J. Sekiguchi
Nagoya Math. J.
Vol. 79 (1980), 151–185

EIGENSPACES OF THE LAPLACE-BELTRAMI OPERATOR ON A HYPERBOLOID

JIRO SEKIGUCHI

§0. Introduction: Statement of the problem

Ever since S. Helgason [4] showed that any eigenfunction of the Laplace-Beltrami operator on the unit disk is represented by the Poisson integral of a hyperfunction on the unit circle, much interest has been arisen to the study of the Poisson integral representation of joint eigenfunctions of all invariant differential operators on a symmetric space X. In particular, his original idea of expanding eigenfunctions into K-finite functions has proved to be generalizable up to the case where X is a Riemannian symmetric space of rank one (cf. [4], [5], [11]). Presently, extension to arbitrary rank has been completed by quite a different formalism which views the present problem as a boundary-value problem for the differential equations. It should be recalled that along this line of approach a general theory of the systems of differential equations with regular singularities was successfully established by Kashiwara-Oshima (cf. [6], [7]).

It is then natural to ask that the success may also be extended to the case of more general (not necessarily Riemannian) symmetric spaces. Our specific problem which underlies this paper is this: Can any joint eigen-hyperfunction of all invariant differential operators on X be represented by the Poisson integral of hyperfunctions on the "boundary" of X? Furthermore, we are anxious to know whether the Kashiwara-Oshima theory still plays a central role in this problem.

Our first try is in Oshima-Sekiguchi [14] where it was shown that the answer was quite affirmative. The class of symmetric spaces treated in [14] is, however, somewhat restricted so that we feel it advisable to pursue this problem further. In order to attack this problem, it seems to us to need to prepare some facts concerning the symmetric space or the

Received April 16, 1979.

https://doi.org/10.1017/S0027763000019000 Published online by Cambridge University Press

representation theory and, by this reason, it may be difficult, at present, to generalize the above mentioned result for all symmetric spaces of the non-compact type.

Within the scope of this paper, our aim is less ambitious. We take out three series of symmetric spaces:

$$SO_0(p+1, q+1)/SO(p+1, q)$$
$$SU(p+1, q+1)/S(U(p+1, q) \times U(1))$$
$$Sp(p+1, q+1)/Sp(p+1, q) \times Sp(1)$$

with $p, q \geqq 1$. These symmetric spaces are not contained in the class dealt with in [6] and [14]. Let X be one of the above symmetric spaces. We denote by Δ the Laplace-Beltrami operator on X corresponding to the G-invariant pseudo-Riemannian metric induced by the Killing form of the Lie algebra of G. Here G denotes the motion group of X. The main result of this paper is Theorem 8.4 which says that the above problem is affirmatively solved under a certain mild condition with respect to the eigenvalue of Δ (see § 8).

The construction of this paper is as follows. In § 1, we shall describe somewhat well-known, but rather important facts about the structure of semisimple Lie groups and Lie algebras. The several decompositions mentioned there may play an elementary role to the study of (not necessarily Riemannian) symmetric spaces. In § 2, we deal with the concrete description about the semisimple groups on which we study the eigenfunctions later. We realize the symmetric space in a compact real analytic manifold in § 3 and the Laplace-Beltrami operator is calculated in § 4. Furthermore, the boundary values of eigenfunctions on the symmetric space are defined by use of the Kashiwara-Oshima theory. In § 5, we define the Poisson transformation to X. We shall mention the main result of [6] and calculate a special case of Harish-Chandra's c-function in § 6. In order to prove the Poisson transformation is surjective, a special eigenfunction of Δ is investigated in § 7. § 8 is devoted to proving the main result. We also remark about generalized zonal spherical functions on X.

It is a pleasure to thank Professor T. Oshima for useful discussion. My thanks are also due to Professor M. Sato for his continual encouragement.

§ 1. Preliminaries

1.1. Let $\mathfrak{g}$ be a real semisimple Lie algebra and let θ be a Cartan involution of $\mathfrak{g}$. Then we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where

(1.1)
$$\mathfrak{k} = \{X \in \mathfrak{g}; \theta(X) = X\}, \quad \mathfrak{p} = \{X \in \mathfrak{g}; \theta(X) = -X\}.$$

Let σ be another (non Cartan) involution which commutes with θ. Then we also have the decomposition, $\mathfrak{g} = \mathfrak{k}' + \mathfrak{p}'$, where

(1.1)′
$$\mathfrak{k}' = \{X \in \mathfrak{g}; \sigma(X) = X\}, \quad \mathfrak{p}' = \{X \in \mathfrak{g}; \sigma(X) = -X\}.$$

We denote by $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$ and put $\mathfrak{a}_0 = \mathfrak{p}' \cap \mathfrak{a}$.

Let us first recall the root system Σ of $(\mathfrak{g}, \mathfrak{a})$. The root system Σ is given by

$$\Sigma = \{\lambda; \lambda \text{ is a linear form on } \mathfrak{a} \text{ such that } \lambda \neq 0,\ \mathfrak{g}^\lambda \neq 0\},$$

where

$$\mathfrak{g}^\lambda = \{X \in \mathfrak{g}; [H, X] = \lambda(H)X \text{ for } H \in \mathfrak{a}\}.$$

Let us choose a connected component of the set $\mathfrak{a}'$ of regular elements in $\mathfrak{a}$, and denote it by $\mathfrak{a}^+$. Then we can introduce an ordering on Σ so that $\beta \in \Sigma$ is positive if and only if $\beta(H) > 0$ for all $H \in \mathfrak{a}^+$. If we denote $\mathfrak{n} = \sum_{\beta>0} \mathfrak{g}^\beta$, $\bar{\mathfrak{n}} = \theta(\mathfrak{n})$ and $\mathfrak{m} = \{X \in \mathfrak{k}; [H, X] = 0 \text{ for all } H \in \mathfrak{a}\}$, the well-known decompositions read:

(1.2)
$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n} \text{ (Iwasawa decomposition)}, \quad \mathfrak{g} = \bar{\mathfrak{n}} + \mathfrak{m} + \mathfrak{a} + \mathfrak{n} \text{ (Bruhat decomposition)}.$$

Next we try to define the root system of the pair $(\mathfrak{g}, \mathfrak{a}_0)$. For any linear form λ on $\mathfrak{a}_0$, we set

$$\mathfrak{g}(\lambda) = \{X \in \mathfrak{g}; [H, X] = \lambda(H)X \text{ for } H \in \mathfrak{a}_0\}$$

and

$$\Sigma(\mathfrak{a}_0) = \{\lambda; \lambda \text{ is a linear form on } \mathfrak{a}_0 \text{ such that } \lambda \neq 0,\ \mathfrak{g}(\lambda) \neq 0\}.$$

To be parallel to the preceding case, we need here the following assumptions for the pair $(\mathfrak{g}, \mathfrak{a}_0)$.

(AI) $\Sigma(\mathfrak{a}_0)$ is a root system in $\mathfrak{a}_0^*$, where $\mathfrak{a}_0^*$ is the dual of $\mathfrak{a}_0$.

(AII) $\mathfrak{m}_0 = \{X \in \mathfrak{g}; [H, X] = 0 \text{ for any } H \in \mathfrak{a}_0\}$ is contained in $\mathfrak{k}'$.

https://doi.org/10.1017/S0027763000019000 Published online by Cambridge University Press

It should be remembered that (AII) at least does not hold in general, but it will be shown that both of the assumptions are well met in the cases that we treat in what follows.

Let us next put $\mathfrak{a}_0' = \{H \in \mathfrak{a}_0; \alpha(H) \neq 0 \text{ for all } \alpha \in \Sigma(\mathfrak{a}_0)\}$ and choose a connected component $\mathfrak{a}_0^+$ in $\mathfrak{a}_0'$ in such a way that $\mathfrak{a}_0^+$ is contained in the closure of the previous $\mathfrak{a}^+$. Then we can introduce analogously an ordering on $\Sigma(\mathfrak{a}_0)$ so that $\alpha \in \Sigma(\mathfrak{a}_0)$ is positive if and only if $\alpha(H) > 0$ for all $H \in \mathfrak{a}_0^+$. Denote $\mathfrak{n}_0 = \sum_{\alpha>0} \mathfrak{g}(\alpha)$ (with $\alpha \in \Sigma(\mathfrak{a}_0)$) and $\bar{\mathfrak{n}}_0 = \theta(\mathfrak{n}_0)$. Then we have the following decompositions:

$$(1.2)' \qquad \begin{aligned} \mathfrak{g} &= \mathfrak{k}' + \mathfrak{a}_0 + \mathfrak{n}_0 \\ \mathfrak{g} &= \bar{\mathfrak{n}}_0 + \mathfrak{m}_0 + \mathfrak{a}_0 + \mathfrak{n}_0 \,. \end{aligned}$$

1.2. We now proceed to mention a connected semisimple Lie group G with finite center whose Lie algebra is $\mathfrak{g}$. Let A (resp. A_0), N (resp. N_0) and $\bar{N}$ (resp. $\bar{N}_0$) be the analytic subgroups of G corresponding to $\mathfrak{a}$ (resp. $\mathfrak{a}_0$), $\mathfrak{n}$ (resp. $\mathfrak{n}_0$) and $\bar{\mathfrak{n}}$ (resp. $\bar{\mathfrak{n}}_0$). We also use the notation $A^+ = \exp \mathfrak{a}^+$ (resp. $A_0^+ = \exp \mathfrak{a}_0^+$).

Let K be the maximal compact subgroup of G whose Lie algebra is $\mathfrak{k}$. We denote by M the centralizer of A in K. Then, as is known,

[DI] (Iwasawa decomposition)

The map $K \times A \times N \to G$ given by $(k, a, n) \to kan$ $(k \in K, a \in A, n \in N)$ is a surjective diffeomorphism.

[DII] (Cartan decomposition)

KA^+K is an open dense subset of G such that $KC\ell(A^+)K = G$, where $C\ell(A^+)$ stands for the closure of A^+ in A.

[DIII] (Bruhat decomposition)

$\bar{N}MAN$ is an open dense subset of G.

We now turn to take out a closed subgroup K' of G whose Lie algebra is $\mathfrak{k}'$, this being the subgroup with which we are exclusively concerned in this paper. Corresponding to M, we have M_0 as the centralizer of A_0 in K'. In this case, corresponding to [DI], [DII], [DIII], we may have the following:

[DI′] The map $K' \times A_0 \times N_0 \to G$ given by $(k', a, n) \to k'an$ $(k' \in K', a \in A_0, n \in N_0)$ is an injective diffeomorphism and its image is open dense in G.

[DII′] KA_0^+K' is an open dense subset of G and $G = KC\ell(A_0^+)K'$ where $C\ell(A_0^+)$ denotes the closure of A_0^+ in A_0.

[DIII′] $\bar{N}_0 M_0 A_0 N_0$ is an open dense subset of G.

The first and the last of these decompositions are not necessarily satisfied while the second holds in general (cf. [8]). It may be, however, clear that the presence of such decompositions as these play a judicious role in investigating the eigenfunctions on G/K'. In fact it will turn out that these three give an insight into our problem and really hold in the cases announced in the Introduction.

For any $g \in G$, we define $\bar{n}_B(g) \in \bar{N}_0$ and $H_B(g) \in \mathfrak{a}_0$ by

$$g = \bar{n}_B(g) m \exp(H_B(g)) n$$

with $m \in M_0$ and $n \in N_0$. The existence and the uniqueness of $\bar{n}_B(g)$ and $H_B(g)$ is shown under the assumption of [DIII′]. This notation is used later.

§ 2. Computation of decompositions

2.1. Let us denote by I_n the unit matrix of order n and introduce the following matrices:

$$I_{pq} = \begin{bmatrix} -I_p & \\ & I_q \end{bmatrix} \qquad I'_{pq} = \begin{bmatrix} & & 1 \\ & I_{pq} & \\ 1 & & \end{bmatrix} \qquad {}^{\pm}I''_{pq} = \begin{bmatrix} 1 & & \\ & {}^{\mp}I_{pq} & \\ & & 1 \end{bmatrix}$$

$$J_n = \begin{bmatrix} & I_n \\ -I_n & \end{bmatrix} \qquad K'_{pq} = \begin{bmatrix} I'_{pq} & \\ & I'_{pq} \end{bmatrix} \qquad {}^{\pm}K''_{pq} = \begin{bmatrix} {}^{\pm}I''_{pq} & \\ & {}^{\pm}I''_{pq} \end{bmatrix}$$

We define the following Lie algebras:

$$\begin{aligned} \mathfrak{g}_1 &= \{X \in \mathfrak{sl}(p+q+2, \boldsymbol{R});\, {}^tXI'_{pq} + I'_{pq}X = 0\} \\ \mathfrak{g}_2 &= \{X \in \mathfrak{sl}(p+q+2; \boldsymbol{C});\, {}^t\bar{X}I'_{pq} + I'_{pq}X = 0\} \\ \mathfrak{g}_3 &= \{X \in \mathfrak{sl}(2p+2q+4, \boldsymbol{C});\, {}^tXJ_{p+q+2} + J_{p+q+2}X = 0, \\ &\qquad {}^t\bar{X}K'_{pq} + K'_{pq}X = 0\}\,. \end{aligned} \tag{2.1}$$

Throughout this paper we assume $p, q\ (\geqq 1)$ and put $r = p + q + 1$.

We shall prove by case by case discussions that the assumptions (AI) and (AII) hold for pairs $(\mathfrak{g}_i, {}^{+}\mathfrak{k}'_i)$ and $(\mathfrak{g}_i, {}^{-}\mathfrak{k}'_i)$ for certain subalgebras ${}^{+}\mathfrak{k}'_i$ and ${}^{-}\mathfrak{k}'_i$ of $\mathfrak{g}_i$.

2.1.1 Case I: $\mathfrak{g}_1$.

Let θ be a Cartan involution of $\mathfrak{g}_1$ defined by $\theta(X) = -{}^t\bar{X}$ for X in $\mathfrak{g}_1$. Let $\sigma_1^{\pm}$ be an involution of $\mathfrak{g}_1$ defined by

$$\sigma_1^{\pm}(X) = -{}^{\pm}I''_{pq}\,{}^tX\,{}^{\pm}I''_{pq} \qquad \text{for } X \text{ in } \mathfrak{g}_1 . \tag{2.2}$$

Let E_{ij} denote the matrix of order $r+1$ whose (i,j) entry is 1 and others are 0 $(0 \leqq i, j \leqq r)$. We define the following elements of $\mathfrak{g}_1$:

$$\begin{aligned} H_0 &= E_{00} - E_{rr} , \\ H_i &= E_{i,r-i} + E_{r-i,i} \qquad (i = 1, \cdots, \ell) , \\ X_j &= E_{j0} + \varepsilon_j E_{rj} , \\ Y_j &= {}^tX_j \qquad (j = 1, \cdots, r-1) . \end{aligned} \tag{2.3}$$

Here

$$\varepsilon_j = \begin{cases} 1 \ (1 \leqq j \leqq p) \\ -1 \ (p < j \leqq p+q) \end{cases} \quad \text{and} \quad \ell = \min(p, q) .$$

These matrices are obviously contained in $\mathfrak{g}_1$. The involutions θ and $\sigma_1^{\pm}$ commute with each other. We define $\mathfrak{k}_1, \mathfrak{p}_1, {}^{\pm}\mathfrak{k}'_1, \mathfrak{p}'_1$ as follows:

$$\begin{aligned} \mathfrak{k}_1 &= \{X \in \mathfrak{g}_1; \theta(X) = X\} , \\ \mathfrak{p}_1 &= \{X \in \mathfrak{g}_1; \theta(X) = -X\} , \\ {}^{\pm}\mathfrak{k}'_1 &= \{X \in \mathfrak{g}_1; \sigma_1^{\pm}(X) = X\} , \\ {}^{\pm}\mathfrak{p}'_1 &= \{X \in \mathfrak{g}_1; \sigma_1^{\pm}(X) = -X\} . \end{aligned} \tag{2.4}$$

Then $\mathfrak{a} = \sum_{i=0}^{\ell} RH_i$ is a maximal abelian subspace of $\mathfrak{p}_1$ and $\mathfrak{a}_0 = RH_0$ is a maximal abelian subspace of $\mathfrak{p}_1 \cap {}^{\pm}\mathfrak{p}'_1$ which is contained in $\mathfrak{a}$. Let α be a linear form on $\mathfrak{a}_0$ defined by $\alpha(H_0) = 1$. Then the root spaces of $\pm\alpha$ are expressed as follows.

$$\begin{aligned} \mathfrak{g}(\alpha) &= \sum_{j=1}^{p+q} RY_j , \\ \mathfrak{g}(-\alpha) &= \sum_{j=1}^{p+q} RX_j . \end{aligned} \tag{2.5}$$

It is easy to prove that $\Sigma(\mathfrak{a}_0) = \{\alpha, -\alpha\}$ is a root system of type A_1. Furthermore, $\mathfrak{m}_0 = \{X \in \mathfrak{g}_1; [H, X] = 0 \text{ for } H \in \mathfrak{a}_0\}$ is contained in ${}^{\pm}\mathfrak{k}'_1$. Hence (AI) and (AII) hold for the pairs $(\mathfrak{g}_1, {}^{+}\mathfrak{k}'_1)$ and $(\mathfrak{g}_1, {}^{-}\mathfrak{k}'_1)$.

Take $\mathfrak{a}^+ = \{\sum_{i=0}^{\ell} t_i H_i; t_0 > t_1 > \cdots > t_\ell\}$ and $\mathfrak{a}_0^+ = \{tH_0; t > 0\}$ as the positive Weyl chambers in $\mathfrak{a}'$ and $\mathfrak{a}'_0$ in order to define orderings on Σ and $\Sigma(\mathfrak{a}_0)$, respectively. We put $\mathfrak{n}_0 = \mathfrak{g}(\alpha)$ and $\bar{\mathfrak{n}}_0 = \mathfrak{g}(-\alpha)$. Then we have the following decompositions:

$$\begin{aligned} \mathfrak{g} &= {}^{+}\mathfrak{k}'_1 + \mathfrak{a}_0 + \mathfrak{n}_0 \\ &= {}^{-}\mathfrak{k}' + \mathfrak{a}_0 + \mathfrak{n}_0 . \end{aligned} \tag{2.6}$$

2.1.2. Case II: $\mathfrak{g}_2$.

Let θ be a Cartan involution of $\mathfrak{g}_2$ defined by $\theta(X) = -{}^t\overline{X}$ for X in $\mathfrak{g}_2$. Let $\sigma_2^{\pm}$ be an involution of $\mathfrak{g}_2$ defined by

$$\sigma_2^{\pm}(X) = -{}^{\pm}I''_{pq}\,{}^t\overline{X}\,{}^{\pm}I''_{pq} \qquad \text{for } X \text{ in } \mathfrak{g}_2 . \tag{2.7}$$

Using the same notation E_{ij}, ℓ as in 2.1.1, we define the following matrices.

$$\begin{aligned} X'_j &= \sqrt{-1}\,(E_{j0} - \varepsilon_j E_{rj}) \qquad (j = 1, \cdots, r-1) \\ X_0 &= \sqrt{-1}\,E_{r0} \\ Y'_j &= {}^tX'_j \\ Y_0 &= {}^tX_0 . \end{aligned}$$

Then, as is easily seen, H_i $(0 \leqq i \leqq \ell)$, X_0, Y_0, X_j, X'_j, Y_j, Y'_j $(1 \leqq j \leqq p+q)$ are contained in $\mathfrak{g}_2$. For later convenience, we put $X_{1j} = X_j$, $X_{2j} = X'_j$, $Y_{1j} = Y_j$, $Y_{2j} = Y'_j$ $(1 \leqq j \leqq p+q)$. We define $\mathfrak{k}_2$, $\mathfrak{p}_2$, ${}^{\pm}\mathfrak{k}'_2$, ${}^{\pm}\mathfrak{p}'_2$ as follows:

$$\begin{aligned} \mathfrak{k}_2 &= \{X \in \mathfrak{g}_2 ; \theta(X) = X\} , \\ \mathfrak{p}_2 &= \{X \in \mathfrak{g}_2 ; \theta(X) = -X\} , \\ {}^{\pm}\mathfrak{k}'_2 &= \{X \in \mathfrak{g}_2 ; \sigma_2^{\pm}(X) = X\} , \\ {}^{\pm}\mathfrak{p}'_2 &= \{X \in \mathfrak{g}_2 ; \sigma_2^{\pm}(X) = -X\} . \end{aligned} \tag{2.8}$$

We also use the notation $\mathfrak{a}$, $\mathfrak{a}_0$, α as in 2.1.1. Then the root spaces of $\pm\alpha$, $\pm 2\alpha$ are

$$\begin{aligned} \mathfrak{g}(\alpha) &= \sum_{i=1}^{2}\sum_{j=1}^{p+q} RY_{ij} , \\ \mathfrak{g}(2\alpha) &= RY_0 , \\ \mathfrak{g}(-\alpha) &= \sum_{i=1}^{2}\sum_{j=1}^{p+q} RX_{ij} , \\ \mathfrak{g}(-2\alpha) &= RX_0 . \end{aligned} \tag{2.9}$$

In this case, $\Sigma(\mathfrak{a}_0) = \{\alpha, 2\alpha, -\alpha, -2\alpha\}$ is a root system of type BC_1. Furthermore $\mathfrak{m}_0 = \{X \in \mathfrak{g}_2 ; [X, H] = 0 \text{ for } H \in \mathfrak{a}_0\}$ is contained in ${}^{\pm}\mathfrak{k}'_2$. Hence (AI) and (AII) hold for the pairs $(\mathfrak{g}_2, {}^{+}\mathfrak{k}'_2)$ and $(\mathfrak{g}_2, {}^{-}\mathfrak{k}'_2)$. We put $\mathfrak{n}_0 = \mathfrak{g}(\alpha) + \mathfrak{g}(2\alpha)$ and $\bar{\mathfrak{n}}_0 = \theta(\mathfrak{n}_0)$.

2.1.3. Case III: $\mathfrak{g}_3$.

Let θ be a Cartan involution of $\mathfrak{g}_3$ defined by $\theta(X) = -{}^t\overline{X}$ for X in $\mathfrak{g}_3$. Let $\sigma_3^{\pm}$ be an involution of $\mathfrak{g}_3$ defined by

$$\sigma_3^{\pm}(X) = -{}^{\pm}K''_{pq}\,{}^t\overline{X}\,{}^{\pm}K''_{pq} . \tag{2.10}$$

https://doi.org/10.1017/S0027763000019000 Published online by Cambridge University Press

Retain the notation ε, ℓ in 2.1.1. In this case, we denote by E_{ij} the matrix of order $2(r+1)$ whose (i,j)-entry is 1 and others are 0 $(0 \leqq i, j \leqq 2r+1)$. Define the following matrices:

$$\begin{aligned}
H_0 &= E_{00} - E_{rr} + E_{r+1,r+1} - E_{2r+1,2r+1} \\
H_i &= E_{ir} + E_{r-i,i} - E_{r+1+i,2r+1-i} - E_{2r+1-i,r+1+i} \qquad (i = 1, \cdots, \ell) \\
Y_{1j} &= E_{0j} + \varepsilon_j E_{jr} - E_{r+1,r+1+j} - \varepsilon_j E_{r+1+j,2r+1} \\
Y_{2j} &= \sqrt{-1}\,(E_{0j} - \varepsilon_j E_{jr} - E_{r+1,r+1+j} + \varepsilon_j E_{r+1+j,2r+1}) \\
Y_{3j} &= E_{0,r+1+j} + E_{j+1,r+1} + \varepsilon_j E_{2r+1-j,r} + \varepsilon_j E_{2r+1,r-j} \\
Y_{4j} &= \sqrt{-1}\,(E_{0,r+1+j} + E_{j+1,r+1} - \varepsilon_j E_{2r+1-j,r} - \varepsilon_j E_{2r+1,r-j}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j = 1, \cdots, r-1) \\
Y_{01} &= \sqrt{-1}\,(E_{0r} - E_{2r+1,r+1}) \\
Y_{02} &= E_{0,r+1} - E_{2r+1,r} \\
Y_{03} &= \sqrt{-1}\,(E_{0,r+1} - E_{2r+1,r}) \\
X_{kj} &= {}^tY_{kj} \qquad (k = 0, 1, 2, 3,\ j = 1, \cdots, r-1)\,.
\end{aligned}$$

These are obviously contained in $\mathfrak{g}_3$. We define $\mathfrak{k}_3$, $\mathfrak{p}_3$, ${}^{\pm}\mathfrak{k}'_3$, ${}^{\pm}\mathfrak{p}'_3$ as follows:

$$(2.11)\qquad \begin{aligned}
\mathfrak{k}_3 &= \{X \in \mathfrak{g}_3;\, \theta(X) = X\}\,, \\
\mathfrak{p}_3 &= \{X \in \mathfrak{g}_3;\, \theta(X) = -X\}\,, \\
{}^{\pm}\mathfrak{k}'_3 &= \{X \in \mathfrak{g}_3;\, \sigma_3^{\pm}(X) = X\}\,, \\
{}^{\pm}\mathfrak{p}'_3 &= \{X \in \mathfrak{g}_3;\, \sigma_3^{\pm}(X) = -X\}\,.
\end{aligned}$$

We take $\mathfrak{a} = \sum_{i=0}^{\ell} RH_i$ (resp. $\mathfrak{a}_0 = RH_0$) as a maximal abelian subspace of $\mathfrak{p}_3$ (resp. $\mathfrak{p}_3 \cap {}^{\pm}\mathfrak{p}'_3$). Let α be a linear form on $\mathfrak{a}_0$ defined by $\alpha(H_0) = 1$. Then the root spaces of $\pm\alpha$, $\pm 2\alpha$ are expressed as follows:

$$(2.12)\qquad \begin{aligned}
\mathfrak{g}(\alpha) &= \sum_{i=1}^{4} \sum_{j=1}^{p+q} RY_{ij}\,, \\
\mathfrak{g}(2\alpha) &= \sum_{j=1}^{3} RY_{0j}\,, \\
\mathfrak{g}(-\alpha) &= \sum_{i=1}^{4} \sum_{j=1}^{p+q} RX_{ij}\,, \\
\mathfrak{g}(-2\alpha) &= \sum_{j=1}^{3} RX_{0j}\,.
\end{aligned}$$

In this case, $\Sigma(\mathfrak{a}_0) = \{\alpha, 2\alpha, -\alpha, -2\alpha\}$ is a root system of type BC_1. Furthermore, $\mathfrak{m}_0 = \{X \in \mathfrak{g}_3;\, [X, H] = 0 \text{ for } H \in \mathfrak{a}_0\}$ is contained in ${}^{\pm}\mathfrak{k}'_3$. Hence (AI) and (AII) hold for the pairs $(\mathfrak{g}_3, {}^{+}\mathfrak{k}'_3)$ and $(\mathfrak{g}_3, {}^{-}\mathfrak{k}'_3)$. We put $\mathfrak{n}_0$ and $\bar{\mathfrak{n}}_0$ as in (2.1.2).

2.2. We next investigate the structure of connected linear semisimple Lie groups whose Lie algebras are one of which we defined in 2.1. We first define the following Lie groups:

$$\begin{aligned} &G_1 = \text{The identity component of } \{g \in SL(r+1, \boldsymbol{R});\, {}^t g I'_{pq} g = I'_{pq}\}\,, \\ (2.13) \qquad &G_2 = \{g \in SL(r+1, \boldsymbol{C});\, {}^t \bar{g} I'_{pq} g = I'_{pq}\}\,, \\ &G_3 = \{g \in SL(2r+2; \boldsymbol{C});\, {}^t g J_{r+1} g = J_{r+1},\, {}^t \bar{g} K'_{pq} g = K'_{pq}\}\,. \end{aligned}$$

Let K_i be the maximal compact subgroup of G_i whose Lie algebra is $\mathfrak{k}_i$ $(i = 1, 2, 3)$. Furthermore we define the following closed subgroups of G_i:

$$\begin{aligned} &{}^{\pm}K'_1 = \{g \in G_1;\, {}^t g\, {}^{\pm}I''_{pq} g = {}^{\pm}I''_{pq}\}\,, \\ (2.14) \qquad &{}^{\pm}K'_2 = \{g \in G_2;\, {}^t \bar{g}\, {}^{\pm}I''_{pq} g = {}^{\pm}I''_{pq}\}\,, \\ &{}^{\pm}K'_3 = \{g \in G_3;\, {}^t \bar{g}\, {}^{\pm}K''_{pq} g = {}^{\pm}K''_{pq}\}\,. \end{aligned}$$

We remark that $G_1 \cong SO_0(p+1, q+1)$, $G_2 \cong SU(p+1, q+1)$ and $G_3 \cong Sp(p+1, q+1)$, furthermore ${}^{+}K'_1 \cong SO(p+1, q)$, ${}^{-}K'_1 \cong SO(p, q+1)$, ${}^{+}K'_2 \cong S(U(p+1, q) \times U(1))$, ${}^{-}K'_2 \cong S(U(p, q+1) \times U(1))$, ${}^{+}K'_3 \cong Sp(p+1, q) \times Sp(1)$, ${}^{-}K'_3 \cong Sp(p, q+1) \times Sp(1)$.

Let $A_0 = \exp \mathfrak{a}_0$ and M_0 be the centralizer of A_0 in ${}^{\pm}K'_i$.

Our objective in this paragraph is to compute the concrete expressions for the decompositions [DI′] and [DIII′] in the previous section for the pairs $(G_i, {}^{+}K'_i)$ and $(G_i, {}^{-}K'_i)$.

2.2.1. Case I: G_1.

In the sequel, we use the following notation. An element $g \in G_1$ is written by

$$g = (g_{ij})_{0 \leq i, j \leq r}\,.$$

(1) The decomposition [DI′].

A direct calculation implies that $g = (g_{ij}) \in G_1$ is expressed in the form $k'an$ with $k' \in {}^{\pm}K'_1$, $a \in A_0$, $n \in N_0$ if and only if $g_{00} \pm g_{r0} \neq 0$, and then

$$a = \mathrm{Diag}\,(a_0, I_{r-1}, a_0^{-1})$$

with

$$a_0 = |g_{00} \pm g_{r0}|$$

and

$$n = \begin{bmatrix} 1 & x & -\frac{1}{2}xI_{pq}{}^t x \\ & I_{pq} & -I_{pq}{}^t x \\ & & 1 \end{bmatrix}$$

with

$$x = \frac{h' \pm h''}{g_{00} \pm g_{r0}},$$

where

$$h' = (g_{01}, \cdots, g_{0,r-1})$$
$$h'' = (g_{r1}, \cdots, g_{r,r-1}).$$

(2) The decomposition [DIII′]

First remark that $P_0 = M_0A_0N_0$ is a maximal parabolic subgroup of G. Hence the decomposition of this type is well-known. For later convenience we shall derive the concrete expression. A direct calculation implies that $g = (g_{ij}) \in G$ is contained in $\bar{N}_0M_0A_0N_0$ if and only if $g_{00} \neq 0$ and that, if $g_{00} \neq 0$, then $g = \bar{n}man$ with $\bar{n} \in \bar{N}_0$, $m \in M_0$, $a \in A_0$, $n \in N_0$ and in particular

$$a = \mathrm{Diag}\,(|g_{00}|, I_{r-1}, |g_{00}|^{-1})$$

$$\bar{n} = \left(\begin{array}{c:c:c} 1 & & \\ \hdashline x & I_{p+q} & \\ \hdashline -\frac{1}{2}{}^t xI_{pq}x & -{}^t xI_{pq} & 1 \end{array}\right)$$

where

$${}^t x = (g_{10}/|g_{00}|, \cdots, g_{p+q,0}/|g_{00}|).$$

2.2.2. Case II: G_2.

We always write $g = (g_{ij})_{0 \leq i,j \leq r}$ for $g \in G_2$ as in 2.2.1.

(1) The decomposition [DI′]

If $g = (g_{ij}) \in G_2$ is expressed in the form $k'an$ with $k' \in {}^{\pm}K_2'$, $a \in A_0$, $n \in N_0$, then $g_{00} \pm g_{r0} \neq 0$, and

$$a = \mathrm{Diag}\,(a_0, I_{p+q}, a_0^{-1})$$

$$n = \left(\begin{array}{c:c:c} 1 & z & \sqrt{-1}x - \frac{1}{2}zI_{pq}{}^t\bar{z} \\ \hdashline & I_{p+q} & -I_{pq}{}^t\bar{z} \\ \hdashline & & 1 \end{array}\right)$$

with

https://doi.org/10.1017/S0027763000019000 Published online by Cambridge University Press

$$a_0 = |g_{00} \pm g_{r0}|$$
$$z = \frac{h' \pm h''}{g_{00} \pm g_{r0}}$$
$$x = \operatorname{Im}\left(\frac{g_{0r} \pm g_{rr}}{g_{00} \pm g_{r0}}\right)$$

where h' and h'' are defined as in 2.2.1.

(2) The decomposition [DIII′]

By the same reason as in 2.2.1, $\bar{N}_0 M_0 A_0 N_0$ is open dense in G_2. Actually, $g = (g_{ij}) \in G_2$ is contained in $\bar{N}_0 M_0 A_0 N_0$ if and only if $g_{00} \neq 0$. If $g_{00} \neq 0$, $g = \bar{n}man$ with $\bar{n} \in \bar{N}_0$, $m \in M_0$, $a \in A_0$, $n \in N_0$, and

$$a = \operatorname{Diag}(|g_{00}|, I_{p+q}, |g_{00}|^{-1})$$
$$\bar{n} = \left(\begin{array}{c:c:c} 1 & & \\ \hdashline z & I_{p+q} & \\ \hdashline \sqrt{-1}x - \frac{1}{2}\,{}^t\bar{z}I_{pq}z & -{}^t\bar{z}I_{pq} & 1 \end{array}\right)$$

where

$${}^tz = (g_{10}/g_{00}, \cdots, g_{p+q,0}/g_{00})$$
$$\sqrt{-1}x - \frac{1}{2}\,{}^t\bar{z}I_{pq}z = g_{r0}/g_{00}\,.$$

2.2.3. Case III: G_3.

In this case, we shall always write

$$g = (g_{ij})_{0 \leqq i, j \leqq 2r+1} \in G_3\,.$$

(1) The decomposition [DI′].

If $g = k'an$, with $k' \in {}^{\pm}K_3'$, $a \in A_0$, $n \in N_0$, then $g_{00} \pm g_{r0} \neq 0$, or $g_{r+1,0} \pm g_{2r+1,0} \neq 0$. In this case,

$$a = \operatorname{Diag}(a_0, I_{p+q}, a_0^{-1}, a_0^{-1}, I_{p+q}, a_0)$$

$$n = \left(\begin{array}{c:c:c:c:c:c} 1 & z & \sqrt{-1}x - (zI_{pq}\,{}^t\bar{z} + wI_{pq}\,{}^t\bar{w}) & -\bar{v} & w & \\ \hdashline & 1 & -I_{pq}\,{}^t\bar{z} & {}^tw & & \\ \hdashline & & I_{p+q} & & & \\ \hdashline & & & I_{p+q} & & \\ \hdashline & & -I_{pq}\,{}^t\bar{w} & -{}^tz & 1 & \\ \hdashline & -I_{pq}\bar{w} & v & -\sqrt{-1}x - (zI_{pq}\,{}^t\bar{z} + wI_{pq}\,{}^t\bar{w}) & I_{pq}\bar{z} & 1 \end{array}\right)$$

with

https://doi.org/10.1017/S0027763000019000 Published online by Cambridge University Press

$$
\begin{aligned}
a_0 &= (|g_{00} \pm g_{r0}|^2 + |g_{r+1,0} \pm g_{2r+1,0}|^2)^{1/2} \\
z &= a_0^{-2}\{(h_1' \pm h_1'')(\bar{g}_{00} \pm \bar{g}_{r0}) + (h_3' \pm h_3'')(\bar{g}_{r+1,0} \pm \bar{g}_{2r+1,0})\} \\
x &= a_0^{-2}\,\mathrm{Im}\,\{(g_{0r} \pm g_{rr})(\bar{g}_{00} \pm \bar{g}_{r0}) + (g_{r+1,r} \pm g_{2r+1,r})(\bar{g}_{r+1,0} \pm \bar{g}_{2r+1,0})\} \\
w &= a_0^{-2}\{(h_2' \pm h_2'')(\bar{g}_{00} \pm \bar{g}_{r0}) + (h_4' \pm h_4'')(\bar{g}_{r+1,0} \pm \bar{g}_{2r+1,0})\} \\
v &= -(\bar{g}_{0,2r+1} \pm \bar{g}_{r,2r+1})(g_{00} \pm g_{r0}) \\
&\quad - (\bar{g}_{r+1,2r+1} \pm \bar{g}_{2r+1,2r+1})(g_{r+1,0} \pm g_{2r+1,0})\,,
\end{aligned}
$$

where

$$
\begin{aligned}
h_1' &= (g_{01}, \cdots, g_{0,r-1}) \\
h_1'' &= (g_{r1}, \cdots, g_{r,r-1}) \\
h_2' &= (g_{0,r+2}, \cdots, g_{r,2r}) \\
h_2'' &= (g_{r,r+2}, \cdots, g_{r,2r}) \\
h_3' &= (g_{r+1,1}, \cdots, g_{r+1,r-1}) \\
h_3'' &= (g_{2r+1,1}, \cdots, g_{2r+1,r-1}) \\
h_4' &= (g_{r+1,r+2}, \cdots, g_{r+1,2r}) \\
h_4'' &= (g_{2r+1,r+2}, \cdots, g_{2r+1,2r})\,.
\end{aligned}
$$

(2) The decomposition [DIII′].

By the same reason as in 2.2.1, $\bar{N}_0 M_0 A_0 N_0$ is open dense in G_3. An element $g = (g_{ij}) \in G_3$ is contained in $\bar{N}_0 M_0 A_0 N_0$ if and only if $g_{00} \neq 0$. In this case, if we put $g = \bar{n}man$ with $\bar{n} \in \bar{N}_0$, $m \in M_0$, $a \in A_0$, $n \in N_0$, then

$$
a = \mathrm{Diag}\,(|g_{00}|, I_{p+q}, |g_{00}|^{-1}, |g_{00}|^{-1}, I_{p+q}, |g_{00}|)
$$

$$
\bar{n} = \left(\begin{array}{c:c:c:c:c:c}
1 & & & & & \\ \hdashline
z & 1 & & & & -I_{pq}\bar{w} \\ \hdashline
\sqrt{-1}x - \frac{1}{2}({}^t\bar{z}I_{pq}z + {}^t\bar{w}I_{pq}w) & -{}^t\bar{z}I_{pq} & I_{p+q} & & -{}^t\bar{w}I_{pq} & -\bar{v} \\ \hdashline
v & {}^t w & & I_{p+q} & -{}^t z & -\sqrt{-1}x - ({}^t\bar{z}I_{pq}z + {}^t\bar{w}I_{pq}w) \\ \hdashline
w & & & & 1 & I_{pq}\bar{z} \\ \hdashline
 & & & & & 1
\end{array}\right)
$$

where

$$
\begin{aligned}
&{}^t z = (g_{10}/g_{00}, \cdots, g_{p+q}/g_{00}) \\
&\sqrt{-1}x - \tfrac{1}{2}({}^t\bar{z}I_{pq}z + {}^t\bar{w}I_{pq}w) = g_{r0}/g_{00} \\
&v = g_{r+1,0}/g_{00} \\
&w = (g_{r+2,0}/g_{00}, \cdots, g_{2r,0}/g_{00})\,.
\end{aligned}
$$

We have already remarked in Section 1 that the decomposition [DII′] holds for the pairs discussed in this paragraph. Hence we obtain the following.

PROPOSITION. *The decompositions* [DI′], [DII′], [DIII′] *hold for the pairs* $(G_i, {}^+K_i')$ *and* $(G_i, {}^-K_i')$ $(i = 1, 2, 3)$.

§ 3. A realization of $G_i/{}^+K_i'$ and $G_i/{}^-K_i'$

We shall construct a real analytic manifold in which G_i/K_i' and $G_i/{}^-K_i'$ are realized as an open set. This realization is useful in our study to formulate the problem as the boundary value problem with regular singularities.

We first construct a real analytic manifold $\tilde{X}$ in somewhat general situation. Let G be one of G_i and let $P_0 = M_0A_0N_0$. Furthermore we write ${}^+K'$ (and ${}^-K'$) in place of ${}^+K_i'$ (and ${}^-K_i'$) for simplicity. We denote by $\hat{X}$ the product manifold $G \times \boldsymbol{R}$. Then G acts on $\hat{X}$ in the natural way: $(g, (g', y)) \to (gg', y)$ for $g, g' \in G$ and $y \in \boldsymbol{R}$. For $z = (g, y)$ in $\tilde{X}$, we define $\operatorname{sgn} z$, $a(z)$ and $P(z)$ as follows:

$$
\tag{3.1}
\begin{aligned}
\operatorname{sgn} z &= \begin{cases} 1 & \text{if } y > 0 \\ 0 & \text{if } y = 0 \\ -1 & \text{if } y < 0 \end{cases} \\
a(z) &= \begin{cases} \exp\left(-\tfrac{1}{2}(\log|y|)H_0\right) & \text{if } y \neq 0 \\ 1 \text{ (the identity element)} & \text{if } y = 0 \end{cases} \\
P(z) &= \begin{cases} {}^+K' & \text{if } y > 0 \\ P_0 & \text{if } y = 0\,. \\ {}^-K' & \text{if } y < 0 \end{cases}
\end{aligned}
$$

DEFINITION 3.1. For any $z = (g, y)$ and $z' = (g', y')$ in $\hat{X}$, we define the equivalence relation $z \sim z'$ if and only if $\operatorname{sgn} z = \operatorname{sgn} z'$ and $ga(z)P(z) = g'a(z')P(z')$ in $G/P(z)$.

Let $\tilde{X} = \hat{X}/\sim$ be the quotient space of $\hat{X}$ by this equivalence relation $\sim$ and π the projection of $\hat{X}$ onto $\tilde{X}$. For each g in G, we put $\tilde{U}_g = \pi(g\bar{N}_0 \times \boldsymbol{R})$ and define the following sets:

$$
\begin{aligned}
U_g^+ &= \{\pi(g\bar{n}, y);\, \bar{n} \in \bar{N}_0, y > 0\}\,, \\
U_g^0 &= \{\pi(g\bar{n}, 0);\, \bar{n} \in \bar{N}_0\}\,, \\
U_g^- &= \{\pi(g\bar{n}, y);\, \bar{n} \in \bar{N}_0, y < 0\}\,.
\end{aligned}
$$

Our aim in this section is to prove the following theorem.

THEOREM 3.2. *The space* $\tilde{X}$ *satisfies the following properties.*

(i) $\tilde{X}$ *is a connected, compact, real analytic manifold.*

(ii) $\tilde{X} = \bigcup_{g \in G} \tilde{U}_g$.

(iii) *The action of* G *on* $\tilde{X}$ *is real analytic and, for any* $z \in \tilde{X}$, *the* G-*orbit of* $\pi(z)$ *coincides with* $G/P(z)$. *In particular there exist three orbits* $X^+ = \bigcup_{g \in G} U_g^+$, $X^0 = \bigcup_{g \in G} U_g^0$, $X^- = \bigcup_{g \in G} U_g^-$. X^+, X^-, X^0 *are isomorphic to* $G/{}^+K'$, $G/{}^-K'$, G/P_0, *respectively.*

In order to prove this, we introduce a coordinate system on each local chart $\tilde{U}_g$. In the sequel, we use the notation in § 2. First we examine the case I: $G = G_1$. In this case, we identify $\bar{N}_0$ with $\boldsymbol{R}^{p+q}$ by the following map:

Denote by ${}^t x = (x_1, \cdots, x_{p+q})$ an element of $\boldsymbol{R}^{p+q}$. Then

$$\bar{n}(x) = \exp\left(\sum_{j=1}^{p+q} \sqrt{2}\, x_j X_j\right), \tag{3.2}$$

in other words

$$\bar{n}(x) = \begin{bmatrix} 1 & & \\ \sqrt{2}\, x & I_{p+q} & \\ -{}^t x I_{pq} x & -\sqrt{2}\, {}^t x I_{pq} & 1 \end{bmatrix}.$$

Under this identification, we can define a map Φ_g of $\boldsymbol{R}^{p+q} \times \boldsymbol{R}$ to $\tilde{U}_g$:

$$\begin{array}{rcl} \Phi_g \colon \boldsymbol{R}^{p+q} \times \boldsymbol{R} & \longrightarrow & \tilde{U}_g \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ (x, y) & \longmapsto & \pi(g\bar{n}(x), a(y)) . \end{array} \tag{3.3}$$

Here $a(y) = \exp(-\frac{1}{2}(\log|y|)H_0)$ if $y \neq 0$, and $a(0) = 1$. The decompositions [DI′] and [DIII′] show that Φ_g is a bijection.

In the other cases, Φ_g is analogously defined as in Case I. We now only define the identification of $\bar{N}_0$ with $\boldsymbol{R}^d$ ($d = \dim \bar{N}_0$).

Case II: $G = G_2$.

Denote by $x = (x_{11}, x_{21}, \cdots, x_{1,p+q}, x_{2,p+q}, x_0)$ an element of $\boldsymbol{R}^{2p+2q+1}$. Then

$$\bar{n}(x) = \exp\left(\sum_{i=1}^{2} \sum_{j=1}^{p+q} 2x_{ij} X_{ij} + 2x_0 X_0\right). \tag{3.4}$$

Case III: $G = G_3$.

Denote by $x = (x_{11}, x_{21}, x_{31}, x_{41}, \cdots, x_{1,p+q}, x_{2,p+q}, x_{3,p+q}, x_{4,p+q}, x_{01}, x_{02}, x_{03})$ an element of $\boldsymbol{R}^{4p+4q+3}$. Then

$$\bar{n}(x) = \exp\left(\sum_{i=1}^{4}\sum_{j=1}^{p+q} 2x_{ij}X_{ij} + \sum_{j=1}^{3} 2x_{0j}X_{0j}\right). \tag{3.5}$$

LEMMA 3.3. *Put $d = \dim \bar{N}_0$. Then, for any g_1 and g_2 in G, the mapping*

$$\Phi_{g_2}^{-1}\circ\Phi_{g_1}\colon \Phi_{g_1}^{-1}(\tilde{U}_{g_1}\cap\tilde{U}_{g_2}) \longrightarrow \Phi_{g_2}^{-1}(\tilde{U}_{g_1}\cap\tilde{U}_{g_2}) \tag{3.6}$$

defines an analytic isomorphism between the open subsets of $\boldsymbol{R}^{d+1}$.

Proof. First we put $g_2^{-1}g_1 = g$. We define the functions (x', y') of (x, y) by

$$(x', y') = \Phi_{g_2}^{-1}\circ\Phi_{g_1}(x, y).$$

We shall prove that (x', y') depend analytically on (x, y).

We treat the case I: $G = G_1$. If $y > 0$, this equation means that $g\bar{n}(x)a(y)$ is contained in $n(x')a(y')^{+}K_1'$. Hence, using the result in 2.2.1, we have

$$\begin{aligned} y' &= \frac{y}{|g_{00} + \sqrt{2}\,h'\cdot x - (y + {}^t xI_{pq}x)g_{r0}|^2} \\ \sqrt{2}\,x' &= \frac{h + \sqrt{2}\,'g\cdot x - (y + {}^t xI_{pq}x)h''}{g_{00} + \sqrt{2}\,h'\cdot x - (y + {}^t xI_{pq}x)g_{r0}}. \end{aligned} \tag{3.7}$$

Here

$$\begin{aligned} {}^t h &= (g_{10}, \cdots, g_{r-1,0}) \\ {}^t h'' &= (g_{1r}, \cdots, g_{r-1,r}) \\ 'g &= (g_{ij})_{1\leq i,j\leq r-1}. \end{aligned}$$

On the other hand, if $y = 0$, $g\bar{n}(x)$ is contained in $\bar{n}(x')P_0$. Then, by the same reason as obtaining (3.7), we have

$$\sqrt{2}\,x' = \frac{h + \sqrt{2}\,'g\cdot x - ({}^t xI_{pq}x)h''}{g_{00} + \sqrt{2}\,h'\cdot x - ({}^t xI_{pq}x)g_{r0}}. \tag{3.8}$$

When $y < 0$, we obtain the same equation as (3.7).

The equation (3.7) easily implies that x' and y' are real analytic functions of x, y if $y \neq 0$, and are naturally extended to $\Phi_{g_1}^{-1}(U_{g_1}\cap U_{g_2})$. The restriction of (3.7) to $y = 0$ reads the equation (3.8). This means that the map $(x, y) \to \Phi_{g_2}^{-1}\circ\Phi_{g_1}(x, y)$ is real analytic. Hence Lemma 3.3 is proved in this case.

In the remainder cases, the claim is also proved by the argument similar to the case I. Hence we omit it. Q.E.D.

This lemma has an easy corollary.

COROLLARY 3.4. *Using the notation in Lemma* 3.3, *we have*

$$\frac{\partial y'}{\partial y}\Big|_{y=0} = \exp\{-\alpha(H_B(g_2^{-1}g_1\bar{n}(x)))\}\,. \tag{3.9}$$

Proof. In Case I, (3.9) is a direct consequence of (3.7) and the result in 2.2.1. A direct calculation implies the result for the remainder cases.

Proof of Theorem 3.2. The definition of $\tilde{X}$ obviously implies (ii) and (iii). Since Lemma 3.3 implies that $\{\tilde{U}_g; g \in G\}$ forms a system of local charts of $\tilde{X}, \tilde{X}$ is a real analytic manifold. The decompositions [DI′] and [DIII′] in § 1 derive that $\tilde{U}_g = \boldsymbol{R}^{d+1}$ is an open dense subset of $\tilde{X}$. Hence, for any two points x, x' in $\tilde{X}$, there exist g, g' in G such that $g'x$ and $g'x'$ are contained in $\tilde{U}_g$. This implies that X is connected. Since the decomposition [DII′] shows that $\pi(K \times [-1, 1])$ is compact and open dense in $\tilde{X}, \tilde{X}$ is equal to $\pi(K \times [-1, 1])$ and is compact. Q.E.D.

Remark. Our method of the realization deeply depends on that in T. Oshima [13].

§ 4. The invariant differential operators on $\tilde{X}$

Let $\mathfrak{g}_C$ be the complexification of $\mathfrak{g}$ and let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}_C$. Let $\mathfrak{k}_C$ and ${}^{\pm}\mathfrak{k}'_C$ be the complexifications of $\mathfrak{k}$ and ${}^{\pm}\mathfrak{k}'$, respectively. The Casimir operator ω of $\mathfrak{g}$ is an element in the center of $U(\mathfrak{g})$ which is defined as follows. Let $X_1, \cdots, X_n$ be a base of $\mathfrak{g}$ and put $h_{ij} = B(X_i, X_j)$ $(1 \leqq i, j \leqq n)$. Here $B(X, Y)$ denotes the Killing form on $\mathfrak{g}$. Then the matrix $(h_{ij})_{1 \leqq i, j \leqq n}$ is non-singular and if (h^{ij}) denotes its inverse, we have

$$\omega = \sum_{i,j=1}^{n} h^{ij} X_i X_j\,. \tag{4.1}$$

As is well-known, the Casimir operator ω induces the Laplace-Beltrami operator Δ on G/K' corresponding to the G-invariant pseudo-Riemannian metric induced by the Killing form of $\mathfrak{g}$. Our objective of this section is to derive the concrete expression for Δ on each local chart defined in the last section and to prove that Δ is naturally extended to $\tilde{X}$ and that Δ has a regular singularity along X^0 in the weak sense. Furthermore, we shall define the boundary value of eigenfunctions of Δ on X^+ and X^-.

First we remember that

$$\text{(4.2)}\qquad \begin{array}{lll} \text{Case I:} & B(X, Y) = (p+q)\,\mathrm{Tr}\, XY & \text{for } X, Y \in \mathfrak{g}_1 \\ \text{Case II:} & B(X, Y) = 2(p+q+1)\,\mathrm{Tr}\, XY & \text{for } X, Y \in \mathfrak{g}_2 \\ \text{Case III:} & B(X, Y) = 2(p+q+3)\,\mathrm{Tr}\, XY & \text{for } X, Y \in \mathfrak{g}_3\,. \end{array}$$

LEMMA 4.1. *The concrete expression of the Casimir operator* ω *is as follows*:

(1) *Case* I:

$$\text{(4.3)}\qquad 2(p+q)\omega \equiv H_0^2 + \sum_{j=1}^{p+q} (X_j Y_j + X_j Y_j) \qquad \mathrm{mod}\ U(\mathfrak{g}_1)\,{}^{\pm}\mathfrak{k}_1'\,.$$

(2) *Case* II:

$$\text{(4.4)}\qquad \begin{aligned} 4(p+q+2)\omega \equiv H_0^2 &+ \sum_{j=1}^{p+q} \{(X_{1j}Y_{1j} + Y_{1j}X_{1j}) - (X_{2j}Y_{2j} + Y_{2j}X_{2j})\} \\ &- 2(X_0Y_0 + Y_0X_0) \qquad \mathrm{mod}\ U(\mathfrak{g}_2)\,{}^{\pm}\mathfrak{k}_2'\,. \end{aligned}$$

(3) *Case* III:

$$\text{(4.5)}\qquad \begin{aligned} 8(p+q+3)\omega \equiv H_0^2 &+ \sum_{j=1}^{p+q} \{(X_{1j}Y_{1j} + Y_{1j}X_{1j}) - (X_{2j}Y_{2j} + Y_{2j}X_{2j})\} \\ &+ \sum_{j=1}^{p+q} \{(X_{3j}Y_{3j} + Y_{3j}X_{3j}) - (X_{4j}Y_{4j} + Y_{4j}X_{4j})\} \\ &- 2(X_{01}Y_{01} + Y_{01}X_{01}) + 2(X_{02}Y_{02} + Y_{02}X_{02}) \\ &- 2(X_{03}Y_{03} + Y_{03}X_{03}) \qquad \mathrm{mod}\ U(\mathfrak{g}_3)\,{}^{\pm}\mathfrak{k}_3'\,. \end{aligned}$$

These are proved by direct calculation from the definition (4.1) of ω.

If we permute p and q, then X^+ is changed to X^- and X^- to X^+. Hence, we may treat one of X^+ and X^- without loss of generality. In the sequel, we mainly deal with $X^+ = G/{}^+K'$.

We identify $U(\mathfrak{g})$ with the totality of left G-invariant differential operators on G as follows:

For any $f(g)$ in $\mathscr{B}(G)$ and Y in $\mathfrak{g}$,

$$\text{(4.6)}\qquad (Yf)(g) = \frac{d}{dt} f(ge^{tY})|_{t=0}\,.$$

Here $\mathscr{B}(G)$ denotes the space of hyperfunctions on G.

LEMMA 4.2. *Let* Δ *denote a differential operator on* $\tilde{X}$ *which is expressed on* $\tilde{U}_g$ *as follows*:

(1) *Case* I:

$$\text{(4.7)}\qquad \Delta = 4\left(y\frac{\partial}{\partial y}\right)^2 - 2(p+q)y\frac{\partial}{\partial y} - y\left(\sum_{j=1}^{p+q} \varepsilon_j \frac{\partial^2}{\partial x_j^2}\right)\,.$$

https://doi.org/10.1017/S0027763000019000 Published online by Cambridge University Press

(2) *Case* II:

$$\varDelta = 4\left(y\frac{\partial}{\partial y}\right)^2 - 4(p+q+1)y\frac{\partial}{\partial y} - y\sum_{j=1}^{p+q}\left\{\left(\frac{\partial}{\partial x_{1j}} - \varepsilon_j x_{2j}\frac{\partial}{\partial x_0}\right)^2 + \left(\frac{\partial}{\partial x_{2j}} + \varepsilon_j x_{1j}\frac{\partial}{\partial x_0}\right)^2\right\} + y^2\frac{\partial^2}{\partial x_0^2}\,. \tag{4.8}$$

(3) *Case* III:

$$\begin{aligned}\varDelta = {} & 4\left(y\frac{\partial}{\partial y}\right)^2 - 4(2p+2q+3)y\frac{\partial}{\partial y} \\ & - y\sum_{j=1}^{p+q}\varepsilon_j\left\{\left(\frac{\partial}{\partial x_{1j}} - \varepsilon_j x_{2j}\frac{\partial}{\partial x_{01}} - x_{3j}\frac{\partial}{\partial x_{02}} + x_{4j}\frac{\partial}{\partial x_{03}}\right)^2\right. \\ & + \left(\frac{\partial}{\partial x_{2j}} + \varepsilon_j x_{1j}\frac{\partial}{\partial x_{01}} + x_{4j}\frac{\partial}{\partial x_{02}} + x_{3j}\frac{\partial}{\partial x_{03}}\right)^2 \\ & + \left(\frac{\partial}{\partial x_{3j}} - \varepsilon_j x_{4j}\frac{\partial}{\partial x_{01}} + x_{1j}\frac{\partial}{\partial x_{02}} - x_{2j}\frac{\partial}{\partial x_{03}}\right)^2 \\ & \left. + \left(\frac{\partial}{\partial x_{4j}} + \varepsilon_j x_{3j}\frac{\partial}{\partial x_{01}} - x_{2j}\frac{\partial}{\partial x_{02}} - x_{1j}\frac{\partial}{\partial x_{03}}\right)^2\right\} \\ & + y^2\left(\frac{\partial^2}{\partial x_{01}^2} + \frac{\partial^2}{\partial x_{02}^2} + \frac{\partial^2}{\partial x_{03}^2}\right).\end{aligned} \tag{4.9}$$

Then $\varDelta$ commutes with the action of G and the restriction of $\varDelta$ to X^+ and X^- are equal to the Laplace-Beltrami operator on X^+ and X^- corresponding to the G-invariant pseudo-Riemannian metric (up to a constant factor), respectively.

Proof. For any $u \in \mathscr{B}(G/{}^+K')$, we put

$$u^g(x, y) = u(\pi(g\bar{n}(x), y)) \qquad \text{on } U_g^+\,. \tag{4.10}$$

We shall calculate the local expression for ω on U_g^+ by use of this coordinate system. We only examine the case II because the expressions in the other two cases are obtained by the argument similar to this case.

A direct calculation derives that

$$\begin{aligned}4(p+q+2)\omega \equiv {} & H_0^2 + 2(p+q+1)H_0 \\ & + 2\sum_{j=1}^{p+q}(X_{1j}Y_{1j} - X_{2j}Y_{2j}) - 4X_0Y_0 \qquad \operatorname{mod} U(\mathfrak{g}_2)\,{}^+\mathfrak{k}_2'\,.\end{aligned} \tag{4.11}$$

Since $X_0 + Y_0$, $Y_{1j} + \varepsilon_j X_{1j}$, $Y_{2j} - \varepsilon_j X_{2j}$ are contained in ${}^+\mathfrak{k}_2'$, we get

$$\begin{aligned}4(p+q+2)\omega \equiv {} & H_0^2 + 2(p+q+1)H_0 \\ & - 2\sum_{i=1}^{2}\sum_{j=1}^{p+q}\varepsilon_j X_{ij}^2 + 4X_0^2 \qquad \operatorname{mod} U(\mathfrak{g}_2)\,{}^+\mathfrak{k}_2'\,.\end{aligned} \tag{4.12}$$

https://doi.org/10.1017/S0027763000019000 Published online by Cambridge University Press

On the other hand, the definitions (4.6) and (4.10) imply that

$$(4.13)\quad \begin{aligned} (H_0u^g)(x, y) &= -2y\frac{\partial}{\partial y}u^g(x, y)\,, \\ (X_{1j}u^g)(x, y) &= \frac{1}{\sqrt{2}}\sqrt{y}\Big(\frac{\partial}{\partial x_{1j}} - \varepsilon_j x_{2j}\frac{\partial}{\partial x_0}\Big)u^g(x, y)\,, \\ (X_{2j}u^g)(x, y) &= \frac{1}{\sqrt{2}}\sqrt{y}\Big(\frac{\partial}{\partial x_{2j}} + \varepsilon_j x_{1j}\frac{\partial}{\partial x_0}\Big)u^g(x, y)\,, \\ (X_0u^g)(x, y) &= \frac{1}{2}y\frac{\partial}{\partial x_0}u^g(x, y)\,. \end{aligned}$$

The equations (4.12) and (4.13) yield (4.8). This means that the restriction of Δ to X^+ is equal to the Laplace-Beltrami operator on X^+ with respect to the G-invariant pseudo-Riemannian metric. The restriction of Δ to X^- is also satisfied with the property mentioned in the lemma. Hence Δ is defined globally on $\tilde{X}$ and obviously commutes with the action of G. Q.E.D.

If we denote $I_{\mathfrak{g}}^{\pm}$ the centralizer of ${}^{\pm}k'$ in $U(\mathfrak{g})$, then $I_{\mathfrak{g}}^{\pm}/(I_{\mathfrak{g}}^{\pm} \cap U(\mathfrak{g})\,{}^{\pm}\mathfrak{k}')$ is generated by ω (mod $U(\mathfrak{g})\,{}^{\pm}\mathfrak{k}'$) by Theorem 8 in [3]. Hence every invariant differential operator on $G/{}^{\pm}K'$ is a polynomial of $\Delta|_{X^{\pm}}$, and therefore every differential operator on $\tilde{X}$ which commutes with the action of G is a polynomial of Δ.

We shall study the following differential equation on $G/{}^{+}K' = X^{+}$:

$$(4.14)\qquad \mathcal{M}_s\colon \Delta u = \Big(s + \frac{m_1 + 2m_2}{2}\Big)\Big(s - \frac{m_1 + 2m_2}{2}\Big)u\,.$$

Here $m_1 = \dim \mathfrak{g}(\alpha)$ and $m_2 = \dim \mathfrak{g}(2\alpha)$.

LEMMA 4.3. *The differential equation $\mathcal{M}_s$ has a regular singularity along X_0 in the weak sense. The characteristic exponents of Δ are*

$$\frac{1}{2}\Big(s + \frac{m_1 + 2m_2}{2}\Big) \quad \textit{and} \quad \frac{1}{2}\Big(-s + \frac{m_1 + 2m_2}{2}\Big)\,.$$

Proof. This follows from the expressions (4.7), (4.8), (4.9) and Definition 4.3 in [7]. Q.E.D.

Let us denote by $\mathscr{B}(G/K'\colon \mathcal{M}_s)$ the space of hyperfunctions on G/K' which are solutions of $\mathcal{M}_s$. (We write G/K' for $G/{}^{+}K'$ since we mainly treat $G/{}^{+}K'$.) Furthermore we define

$$(4.15)\quad \begin{aligned} \mathscr{B}(G/P_0; s) = \{f(g) \in \mathscr{B}(G); f(gman) = f(g)e^{(s-(m_1+2m_2)/2)\alpha(\log a)} \\ \text{for } g \in G,\ m \in M_0,\ a \in A_0,\ n \in N_0\}\,. \end{aligned}$$

For any $u \in \mathscr{B}(G/K'; \mathscr{M}_s)$, we put $(\pi_s(g)u)(g') = u(g^{-1}g')$. Since Δ commutes with the action of $G, \pi_s(g)u$ is also contained in $\mathscr{B}(G/K'; \mathscr{M}_s)$. Likewise we put $(\tau_s(g)f)(g') = f(g^{-1}g')$ for any f in $\mathscr{B}(G/P_0; s)$.

We shall define a G-homomorphism of $\mathscr{B}(G/K'; \mathscr{M}_s)$ to $\mathscr{B}(G/P_0; s)$. If $2s \notin Z$, for any $u \in \mathscr{B}(G/K'; \mathscr{M}_s)$ we can take the boundary values of $u|_{U_g^+}$ corresponding to the characteristic exponents $\frac{1}{2}(-s + (m_1 + 2m_2)/2)$ $(g \in G)$ by Corollary 4.7, Corollary 4.4 in [4] and Lemma 4.3. In the local chart $\tilde{U}_g$, let us denote by $\beta_s^g u^g$ the boundary value of u corresponding to the exponent $\frac{1}{2}(-s + (m_1 + 2m_2)/2)$. Then, it follows from Definition 4.8 and Definition 5.7 in [4] that there exists a hyperfunction $F_s^g(x)$ on U_g^0 such that

$$(4.16) \qquad \beta_s^g u^g = F_s^g(x) y_+^{\frac{1}{2}(-s+(m_1+2m_2)/2)} .$$

Since the notion of the boundary value does not depend on the local coordinate systems by Theorem 5.8 in [4], we have for any g, g' in G,

$$(4.17) \qquad F_s^g(x) y_+^{\frac{1}{2}(-s+(m_1+2m_2)/2)} = F_s^{g'}(x') y_+'^{\frac{1}{2}(-s+(m_1+2m_2)/2)}$$

on $U_g^0 \cap U_{g'}^0$, where (x, y) (resp. (x', y')) denotes the local coordinate system on $\tilde{U}_g$ (resp. $\tilde{U}_{g'}$) used in § 3 and the correspondence between (x, y) and (x', y') are defined by

$$(4.18) \qquad \pi(g\bar{n}(x), y) = \pi(g'\bar{n}(x'), y') .$$

Then (4.17) and Corollary 3.4 imply that

$$(4.19) \qquad F_s^g(\bar{n}) = F_s^{g'}(\bar{n}_B(g'^{-1}g\bar{n})) \exp\left\{\left(s - \frac{m_1 + 2m_2}{2}\right)\alpha(H_B(g'^{-1}g\bar{n}))\right\}$$

for $\bar{n}$ in $\bar{N}_0P_0 \cap g^{-1}g'\bar{N}_0P_0$. Here we identify $F_s^g(x)$ (resp. $F_s^{g'}(x')$) with $F_s^g(\bar{n}(x))$ (resp. $F_s^{g'}(\bar{n}(x'))$). Using $F_s^g(\bar{n})$, we shall define a hyperfunction F_s on G by

$$(4.20) \qquad F_s(g_0) = F_s^g(\bar{n}_B(g'^{-1}g) \exp\left\{\left(s - \frac{m_1 + 2m_2}{2}\right)\alpha(H_B(g^{-1}g_0))\right\}$$

if g_0 is contained in $g\bar{N}_0P_0$.

LEMMA 4.4. *Under the assumption $2s \notin Z$, F_s is independent of the choice of g and belongs to $\mathscr{B}(G/P_0; s)$.*

Proof. In order to prove that (4.20) is independent of the choice of g, it is sufficient to prove that, for any g, g' in G, and g_0 in $g\bar{N}_0P_0 \cap g'\bar{N}_0P_0$,

$$(4.21)\quad \begin{aligned} &F_s^g(\bar{n}_B(g^{-1}g_0)) \exp\left\{\left(s - \frac{m_1 + 2m_2}{2}\right)\alpha(H_B(g^{-1}g_0))\right\} \\ &= F_s^{g'}(\bar{n}_B(g'^{-1}g_0)) \exp\left\{\left(s - \frac{m_1 + 2m_2}{2}\right)\alpha(H_B(g'^{-1}g_0))\right\} . \end{aligned}$$

But (4.21) follows from (4.20) and an elementary property of the decomposition [DIII′]. The rest of the statement is now obvious. Q.E.D.

We put $F_s = \beta_s u$ and call $\beta_s u$ the boundary value of u corresponding to the characteristic exponent $\frac{1}{2}(-s + (m_1 + 2m_2)/2)$.

LEMMA 4.5. *If $2s \notin Z$, $\tau_s(g)(\beta_s u) = \beta_s(\pi_s(g)u)$. for any u in $\mathscr{B}(G/K'; \mathscr{M}_s)$ and g in G.*

Proof. Retain the above notation. We may examine on $\tilde{U}_1$ without loss of generality (1 denotes the identity element in G). Let (x, y) be the local coordinate system on $\tilde{U}_1$. For any $g \in G$, let (x', y') be the local coordinate system of $\tilde{U}_g$. Then,

$$(\pi_s(g^{-1})u^1)(x, y) = u^g(x, y) ,$$

which implies that

$$\beta_s^1(\pi_s(g^{-1})u^1) = F_s^1(\bar{n}_B(g\bar{n})) \exp\left\{\left(s - \frac{m_1 + 2m_2}{2}\right)\alpha(H_B(g\bar{n}))\right\} y_+^{\frac{1}{2}(-s+(m_1+2m_2)/2)} .$$

On the other hand, it follows from (4.21)

$$F_s(g\bar{n}) = F_s^1(\bar{n}_B(g\bar{n})) \exp\left\{\left(s - \frac{m_1 + 2m_2}{2}\right)\alpha(H_B(g\bar{n}))\right\} .$$

This equation implies the result. Q.E.D.

THEOREM 4.6. *If $2s \notin Z$, β_s defines a G-homomorphism of $\mathscr{B}(G/K'; \mathscr{M}_s)$ to $\mathscr{B}(G/P_0; s)$.*

Proof. This is obvious from Lemmas 4.4 and 4.5. Q.E.D.

§ 5. The Poisson transformation

We shall begin by defining a left K'-invariant section of $\mathscr{B}(G/P_0; s)$:

$$(5.1)\quad h_s(g) = \begin{cases} e^{(s-(m_1+2m_2)/2)t} & \text{if } g = ka_t n \in K'A_0N_0 \\ 0 & \text{if } g \notin K'A_0N_0 . \end{cases}$$

for g in G. Here we have defined $a_t = \exp(tH_0)$.

LEMMA 5.1. *$h_s(g)$ is a hyperfunction on G with a meromorphic parameter s and defines a left K'-invariant section of $\mathscr{B}(G/P_0; s)$. The poles of $h_s(g)$ is contained in*

$$\left\{\frac{r-1}{2}-1, \frac{r-1}{2}-3, \frac{r-1}{2}-5, \cdots\right\} \qquad (\textit{Case I})$$
$$\{r-2, r-4, r-6, \cdots\} \qquad (\textit{Case II})$$
$$\{2r-3, 2r-5, 2r-7, \cdots\} \qquad (\textit{Case III}) .$$

Proof. Let us first note that the expressions for $h_s(g)$ are concretely calculated by use of the decomposition [DI′] realized in § 2 as follows:

$$\text{(5.2)}\qquad \begin{aligned} &\text{Case I:} && h_s(g) = |g_{00} + g_{r0}|^{s-(r-1)/2} \\ &\text{Case II:} && h_s(g) = |g_{00} + g_{r0}|^{s-r} \\ &\text{Case III:} && h_s(g) = (|g_{00} + g_{r0}|^2 + |g_{r+1,0} + g_{2r+1,0}|^2)^{\frac{1}{2}(s-2r-1)} . \end{aligned}$$

Then we have the following:

$$h_s(g) \text{ is locally identified with } \begin{cases} |x|^{s-(r-1)/2} & \text{(Case I)} \\ |x_1^2 + x_1^2|^{\frac{1}{2}(s-r)} & \text{(Case II)} \\ |x_1^2 + x_2^2 + x_3^2 + x_4^2|^{\frac{1}{2}(s-2r-1)} & \text{(Case III)} \end{cases}$$

by suitable coordinate transformations, because, in Case I, g_{00} and g_{r0} are real variables, on the other hand, in Case II, g_{00} and g_{r0} are complex variables and further in Case III, in addition to g_{00} and $g_{r0}, g_{r+1,0}$ and $g_{2r+1,0}$ are also complex variables. As is known, $|x|^s$ (resp. $|x_1^2 + x_2^2|^{s/2}$ and $|x_1^2 + x_2^2 + x_3^2 + x_4^2|^{s/2}$) defines a hyperfunction whose poles are contained in the set $\{-1, -3, -5, \cdots\}$ (resp. $\{-2, -4, -6, \cdots\}$ and $\{-4, -6, -8, \cdots\}$) (see, for example [2]). Hence the last half is proved. The rest of the statement is obvious from the definition. Q.E.D.

By virtue of (5.1), we safely define the following function on G:

$$\text{(5.3)}\qquad P_s(g) = h_s(g^{-1}) ,$$

which we hereafter call the Poisson kernel on G/K': We always choose s so that P_s is free from the poles in the s-plane. Then P_s satisfies

LEMMA 5.2.

$$\Delta P_s = \left(s + \frac{m_1 + 2m_2}{2}\right)\left(s - \frac{m_1 + 2m_2}{2}\right)P_s \qquad on\ X^+ .$$

Proof. In the local coordinate system (x, y) on $\tilde{U}_1$, $P_s(\pi(\bar{n}(x), y))$ takes respectively in the forms:

$$\text{Case I:}\quad P_s(\pi(\bar{n}(x), y)) = \frac{|y|^{\frac{1}{2}((p+q)/2+s)}}{|y + \sum_{j=1}^{p+q} \varepsilon_j x_j^2|^{(p+q)/2+s}}$$

$$\text{Case II:}\quad P_s(\pi(\bar{n}(x), y))$$

$$(5.4)\qquad = \left(\frac{|y|}{(y + \sum_{i=1}^{2} \sum_{j=1}^{p+q} \varepsilon_j x_{ij}^2)^2 + 4x_0^2}\right)^{\frac{1}{2}(s+p+q+1)}$$

$$\text{Case III:}\quad P_s(\pi(\bar{n}(x), y))$$

$$= \left(\frac{|y|}{(y + \sum_{i=1}^{4} \sum_{j=1}^{p+q} \varepsilon_j x_{ij}^2)^2 + 4\sum_{j=1}^{3} x_{0j}^2}\right)^{\frac{1}{2}(s+2p+2q+3)} .$$

In each case, at least if $\operatorname{Re}(s + (m_1 + 2m_2)/2) < -4$, $P_s(\pi(\bar{n}(x), y))$ is of class C^2 so that we are allowed to operate Δ on the C^2 function $P_s(\pi(\bar{n}(x), y))$. Direct calculation by use of (4.7), (4.8), (4.9) shows

$$(5.5)\qquad \Delta P_s(\pi(\bar{n}(x), y)) = \left(s + \frac{m_1 + 2m_2}{2}\right)\left(s - \frac{m_1 + 2m_2}{2}\right)P_s(\pi(\bar{n}(x), y)) .$$

Equation (5.5) is analytically continued to hold for general s outside of the poles of P_s. Since Δ commutes with the action of G, (5.5) therefore holds globally on $X^+ = G/{}^+K'$; hence the assertion. Q.E.D.

DEFINITION 5.3. The Poisson transformation $\mathscr{P}_s$ is the integral transformation of $\mathscr{B}(G/P_0; s)$ into $\mathscr{B}(G/K')$ defined by

$$(5.6)\qquad (\mathscr{P}_s f)(g) = \int_K f(k) P_s(k^{-1}g)dk \qquad \text{for } f \text{ in } \mathscr{B}(G/P_0; s) .$$

Here dk denotes the Haar measure on K normalized by

$$\int_K dk = 1 .$$

THEOREM 5.4. *$\mathscr{P}_s$ is a G-homomorphism of $\mathscr{B}(G/P_0; s)$ into $\mathscr{B}(G/K'; \mathscr{M}_s)$.*

Proof. It follows from Lemma 5.2 that $\mathscr{P}_s f$ is contained in $\mathscr{B}(G/K'; \mathscr{M}_s)$ for f in $\mathscr{B}(G/P_0; s)$.

On the other hand

$$(5.7)\qquad \mathscr{P}_s(\tau_s(g)f) = \pi_s(g)(\mathscr{P}_s f) ,$$

because

$$(5.8)\qquad (\mathscr{P}_s f)(g) = \int_K f(gk) P_s(k^{-1})dk$$

for f in $\mathscr{B}(G/P_0; s)$. The formula (5.8) is an easy consequence of [16, Lemma 7.7.6]. Q.E.D.

§ 6. c-function

This section is devoted to a review on the main result in [6] and a remark about Harish-Chandra's c-function.

Let G be one of the Lie groups as defined in § 2, where we defined an ordering in the root system Σ of $(\mathfrak{g}, \mathfrak{a})$. We put

$$\mathfrak{n} = \sum_{\beta \in \Sigma, \beta > 0} \mathfrak{g}^{\beta}, \qquad \bar{\mathfrak{n}} = \theta(\mathfrak{n}).$$

Let $A, N, \bar{N}$ be the analytic subgroups in G corresponding to $\mathfrak{a}, \mathfrak{n}, \bar{\mathfrak{n}}$ respectively. We denote by $\mathfrak{a}^*$ and $\mathfrak{a}_C^*$ the dual of $\mathfrak{a}$ and its complexification. Furthermore, we put $P = MAN$. Then P is a minimal parabolic subgroup of G which contains P_0. We define as usual (*cf.* [6])

$$\begin{aligned}(6.1)\qquad \mathscr{B}(G/P; L_\lambda) = \{f \in \mathscr{B}(G); f(gman) = f(g) \exp\{(\lambda - \rho)(\log a)\} \\ \text{for } g \in G,\ m \in M,\ a \in A,\ n \in N\}\end{aligned}$$

for λ in $\mathfrak{a}_C^*$. Here ρ denotes a linear form on $\mathfrak{a}$ defined by

$$(6.2)\qquad 2\rho(H) = \operatorname{tr} \operatorname{ad}(H)|_{\mathfrak{n}} \qquad \text{for } H \in \mathfrak{a}.$$

As a prototype of (5.6), we here need the Poisson transformation $\mathscr{P}_\lambda$ of $\mathscr{B}(G/P; L_\lambda)$ into $\mathscr{A}(G/K)$ as follows.

$$(6.3)\qquad (\mathscr{P}_\lambda f)(g) = \int_K f(gk) dk$$

for f in $\mathscr{B}(G/P; L_\lambda)$.

Let $D(G/K)$ denote the algebra of invariant differential operators on G/K. For any algebra-homomorphism χ of $D(G/K)$ into C, we denote by $\mathscr{A}(G/K; \mathscr{M}(\chi))$ the space of all analytic functions on G/K satisfying the system of the differential equations

$$(6.4)\qquad \mathscr{M}(\chi): Du = \chi(D)u \qquad \text{for any } D \in D(G/K).$$

We define the following functions on $\mathfrak{a}_C^*$,

$$c(\lambda) = I(\lambda)/I(\rho),$$

where

$$I(\lambda) = \prod_{\alpha \in \Sigma^+} B\left(\frac{m_\alpha}{2}, \frac{1}{4} m_{\alpha/2} + \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\right)$$

and

https://doi.org/10.1017/S0027763000019000 Published online by Cambridge University Press

$$e(\lambda) = \prod_{\alpha \in \Sigma^+, \frac{1}{2}\alpha \notin \Sigma^+} \left\{ \Gamma\left(\frac{1}{2}\left(\frac{m_\alpha}{2} + 1 + \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\right)\right) \Gamma\left(\frac{1}{2}\left(\frac{m_\alpha}{2} + m_{2\alpha} + \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\right)\right) \right\}^{-1} .$$

Here $\Gamma(x)$ (resp. $B(x, y)$) is the Gamma (resp. Beta) function and m_α is the multiplicity of the root α in Σ. Then we have the following theorem.

THEOREM 6.1 (Kashiwara-Kowata-Minemura-Okamoto-Oshima-Tanaka [6]). *Assume λ in $\mathfrak{a}_C^*$ satisfies the condition $e(\lambda) \neq 0$. Then $\mathscr{P}_\lambda$ is a G-isomorphism of $\mathscr{B}(G/P; L_\lambda)$ onto $\mathscr{A}(G/K; \mathscr{M}(\chi_\lambda))$ where χ_λ is the algebra-homomorphism of $D(G/K)$ into C which is uniquely determined by λ.*

Let β_λ be an inverse of $\mathscr{P}_\lambda$ subjected to

$$\mathscr{P}_\lambda \beta_\lambda = c(\lambda) I .$$

We remark that $c(\lambda)$ is Harish-Chandra's c-function.

We now restrict our attention to a special case of $c(\lambda)$; the case being deeply connected with our aim. Let us define a linear form e_i on $\mathfrak{a}$ by

$$e_i(H_j) = \delta_{ij} \qquad (\delta_{ij} \text{ is Kronecker's delta}) . \tag{6.5}$$

Then

$$\mathfrak{a}_C^* = \sum_{i=0}^{\ell} C e_i \quad \text{and} \quad \mathfrak{a}_{0,C}^* = C e_0 ,$$

where $\mathfrak{a}_{0,C}^*$ denotes the complexification of the dual of $\mathfrak{a}_0$. We also denote

$$\mathfrak{a}_1 = \sum_{i=1}^{\ell} R H_i \quad \text{and} \quad \mathfrak{a}_{1,C}^* = \sum_{i=1}^{\ell} C e_i .$$

Corresponding to (6.2), we can define a linear form ρ_0 on $\mathfrak{a}_0$ by

$$2\rho_0(H) = \operatorname{tr} \operatorname{ad}(H)|_{\mathfrak{n}_0} \qquad \text{for } H \in \mathfrak{a}_0 . \tag{6.6}$$

Then a direct calculation implies

$$\begin{aligned} \rho &= \frac{1}{2} \sum_{i=0}^{\ell} \{m_1 + 2m_2 - 2(m_2 + 1)i\} e_i \\ \rho_0 &= \frac{1}{2}(m_1 + 2m_2) e_0 . \end{aligned} \tag{6.7}$$

In view of this, we put specifically

$$\lambda(s) = s e_0 + \frac{1}{2} \sum_{i=1}^{\ell} \{m_1 + 2m_2 - 2(m_2 + 1)i\} e_i \tag{6.8}$$

https://doi.org/10.1017/S0027763000019000 Published online by Cambridge University Press

for $s \in C$. Then there follows a natural imbedding

$$\mathscr{B}(G/P_0; s) \subset \mathscr{B}(G/P; \lambda(s))\,. \tag{6.9}$$

Furthermore

LEMMA 6.2. (i)

$$\begin{aligned} c(\lambda(s)) = 2^{(m_1+2m_2)/2-s} &\times \Gamma\Big(\frac{(l+1)(m_2+1)}{2}\Big)\Gamma\Big(\frac{m_1+2m_2+2+(m_2+1)|p-q|}{4}\Big) \\ &\times \Gamma(s)\Gamma\Big(\frac{1}{2}\Big(s+1-\frac{m_1}{2}\Big)\Big) \\ &\div \Gamma\Big(\frac{m_2+1}{2}\Big)\Gamma\Big(\frac{1}{2}\Big(s+1+\frac{(m_2+1)|p-q|}{2}\Big)\Big) \\ &\times \Gamma\Big(\frac{1}{2}\Big(s+\frac{m_1+2m_2}{2}\Big)\Big)\Gamma\Big(\frac{1}{2}\Big(s+1-\frac{m_1}{2}+(m_2+1)\ell\Big)\Big) \end{aligned} \tag{6.10}$$

(ii) *If* $2s \notin Z$, *then* $e(\lambda(s)) \neq 0$.

Proof. First we remark that the positive roots of Σ are in the forms:

$$e_i \pm e_j \quad (0 \leqq i < j \leqq \ell)$$
$$e_i,\, 2e_i \quad (0 \leqq i \leqq \ell)\,.$$

We write the table of roots and their multiplicities.

Table I

roots	multiplicity
$e_i \pm e_j$ $(i \neq j)$	$1 + m_2$
e_i	$(1+m_2)\lvert p-q\rvert$
$2e_i$	m_2

(In this table, we assumed that, if the multiplicity is zero, the corresponding root does not exist.)

We suppose first that m_2 and $p - q$ are not equal to zero. Then the definition of $I(\lambda)$ and (6.8) imply that $I(\lambda(s))$ is equal to

$$B\Big(\frac{(1+m_2)|p-q|}{2}, s\Big)B\Big(\frac{m_2}{2}, \frac{2s+(1+m_2)|p-q|}{4}\Big) \quad \text{(continued)}$$

$$(6.11)\qquad \times \prod_{i=1}^{\ell}\left\{B\left(\frac{m_2+1}{2}, \frac{s+\frac{1}{2}m_1+m_2-(m_2+1)i}{2}\right) \times B\left(\frac{m_2+1}{2}, \frac{s-\frac{1}{2}m_1-m_2+(m_2+1)i}{2}\right)\right\}$$

up to a constant factor. (6.11) is easily changed to

$$(6.12)\qquad \begin{aligned}&\Gamma(s)\Gamma(\tfrac{1}{4}(2s+(m_2+1)|p-q|))\Gamma(\tfrac{1}{2}(s-\tfrac{1}{2}m_1+1))\\ &\div \{\Gamma(\tfrac{1}{2}(2s+(m_2+1)|p-q|))\Gamma(\tfrac{1}{2}(s+\tfrac{1}{2}m_1+m_2))\\ &\qquad\times \Gamma(\tfrac{1}{2}(s-\tfrac{1}{2}(m_2+1)|p-q|+1))\}\,.\end{aligned}$$

Hence, by the dublication formula of the Gamma function, we get

$$(6.13)\qquad \begin{aligned}I(\lambda(s)) &= c_0\cdot 2^{-s}\\ &\times \Gamma(s)\Gamma(\tfrac{1}{2}(s-\tfrac{1}{2}m_1+1))\\ &\div \{\Gamma(\tfrac{1}{4}(2s+(m_2+1)|p-q|+2))\Gamma(\tfrac{1}{2}(s+\tfrac{1}{2}m_1+m_2))\\ &\qquad\times \Gamma(\tfrac{1}{2}(s-\tfrac{1}{2}(m_2+1)|p-q|+1))\}\end{aligned}$$

with a constant c_0 independent of s. Due to $\rho = \lambda((m_1+2m_2)/2)$, we obtain (i) under the assumption $m_2 \neq 0$ and $p \neq q$. In case when $p = q$ or $m_2 = 0$, the formula (6.13) also holds by an easy modification. Hence the equation (i) is completely proved.

The claim (ii) is proved easily by use of the expression (6.10) and the definition of $e(\lambda)$. Q.E.D.

§ 7. A special eigenfunction on G/K'

We here prepare a theorem concerning the left K-invariant eigenfunction in $\mathscr{B}(G/K'; \mathscr{M}_s)$. This theorem is useful in the next section in which we will examine the relation between β_s and $\tilde{\mathscr{P}}_s$. We use the notation $p' = m_2(p+1)+p$ and $q' = m_2(q+1)+q$ in the sequel.

First we take into account of the image by $\tilde{\mathscr{P}}_s$ of a left K-invariant section $h_s^K(g)$ of $\mathscr{B}(G/P_0; s)$ with the condition $h_s^K(1) = 1$. Put

$$(7.1)\qquad \varphi_s(g) = \int_K h_s^K(g)P_s(k^{-1}g)dk\,.$$

Our aim in this section is to prove the following theorem.

THEOREM 7.1. *Under the assumption $2s \notin \boldsymbol{Z}$, we have*

$$\varphi_s(a_t) = \frac{\Gamma((q'+1)/2)\Gamma(\frac{1}{2}(s+m_2+1-(p'+q')/2))}{\Gamma((m_2+1)/2)\Gamma(\frac{1}{2}(s+1+(q'-p')/2))} \quad \text{(continued)}$$

$$(7.2)\qquad \times (\cosh t)^{-(s+(p'+q')/2)} \times F\Big(\frac{1}{2}\Big(s+\frac{p'+q'}{2}\Big), \frac{1}{2}\Big(s+1+\frac{p'-q'}{2}\Big), \frac{p'+1}{2}; (\tanh t)^2\Big),$$

$$(7.3)\qquad \begin{cases} \lim_{t\to+\infty} e^{((p'+q')/2+s)t}\varphi_s(a_t) = 2^{s+(p'+q')/2} \\ \quad \times \cos\Big(\frac{\pi}{2}\Big(s+\frac{q'-p'}{2}\Big)\Big)\Gamma\Big(\frac{p'+1}{2}\Big)\Gamma\Big(\frac{q'+1}{2}\Big) \\ \qquad\qquad \times \Gamma(-s)\Gamma\Big(\frac{1}{2}\Big(s+m_2+1-\frac{p'+q'}{2}\Big)\Big) \\ \quad \div \Big\{\pi\Gamma\Big(\frac{m_2+1}{2}\Big)\Gamma\Big(\frac{1}{2}\Big(\frac{p'+q'}{2}-s\Big)\Big)\Big\} \qquad \textit{if } \operatorname{Re} s < 0, \\ \lim_{t\to+\infty} e^{((p'+q')/2-s)t}\varphi_s(a_t) = 2^{(p'+q')/2-s} \\ \quad \times \Gamma\Big(\frac{p'+1}{2}\Big)\Gamma\Big(\frac{q'+1}{2}\Big)\Gamma(s)\Gamma\Big(\frac{1}{2}\Big(s+m_2+1-\frac{p'+q'}{2}\Big)\Big) \\ \quad \div \Big\{\Gamma\Big(\frac{m_2+1}{2}\Big)\Gamma\Big(\frac{1}{2}\Big(s+\frac{p'+q'}{2}\Big)\Big) \\ \qquad \times \Gamma\Big(\frac{1}{2}\Big(s+1+\frac{q'-p'}{2}\Big)\Big)\Gamma\Big(\frac{1}{2}\Big(s+1+\frac{p'-q'}{2}\Big)\Big)\Big\} \\ \qquad\qquad \textit{if } \operatorname{Re} s > 0. \end{cases}$$

Here we put $a_t = \exp(tH_0)$ and $F(\alpha, \beta, \gamma; x)$ denotes the Gaussian hypergeometric function.

In order to prove this theorem, we prepare two lemmas.

LEMMA 7.2. *We fix $a = a_t$ $(t \neq 0)$. Then*

$$(7.4)\qquad 2(p'+q'+2m_2)\omega \equiv H_0^2 + (p'\coth t + q'\tanh t)H_0 \mod \mathfrak{k}^{a^{-1}}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}'.$$

Proof. Let us prove (7.4) only when $G = G_2$ (Remaining are also provable by the argument similar to what follows). We make use of the notation introduced in § 2.

Since

$$(7.5)\qquad \begin{aligned} &X_{1j} - Y_{1j} \in \mathfrak{k}_2 \\ &X_{1j} - \varepsilon_j Y_{1j} \in \mathfrak{k}_2' \\ &X_{1j} = \frac{e^t}{e^t - \varepsilon_j e^{-t}}(Y_{1j} - \varepsilon_j X_{1j}) - \frac{1}{e^t - \varepsilon_j e^{-t}}(Y_{1j} - X_{1j}) \\ &[X_{1j}, \operatorname{Ad}(a^{-1})(Y_{1j} - X_{1j})] = -e^{-t}H_0 \end{aligned}$$

for $1 \leqq j \leqq p+q$, we obtain

$$X_{1j}^2 \equiv \frac{e^{-t}}{e^t - \varepsilon_j e^{-t}} H_0 \qquad \bmod \mathfrak{k}_2^{a^{-1}} U(\mathfrak{g}_2) + U(\mathfrak{g}_2)\mathfrak{k}_2' . \tag{7.6}$$

By the same token

$$X_{2j}^2 \equiv \frac{e^{-t}}{e^t - \varepsilon_j e^{-t}} H_0 \qquad \bmod \mathfrak{k}_2^{a^{-1}} U(\mathfrak{g}_2) + U(\mathfrak{g}_2)\mathfrak{k}_2' . \tag{7.7}$$

for $1 \leqq j \leqq p+q$. Furthermore, it is easy to see that

$$\begin{aligned} &X_0 + Y_0 \in \mathfrak{k}_2 \cap \mathfrak{k}_2' \\ &X_0 = \frac{-e^{-2t}}{e^{2t} - e^{-2t}}(Y_0 - X_0) + \frac{1}{e^{2t} - e^{-2t}} \operatorname{Ad}(a^{-1})(Y_0 + X_0) \\ &[X_0, \operatorname{Ad}(a^{-1})(Y_0 + X_0)] = e^{-2t} H_0 . \end{aligned} \tag{7.8}$$

Hence

$$X_0 \equiv \frac{e^{-2t}}{e^{2t} - e^{-2t}} H_0 \qquad \bmod \mathfrak{k}_2^{a^{-1}} U(\mathfrak{g}_2) + U(\mathfrak{g}_2)\mathfrak{k}_2' . \tag{7.9}$$

These results as well as (4.4) combine to give

$$\begin{aligned} 4(p+q+2)\omega \equiv{} & H_0^2 + 2(p+q+1)H_0 \\ & + 4p\frac{e^{-t}}{e^t - e^{-t}}H_0 - 4q\frac{e^{-t}}{e^t + e^{-t}}H_0 + 4\frac{e^{-2t}}{e^{2t} - e^{-2t}}H_0 \\ & \qquad \bmod \mathfrak{k}_2^{a^{-1}} U(\mathfrak{g}_2) + U(\mathfrak{g}_2)\mathfrak{k}_2' \\ ={} & H_0^2 + (p' \coth t + q' \tanh t)H_0 . \end{aligned} \tag{7.9}$$

Q.E.D.

Let $u(g)$ be a left K-invariant and right K'-invariant real analytic function on G. We mainly consider the restriction of $u(g)$ on A_0. Since

$$(H_0 u)(a_t) = \frac{d}{ds} u(a_{t+s})\Big|_{s=0} = \frac{du}{dt}(a_t) , \tag{7.10}$$

we have the following expression for ω by Lemma 7.2.

$$\begin{aligned} &2(p' + q' + 2m_2)(\omega u)(a_t) \\ &\quad = \left\{\left(\frac{d}{dt}\right)^2 + (p' \coth t + q' \tanh t)\frac{d}{dt}\right\} u(a_t) . \end{aligned} \tag{7.11}$$

Let us produce the differential equation:

$$
(7.12)\quad \begin{aligned} &\left\{\left(\frac{d}{dt}\right)^2 + (p' \coth t + q' \tanh t)\frac{d}{dt}\right\}u(a_t) \\ &= \left(s + \frac{p'+q'}{2}\right)\left(s - \frac{p'+q'}{2}\right)u(a_t)\,, \end{aligned}
$$

where p' and q' are subjected to $p' + q' = m_1 + 2m_2$. If we put $z = (\tanh t)^2$ and $f(z) = (\cosh t)^{s+(p'+q')/2}u(a_t)$, then (7.12) is transformed into

$$
(7.13)\quad \begin{aligned} &\left\{z(1-z)\left(\frac{d}{dz}\right)^2 + \left(\frac{p'+1}{2} - \frac{2s+p'+3}{2}z\right)\frac{d}{dz}\right. \\ &\left. - \frac{1}{4}\left(s + \frac{p'+q'}{2}\right)\left(s + 1 + \frac{p'-q'}{2}\right)\right\}f(z) = 0\,. \end{aligned}
$$

This is a hypergeometric differential equation which just governs

$$
F\left(\frac{1}{2}\left(s + \frac{p'+q'}{2}\right), \frac{1}{2}\left(s + 1 + \frac{p'+q'}{2}\right), \frac{p'+1}{2}; z\right).
$$

Another independent solution of (7.13) is not real analytic in a neighbourhood of $z = 0$ under the condition $p' \geqq 1$. Hence

$$
(7.14)\quad \begin{aligned} u_s(a_t) = (\cosh t)^{-(s+(p'+q')/2)}F\Bigg(&\frac{1}{2}\left(s + \frac{p'+q'}{2}\right), \\ &\frac{1}{2}\left(s + 1 + \frac{p'-q'}{2}\right), \frac{p'+1}{2}; (\tanh t)^2\Bigg) \end{aligned}
$$

is the unique real analytic solution of the equation (7.12) up to a constant factor. Applying the following well-known formulas

$$
(7.15)\quad \begin{aligned} &\lim_{x\to 1-0} F(\alpha, \beta, \gamma; x) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \quad \text{if } \begin{array}{l}\operatorname{Re}\gamma > 0 \\ \operatorname{Re}(\gamma-\alpha-\beta) > 0\end{array} \\ &\lim_{x\to 1-0} (1-x)^{\alpha+\beta-\gamma}F(\alpha, \beta, \gamma; x) = \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \\ &\qquad\qquad \text{if } \begin{array}{l}\operatorname{Re}\gamma > 0 \\ \operatorname{Re}(\alpha+\beta-\gamma) > 0\,,\end{array} \end{aligned}
$$

we obtain

$$
\begin{cases} \lim\limits_{t\to+\infty} e^{(s+(p'+q')/2)t}u_s(a_t) \\ \quad = 2^{s+(p'+q')/2}\dfrac{\Gamma((p'+1)/2)\Gamma(-s)}{\Gamma(\frac{1}{2}((p'-q')/2+1-s))\Gamma(\frac{1}{2}((p'+q')/2-s))} \\ \qquad\qquad \text{if} \quad \operatorname{Re} s < 0 \end{cases}
$$

$$(7.16)\quad \begin{cases} \lim_{t\to+\infty} e^{((p'+q')/2-s)t}u_s(a_t) \\ \quad = 2^{(p'+q')/2-s}\dfrac{\Gamma((p'+1)/2)\Gamma(s)}{\Gamma(\frac{1}{2}(s+(p'+q')/2))\Gamma(\frac{1}{2}(s+1+(p'-q')/2))} \\ \qquad\qquad\qquad\qquad \text{if} \quad \operatorname{Re} s > 0 . \end{cases}$$

The decomposition [DII′] in § 1 guarantees that u_s is uniquely extended to a real analytic function on G by defining $u_s(kak') = u_s(a)$ for $k \in K$, $a \in A_0$, $k' \in K'$.

LEMMA 7.3. *If $u(g)$ is contained in $\mathscr{B}(G/K'; \mathscr{M}_s)$ and is left K-invariant, then $u(g)$ is real analytic and is a constant multiple of $u_s(g)$.*

Proof. Since the Casimir operator ω is contained in the center of $U(\mathfrak{g})$, we may regard $u(g)$ as a function which is an eigenfunction of the Laplace-Beltrami operator on $K\backslash G$. Then $u(g)$ must be real analytic because the Laplace-Beltrami operator on $K\backslash G$ is elliptic. Furthermore, it is easy to see that $u(a_t)$ satisfies the differential equation (7.12). Since we have already proved that $u_s(a_t)$ is the unique analytic solution of (7.12) up to a constant factor, we get $u(a_t) = cu_s(a_t)$ with a constant c. Q.E.D.

Proof of Theorem 7.1. Since $\varphi_s(g)$ satisfies the assumption of Lemma 7.3,

$$(7.17)\qquad \varphi_s(g) = cu_s(g)$$

with a constant c. On the other hand, (6.9) shows that h_s is contained in $\mathscr{B}(G/P; L_{\lambda(s)})$ and therefore $\varphi_s(g^{-1})$ can be regarded as the Poissin transform of h_s to G/K. Hence, under the assumption $2s \notin \boldsymbol{Z}$, Theorem 6.1 implies

$$(7.18)\qquad \lim_{t\to+\infty} e^{((p'+q')/2-s)t}\varphi_s(a_t) = c(\lambda(s)) \qquad \text{if} \quad \operatorname{Re} s > 0 ,$$

because Lemma 7.3 shows that $\varphi_s(a_t)$ satisfies the assumption of Theorem 5.14 in [7] and because Lemma 6.2 shows that $e(\lambda(s)) \neq 0$. The equations (7.18), (7.16), (6.10) determine the constant c. Q.E.D.

§ 8. Integral representation

Let $\delta(kM_0)$ be Dirac's delta function on K/M_0 supported at the origin. Furthermore, put

$$(8.1)\qquad \delta_s(g) = \delta(kM_0) \exp\left\{\left(s - \frac{p'+q'}{2}\right)\alpha(\log a)\right\}$$

for $g = kman$ with $k \in K$, $m \in M_0$, $a \in A_0$, $n \in N_0$. Then $\delta_s(g)$ is contained in $B(G/P_0; s)$.

LEMMA 8.1. *Assume $2s \notin Z$. If $f(g) \in \mathscr{B}(G/P_0; s)$ has the property*

$$f(mang) = f(g) \exp\left\{\left(\frac{p'+q'}{2} + s\right)\alpha(\log a)\right\} \quad \text{for } g \in G,\ man \in P_0\,, \tag{8.2}$$

then $f(g)$ is equal to $\delta_s(g)$ up to a factor.

This lemma is proved by the argument similar to the proposition in [6, Appendix I]. Hence we omit the proof.

We put

$$\begin{aligned} c(s) = {} & 2^{s+(p'+q')/2} \\ & \times \cos\frac{\pi}{2}\left(s + \frac{q'-p'}{2}\right)\Gamma\left(\frac{1}{2}\left(s + m_2 + 1 - \frac{p'+q'}{2}\right)\right)\Gamma(-s) \\ & \times \Gamma\left(\frac{p'+1}{2}\right)\Gamma\left(\frac{q'+1}{2}\right) \\ & \div \pi\Gamma\left(\frac{1}{2}\left(\frac{p'+q'}{2} - s\right)\right)\Gamma\left(\frac{m_2+1}{2}\right). \end{aligned} \tag{8.4}$$

LEMMA 8.2. *$\beta_s P_s = c(s)\delta_s$ if $2s \notin Z$.*

Proof. Since Lemma 4.5 shows that $\beta_s P_s$ satisfies the condition (8.2), we conclude by Lemma 8.1 that

$$\beta_s P_s = c\delta_s$$

with a constant c. In order to determine the constant c, it is sufficient to take the boundary value of φ_s. From the definition of φ_s, Lemma 4.5 also shows

$$\begin{aligned} \beta_s\varphi_s &= \beta_s\bar{\mathscr{P}}_s(h_s^K) \\ &= \int_K \tau_s(k)(\beta_s P_s)dk \\ &= \int_K c\tau_s(k)\delta_s dk \\ &= ch_s^K\,. \end{aligned}$$

On the other hand, since $\varphi_s(a_t)$ is expressed in terms of the hypergeometric function (see Theorem 7.1), the assumption of Proposition 5.14 in [7] is fulfilled for $\varphi_s(g)$ under the condition $2s \notin Z$ and furthermore

https://doi.org/10.1017/S0027763000019000 Published online by Cambridge University Press

(8.5) $$c = \lim_{t\to\infty} e^{(s-(p'+q')/2)t}\varphi_s(a_t) \quad \text{if} \quad \operatorname{Re} s < 0 .$$

Then by Theorem 7.1, we can conclude $c = c(s)$. Q.E.D.

LEMMA 8.3. *β_s is injective if $2s \notin Z$.*

Proof. For any u in $\mathscr{B}(G/K'; \mathscr{M}_s)$, we consider the integral

(8.6) $$U_g(g') = \int_K u(gkg')dk$$

in order to prove this lemma. Since U_g is contained in $\mathscr{B}(G/K'; \mathscr{M}_s)$ and left K-invariant, Lemma 7.3 and Theorem 7.1 yield

(8.7) $$U_g(g') = v(g)\varphi_s(g')$$

with a constant $v(g)$ only depending on g. Taking the boundary value of U_g corresponding to the characteristic exponent $\frac{1}{2}(-s + (p' + q')/2)$, we get

$$\begin{aligned} v(g)c(s)h_s^K &= v(g)\beta_s\varphi_s \\ &= \beta_s U_g \\ &= \int_K \tau_s((gk)^{-1})(\beta_s u)dk \\ &= \left\{\int_K (\beta_s u)(gk)\, dk\right\} h_s^K . \end{aligned}$$

(As to the definition of h_s^K, see § 7.) Hence we obtain the integral representation of $v(g)$:

(8.8) $$c(s)v(g) = \int_K (\beta_s u)(gk)dk .$$

We remark that this equation also holds if we replace s by $-s$.

We now assume that $\beta_s u = 0$ for a u in $\mathscr{B}(G/K'; \mathscr{M}_s)$. Then (8.8) shows $v(g) = 0$ because $c(s) \neq 0$ under the assumption $2s \notin Z$. Hence, by the above remark, we get

(8.9) $$\int_K (\beta_{-s} u)(gk)dk = 0 .$$

Since the left hand side of (8.9) is regarded as the Poisson transform of $\beta_{-s}u$ to G/K, (8.9), (6.9) and Theorem 6.1 imply

(8.10) $$\beta_{-s}u = 0 .$$

Hence all the boundary values of u vanish and therefore we can conclude $u=0$ from Proposition 2.15 in [14] and the G-equivariance of the map β_s. Q.E.D.

THEOREM 8.4. *If* $2s \notin Z$, *the Poisson transformation* $\tilde{\mathscr{P}}_s$ *is a G-isomorphism of* $\mathscr{B}(G/P_0; s)$ *onto* $\mathscr{B}(G/K'; \mathscr{M}_s)$.

Proof. This theorem is an easy consequence of Lemmas 8.2 and 8.3. Actually for any u in $\mathscr{B}(G/K'; \mathscr{M}_s)$ we consider $u' = u - (1/c(s))\tilde{\mathscr{P}}_s\beta_s u$. Then Theorem 5.4 shows that u' is contained in $\mathscr{B}(G/K'; \mathscr{M}_s)$ and $\beta_s u' = 0$ from its definition. Hence Lemma 8.3 implies $u' = 0$. This means that $c(s)u = \tilde{\mathscr{P}}_s\beta_s u$. This and Lemma 8.2 shows that $\tilde{\mathscr{P}}_s$ and β_s are mutually inverse mappings up to a non-zero constant factor. Hence the theorem.

PROPOSITION 8.5. *Let us define*

$$\mathscr{B}_{K'}(G/K'; \mathscr{M}_s) = \{u \in \mathscr{B}(G/K'; \mathscr{M}_s); \pi_s(k)u = u \text{ for any } k \in K'\} .$$

If $2s \notin Z$, *then*

$$\dim_C \mathscr{B}_{K'}(G/K'; \mathscr{M}_s) = 1 .$$

Proof. Let u be a function of $\mathscr{B}_{K'}(G/K'; \mathscr{M}_s)$. Then Theorem 8.4 implies that $u = \tilde{\mathscr{P}}_s f$ for an element $f \in \mathscr{B}(G/P_0; s)$. Since the map β_s is G-equivariant, we have $\tau_s(k)f = f$ for any $k \in K'$. Then Lemma 5.1 shows that f is equal to h_s up to a factor. Hence the proposition.

Remark. A function u of $\mathscr{B}_{K'}(G/K'; \mathscr{M}_s)$ is an analogue of the notion of a zonal spherical function on G/K.

Added in proof. Recently Professor T. Oshima announced without proof that a similar result as Theorem 8.4 is obtained for an arbitrary affine symmetric space in T. Oshima, "Poisson transformations on affine symmetric spaces", Proc. of Japan Acad. 55, Ser. A (1979).

REFERENCES

[1] J. Faraut, Distributions spheriques sur les espace pseudo-Riemanniens et les hyperboloides (preprint).
[2] I. M. Gel'fand and G. E. Shilov, Generalized functions, vol. I, 1964, Academic Press, New York and London.
[3] Harish-Chandra, Spherical functions on a semisimple Lie group I, Amer. J. Math., **80** (1958), 241–310.
[4] S. Helgason, A duality for symmetric spaces with applications to group representations, Advances in Math., **5** (1970), 1–154.

[5] S. Helgason, Invariant differential equations on homogeneous manifolds, Bull. Amer. Math. Soc. vol. **83** (1977), 751–774.
[6] M. Kashiwara, K. Kowata, K. Minemura, K. Okamoto, T. Oshima, and M. Tanaka, Eigenfunctions of invariant differential operators on a symmetric space, Ann. of Math., **107** (1978), 1–39.
[7] M. Kashiwara and T. Oshima, Systems of differential equations with regular singularities and their boundary value problems, Ann. of Math., **106** (1977), 145–200.
[8] S. S. Koh, On affine symmetric spaces, Trans. Amer. Math. Soc., **119** (1965), 291–309.
[9] T. Matsuki, The orbits of affine symmetric spaces under the action of minimal parabolic subgroups, J. Math. Soc. Japan, **31**, No. 2 (1979), 331–357.
[10] S. Matsumoto, K. Hiraoka and K. Okamoto, Eigenfunctions of the laplacian on a real hyperboloid of one sheet, Hiroshima Math. J., **7** (1978), 855–864.
[11] K. Minemura, Eigenfunctions of the laplacian on a real hyperbolic space, J. Math. Soc. Japan, **27** (1975), 82–105.
[12] T. Oshima, Boundary value problem for symmetric spaces corresponding to various boundaries, RIMS Kōkyūroku, **281** (1976), 211–226 (in Japanese).
[13] ——, On a realization of Riemannian symmetric spaces, J. Math. Soc. Japan, **30** (1978), 117–132.
[14] T. Oshima and J. Sekiguchi, Eigenspace of invariant differential operators on an affine symmetric space, preprint.
[15] W. Rossmann, Analysis on real hyperbolic spaces, J. Functional Analysis, **30** (1978), 448–477.
[16] N. R. Wallach, Harmonic analysis on homogeneous spaces, Marcel Dekker, Inc., New York.

Department of Mathematics
Tokyo Metropolitan University