $q_{a+2}(x)$ is $-(d + 2a)\alpha/((a+1)(a+2)) \neq 0$. Therefore, $d_{a+2} = d + a - 2 > d_{a+1} - e_{a+2} - 1$. It now follows, inductively, that $d_{a+j} = d + a + j - 4$ for all $j \geq 3$, which is impossible. This completes the proof that P. Spec and Spec R are as stated above.

Example 5.12. Here we consider the Poisson bracket on B arising from [**10**, Example 2.15], where $\delta = 2y\partial_x + (y^2 + x)\partial_y$ so that $\{y, z\} = -(y^2 + x)$, $\{z, x\} = 2y$ and $\{x, y\} = 0$. In [**10**], it is shown that the only non-zero δ-prime ideals of A are the maximal ideal $M := xA + yA$ and the height one prime $Q := (y^2 + x + 1)A$. Note that $Q \nsubseteq M$ and $\delta(A) \subseteq M$ but $\delta(A) \nsubseteq Q$. By Theorem 3.2,

$$\text{P. Spec}\, B = \{0, (y^2 + x + 1)B, xB + yB\} \cup \{xB + yB + (z - \alpha)B : \alpha \in \mathbb{C}\}.$$

If $R = A[z; \delta]$ then

$$\text{Spec}\, R = \{0, (y^2 + x + 1)R, xR + yR\} \cup \{xR + yR + (z - \alpha)R : \alpha \in \mathbb{C}\}.$$

Note that $\text{P. Spec}_2\, B = \{0, (y^2 + x + 1)B\}$ and $\text{Spec}_2\, R = \{0, (y^2 + x + 1)R\}$. In contrast to Example 5.11, there is a unique non-zero Poisson prime ideal that is not residually null.

REMARK 5.13. In both Examples 5.11 and 5.12, the Poisson algebra B has a Poisson prime ideal $P = xB + yB$ which has height two as a prime ideal but is minimal as a non-zero Poisson prime ideal. In both cases P is residually null. To obtain examples of this phenomenon in which P is not residually null, pass to $B' = B[u, v] = \mathbb{C}[x, y, z, u, v]$ with the Poisson bracket such that $\{u, b\} = \{v, b\} = 0$ for all $b \in B$ and $\{u, v\} = 1$. This is the tensor product, as Poisson algebras, of B and a copy of the coordinate ring of the symplectic plane. Then $xB' + yB'$ again has height two as a prime ideal and is minimal as a non-zero Poisson prime ideal but it is not residually null, having factor isomorphic to $\mathbb{C}[z, u, v]$ with $\{u, v\} = 1$ and $\{u, z\} = \{v, z\} = 0$.

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