ON ORIENTATIONS, CONNECTIVITY AND ODD-VERTEX-PAIRINGS IN FINITE GRAPHS

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1. Introduction. The integer part of a non-negative real number p will be denoted by $[p]$. For any integer n, n^* will denote the greatest even integer less than or equal to n, that is, $n^* = n$ or $n - 1$ according as n is even or odd respectively.

The *order* of a set A, denoted by $|A|$, is the number of elements in A. The set whose elements are $a_1, a_2, \ldots, a_n$ will be denoted by $\{a_1, a_2, \ldots, a_n\}$. The empty set will be denoted by Λ. A set will be said to *include* each of its elements. A set *separates* two elements if it includes one but not both of them.

An *unoriented graph* U consists of two disjoint sets $V(U)$, $E(U)$, the elements of $V(U)$ being called *vertices* of U and the elements of $E(U)$ being called *edges* of U, together with a relationship whereby with each edge is associated an unordered pair of distinct vertices which the edge is said to *join*. The letter U, without further introduction, will always denote an unoriented graph. An *oriented graph* N consists of two disjoint sets $V(N)$, $E(N)$, the elements of $V(N)$ being called *vertices* of N and the elements of $E(N)$ being called *edges* of N, together with a relationship whereby with each edge λ is associated an *ordered* pair $(\lambda t, \lambda h)$ of distinct vertices called the *tail* and *head* of λ respectively; the statement that λ *joins* two vertices ξ and η will mean that *either* $\xi = \lambda t$ and $\eta = \lambda h$ *or* $\xi = \lambda h$ and $\eta = \lambda t$. The letter N, without further introduction, will always denote an oriented graph. An *orientation* of U is any one of the $2^{|E(U)|}$ oriented graphs N such that $V(N) = V(U)$, $E(N) = E(U)$ and each edge of N joins the same vertices in N as in U.

Let G be an unoriented or oriented graph. Then G is *finite* or *infinite* according as the set $V(G) \cup E(G)$ is finite or infinite. Henceforward, except when the contrary is explicitly indicated, all graphs mentioned in this paper will be finite, and the word "graph" will mean "finite graph." An edge of G is *incident* with each of the vertices which it joins. If S, T are subsets of $V(G)$, $\bar{S}$ will denote $V(G) - S$, $S \circ T$ will denote the set of those edges which join elements of S to elements of T, and $S\delta$ will denote $S \circ \bar{S}$. The *degree* of S, denoted by $d(S)$, is $|S\delta|$. The *degree* $d(\xi)$ of a vertex ξ of G is the number of edges incident with ξ; thus $d(\xi) = d(\{\xi\})$. A *path* of G is a finite sequence

$$\xi_0, \lambda_1, \xi_1, \lambda_2, \xi_2, \lambda_3, \ldots, \lambda_n, \xi_n$$

in which the ξ_i are vertices of G, the λ_i are edges of G and $\lambda_i \in \{\xi_{i-1}\} \circ \{\xi_i\}$ $(i = 1, 2, \ldots, n)$. A path with first term ξ and last term η is a *$\xi\eta$-path*. A

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collection of paths are *edge-disjoint* if no edge appears in more than one of them. The *connectivity* $c(\xi, \eta)$ of two distinct vertices ξ, η is the minimum of the degrees of the subsets of $V(G)$ which separate them. It can be shown[1] that $c(\xi, \eta)$ is also the maximum number of edge-disjoint $\xi\eta$-paths which can be found in G. G is *k-connected* if $d(S) \geqslant k$ for every non-empty proper subset S of $V(G)$.

A path

$$\xi_0, \lambda_1, \xi_1, \lambda_2, \xi_2, \lambda_3, \ldots, \lambda_n, \xi_n$$

of N is *forwards-directed* if $\lambda_i t = \xi_{i-1}$ (and so necessarily $\lambda_i h = \xi_i$) for $i = 1, 2, \ldots, n$. If $S \subset V(N)$, an edge λ is an *exit* of S if $\lambda t \in S$, $\lambda h \in \bar{S}$, and is an *entry* of S if $\lambda h \in S$, $\lambda t \in \bar{S}$. The number of entries $\langle$exits$\rangle$ of S will be denoted by $e(S)$ $\langle x(S)\rangle$. If ξ, η are distinct vertices of N, $a(\xi, \eta)$ (the *coefficient of accessibility* of η from ξ) is defined to be the minimum of the values of $x(S)$ as S runs through those subsets of $V(N)$ which include ξ but not η. It can be shown[2] that $a(\xi, \eta)$ is also the maximum number of edge-disjoint forwards-directed $\xi\eta$-paths which can be found in N. N is *k-accessible* if $x(S) \geqslant k$ for every non-empty proper subset S of $V(N)$. N is *admissible* if $a(\xi, \eta) \geqslant [\frac{1}{2}c(\xi, \eta)]$ for every ordered pair ξ, η of distinct vertices of N.

Robbins **(4)** proved that every 2-connected unoriented graph has an orientation in which every vertex is accessible from every other. Such an orientation is clearly 1-accessible, since, if $S \subset V(N)$ and $x(S) = 0$, no element of $\bar{S}$ is accessible from any element of S. This suggests the generalization that, for every positive integer k, every $2k$-connected unoriented graph has a k-accessible orientation. ($2k$-connectedness is of course a *necessary* condition for possessing a k-accessible orientation, since $d(S) = x(S) + x(\bar{S})$ for every subset S of the vertices of an oriented graph.) Since an unoriented $\langle$oriented$\rangle$ graph is clearly k-connected $\langle k$-accessible$\rangle$ if and only if $c(\xi, \eta)$ $\langle a(\xi, \eta)\rangle \geqslant k$ for every pair $\langle$ordered pair$\rangle$ of distinct vertices ξ, η, our proposed generalization of Robbins' theorem states that, if $c(\xi, \eta) \geqslant 2k$ for every pair ξ, η of distinct vertices of U, then U has an orientation in which $a(\xi, \eta) \geqslant k$ for every ordered pair ξ, η of distinct vertices (k being a positive integer). This in turn suggests the following sharper result, which it is the object of this paper to prove:

THEOREM 1. *Every unoriented graph has an admissible orientation.*

Robbins' theorem was extended to infinite graphs by Egyed **(2)**. An extension of Theorem 1 to infinite graphs has been obtained, but the details, being somewhat heavy, are deferred to a possible future paper.

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[1]Since this result is relevant to the present paper only as a slight additional motivation for the definition of connectivity, we omit its proof. It can be proved on lines suggested by the proof of Menger's Theorem on pp. 244–247 of **(3)**.

[2]This result is mentioned only as additional motivation for the definition of $a(\xi, \eta)$, and its proof is omitted.

A vertex of U is *even* or *odd* according as its degree is even or odd respectively. A *partition* of a set A is a set of disjoint subsets of A whose union is A. A *pair-set* of A is a set of subsets of order 2 of A. If P is a pair-set of A and $B \subset A$, the subset P_B of P is defined to be the set of those pairs $\{\alpha, \beta\} \in P$ such that B separates α, β. If $S \subset V(U)$ and P is a pair-set of $V(U)$, the *P-reduced degree* $d^P(S)$ of S is $d(S) - |P_S|$. The *P-reduced connectivity* $c^P(\xi, \eta)$ of two distinct vertices ξ, η is the minimum of the P-reduced degrees of the subsets of $V(U)$ which separate them. An *odd-vertex-pairing* of U is a partition of the set of odd vertices of U into subsets of order 2; such a partition exists since, by (**3**, chapter II, Theorem 3), the number of odd vertices of U is even[3]. We shall show in §2 that, if P is an odd-vertex-pairing of U and ξ, η are distinct vertices of U, then $c^P(\xi, \eta) \leqslant c(\xi, \eta)^*$. P will be called *optimal* if $c^P(\xi, \eta) = c(\xi, \eta)^*$ for every pair ξ, η of distinct vertices of U. Our proof of Theorem 1 will depend on the following subsidiary result:

THEOREM 2. *Every unoriented graph has an optimal odd-vertex-pairing.*

2. Proof of Theorem 2.

LEMMA 1. *If α, β, γ are distinct elements of a set A and $B \subset A$, then*

$$|\{\{\alpha, \beta\}\}_B| + |\{\{\beta, \gamma\}\}_B| \geqslant |\{\{\alpha, \gamma\}\}_B|.$$

(In accordance with our definitions, the notation $\{\{\theta, \phi\}\}_B$ means P_B where P is the pair-set whose sole member is $\{\theta, \phi\}$.)

The proof of Lemma 1 is left to the reader.

Definition. Let A be a set, P be a pair-set of A and B, C be subsets of A. Then $P(B, C)$ will denote the number of pairs $\{\alpha, \beta\} \in P$ such that one of α, β belongs to B and the other to C.

LEMMA 2. *Let S, T be subsets of $V(U)$ and P be a pair-set of $V(U)$. Then*
(i) $d(S) + d(T) = d(S \cap T) + d(\bar{S} \cap \bar{T}) + 2|(S \cap \bar{T}) \circ (\bar{S} \cap T)|$;
(ii) $d^P(S) + d(T) \geqslant \frac{1}{2}(d^P(S \cap T) + d^P(S \cap \bar{T}) + d^P(\bar{S} \cap T) + d^P(\bar{S} \cap \bar{T}))$;
(iii) *if $P_T = \{\{\theta, \phi\}\}$, where $\theta \in T$ and $\phi \in \bar{T}$, then*

$$d^P(S) + d(T) \geqslant d^P(S \cap \bar{T}) + d^P(\bar{S} \cap T) - 1,$$

and this inequality can only become an equality if $\theta \in S \cap T$, $\phi \in \bar{S} \cap \bar{T}$.

Proof. Write

$$S \cap T = Z_1,\ S \cap \bar{T} = Z_2,\ \bar{S} \cap T = Z_3,\ \bar{S} \cap \bar{T} = Z_4,$$
$$d_{ij}(= d_{ji}) = |Z_i \circ Z_j|,\ p_{ij}(= p_{ji}) = P(Z_i, Z_j).$$

Then (i) and (ii) are easily proved by expressing all terms on each side of (i) and (ii) in terms of the d_{ij} and p_{ij}—for example,

[3]This result also follows by putting $S = V(U)$ in Lemma 3 of this paper.

$$d(S) = d_{13} + d_{14} + d_{23} + d_{24}$$
$$d^P(S) = d_{13} + d_{14} + d_{23} + d_{24} - p_{13} - p_{14} - p_{23} - p_{24}.$$

It can also be shown by this method that

$$d^P(S) + d(T) - d^P(S \cap \bar{T}) - d^P(\bar{S} \cap T) \geqslant |P_T| - 2P(S \cap T, \bar{S} \cap \bar{T}),$$

which clearly implies (iii).

Definitions. If $S \subset V(U)$, $o(S)$ will denote the number of odd elements of S. An odd-vertex-pairing P of U is *S-optimal* if $c^P(\xi, \eta) = c(\xi, \eta)^*$ for every pair ξ, η of distinct elements of S. We define $c(S)$ to be 0 if $S = \Lambda$ or $V(U)$, and to be

$$\max_{\xi \in S, \eta \in \bar{S}} c(\xi, \eta)$$

otherwise.

If m, n are integers, the statement "$m \equiv n \pmod 2$" will be abbreviated to "$m \equiv n$".

Lemma 3. *If $S \subset V(U)$, $d(S) \equiv o(S)$.*

Proof. If Σ denotes the sum of the degrees of the elements of S, an edge contributes 2, 1 or 0 to Σ according as it belongs to $S \circ S$, $S\delta$ or $\bar{S} \circ \bar{S}$ respectively. Therefore $\Sigma \equiv |S\delta| = d(S)$. But clearly $\Sigma \equiv o(S)$.

Corollary 3A. *If P is an odd-vertex-pairing of U, $d^P(S)$ is even.*

Proof. Clearly $|P_S| \equiv o(S)$; therefore, by Lemma 3, $|P_S| \equiv d(S)$.

Corollary 3B. *If ξ, η are distinct vertices of U, $c^P(\xi, \eta) \leqslant c(\xi, \eta)^*$.*

Proof. Clearly $c^P(\xi, \eta) \leqslant c(\xi, \eta)$. But $c^P(\xi, \eta)$ is even, by Corollary 3A. Therefore $c^P(\xi, \eta) \leqslant c(\xi, \eta)^*$.

Corollary 3C. *If $Y \subset V(U)$, P is Y-optimal if and only if $d^P(S) \geqslant c(\xi, \eta)^*$ for every triple S, ξ, η such that $S \subset V(U)$, $\xi \in S \cap Y$ and $\eta \in \bar{S} \cap Y$.*

Proof. The given condition is equivalent to the assertion that, for every pair ξ, η of distinct elements of Y, $c^P(\xi, \eta) \geqslant c(\xi, \eta)^*$; and this inequality is equivalent to equality by Corollary 3B.

Corollary 3D. *P is optimal if and only if $d^P(S) \geqslant c(S)^*$ for every subset S of $V(U)$.*

Proof. Take $Y = V(U)$ in Corollary 3C.

Notational Conventions. When, to avoid ambiguity, it is necessary to specify the graph relative to which a graph-theoretical symbol is defined, the letter denoting the graph will be attached to the symbol in some convenient way. For example, if ξ is a common vertex of two graphs G and H, $d_G(\xi)$ will denote the degree of ξ in G. We shall, however, make the convention that, whenever two or more graphs are under consideration and one of them is denoted

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by the letter U, all graph-theoretical symbols relate to U unless the contrary is specified—for example, $d(\xi)$, if otherwise ambiguous, means $d_U(\xi)$.

Definitions. A *subgraph* of U is an unoriented graph H such that $V(H) \subset V(U)$, $E(H) \subset E(U)$ and each edge of H joins the same vertices in H as in U. If X is a subset of $V(U)$, U_X will denote the unoriented graph defined by

(i) $V(U_X) = \bar{X} \cup \{X'\}$, $E(U_X) = \bar{X} \circ V(U)$, where X' is a newly introduced vertex and is not an element of the set $V(U) \cup E(U)$;

(ii) each element of $\bar{X} \circ \bar{X}$ joins the same vertices in U_X as in U;

(iii) if $\xi \in \bar{X}$ and $\lambda \in \{\xi\} \circ X$, then λ joins ξ and X' in U_X.

Thus U_X is obtained from U by contracting to a single vertex X' the subgraph of U formed by the elements of X and those of $X \circ X$.

LEMMA 4. *If $Z \subset X \subset V(U)$ and P is an optimal odd-vertex-pairing of $U_{\bar{X}}$, then $d(Z) \geqslant |P_Z| + c(Z)^*$.*

Proof. Let $U_{\bar{X}} = H$. Since $d(Z) = d_H(Z) \geqslant |P_Z| + c_H(Z)^*$ by Corollary 3D, it suffices to prove that $c_H(Z) \geqslant c(Z)$. This will clearly follow if we show that

(i) if ξ, η are distinct elements of X, then $c_H(\xi, \eta) \geqslant c(\xi, \eta)$,

(ii) if $\zeta \in X$, $\tau \in \bar{X}$, then $c_H(\zeta, \bar{X}') \geqslant c(\zeta, \tau)$.

Let W denote an arbitrary subset of $V(H)$, and T denote whichever of W, $V(H) - W$ does not include $\bar{X}'$. Then (i) follows from the fact that, if W separates ξ and η, $d_H(W) = d(T) \geqslant c(\xi, \eta)$, and (ii) from the fact that, if W separates ζ and $\bar{X}'$, $d_H(W) = d(T) \geqslant c(\zeta, \tau)$.

LEMMA 5. *If S, $T \subset V(U)$, then either $c(S \cap T)^* \geqslant c(S)^* = c(\bar{S})^*$ or $c(S \cap \bar{T})^* \geqslant c(S)^* = c(\bar{S})^*$.*

Proof. If $S = \Lambda$ or $V(U)$, the result is trivial. If not, select $\xi \in S$, $\eta \in \bar{S}$ such that $c(\xi, \eta) = c(S)$. If $\xi \in T$, $S \cap T$ separates ξ, η and so

$$c(S \cap T) \geqslant c(\xi, \eta) = c(S),$$

whence $c(S \cap T)^* \geqslant c(S)^*$. Similarly, if $\xi \in \bar{T}$, then $c(S \cap \bar{T})^* \geqslant c(S)^*$. It is obvious that $c(S)^* = c(\bar{S})^*$.

LEMMA 6. *If S, $T \subset V(U)$, then either*

$$c(S \cap T)^* + c(\bar{S} \cap \bar{T})^* \geqslant c(S)^* + c(T)^* \tag{1}$$

or

$$c(S \cap \bar{T})^* + c(\bar{S} \cap T)^* \geqslant c(S)^* + c(T)^*. \tag{2}$$

Proof. We may assume without loss of generality that $c(S)^* \leqslant c(T)^*$. Then, if $c(S \cap T)^* < c(S)^*$, we have $c(S \cap T)^* < c(T)^*$ also; these two inequalities and Lemma 5 give $c(S \cap \bar{T})^* \geqslant c(S)^*$ and $c(\bar{S} \cap T)^* \geqslant c(T)^*$, whence (2) is true. Similarly, if $c(\bar{S} \cap T)^* < c(S)^*$, then $c(\bar{S} \cap T)^* < c(T)^*$;

these inequalities and Lemma 5 give $c(\bar{S} \cap \bar{T})^* \geqslant c(S)^*$, $c(S \cap T)^* \geqslant c(T)^*$, whence (1) is true. If, finally, $c(S \cap T)^* \geqslant c(S)^*$ and $c(\bar{S} \cap \bar{T})^* \geqslant c(S)^*$, then, since either $c(\bar{S} \cap \bar{T})^* \geqslant c(T)^*$ or $c(S \cap \bar{T})^* \geqslant c(T)^*$ by Lemma 5, either (1) or (2) respectively is true.

Definitions. Let $\xi, \eta \in V(U)$. Then a subset X of $V(U)$ is *$\xi\eta$-critical* if X separates ξ, η and $d(X) = c(\xi, \eta) \equiv 0$. A subset of $V(U)$ is *critical* if it is $\xi\eta$-critical for some pair ξ, η of vertices of U.

LEMMA 7. *If X is a critical subset of $V(U)$, then $d(X) = c(X)^*$.*

Proof. Since X is critical, it is $\xi_0\eta_0$-critical for some $\xi_0 \in X, \eta_0 \in \bar{X}$. Therefore $c(\xi_0, \eta_0) = d(X)$. But $c(\xi, \eta) \leqslant d(X)$ for every $\xi \in X, \eta \in \bar{X}$ by the definition of $c(\xi, \eta)$. Therefore $c(X) = d(X)$. Moreover, $d(X) \equiv 0$ since X is $\xi_0\eta_0$-critical. Since $c(X) = d(X) \equiv 0$, it follows that $d(X) = c(X)^*$.

Definitions. The *order* of U, denoted by ord U, is $|V(U) \cup E(U)|$. If $\lambda \in E(U)$, $U - \lambda$ will denote the subgraph of U defined by $V(U - \lambda) = V(U)$, $E(U-\lambda) = E(U) - \{\lambda\}$. If $\xi \in V(U)$, $\epsilon(\xi)$ is defined to be 0 or 1 according as ξ is even or odd respectively. A subset S of $V(U)$ is *vertical* (in the sense of "pertaining to a vertex") if either $|S| = 1$ or $|\bar{S}| = 1$. For every subset S of $V(U)$, $S\delta$ is called a *cincture*[4] of U. Two sets *meet* if they have at least one element in common. A subset S of $V(U)$ *divides* a subset Y of $V(U)$ if both S and $\bar{S}$ meet Y. A subset of $V(U)$ is *Y-minimal* if (i) it divides Y and (ii) its degree is minimal amongst the degrees of those subsets of $V(U)$ which divide Y. A subset of $V(U)$ is *Y-critical* if it is $\xi\eta$-critical for some pair ξ, η of elements of Y. A *Y-critical* cincture is a cincture of the form $X\delta$, where X is a Y-critical subset of $V(U)$.

We shall now suppose that U is a given unoriented graph, and make the inductive hypothesis that *every unoriented graph of lower order than U has an optimal odd-vertex-pairing*. By deducing that U has one also, we shall clearly establish Theorem 2.

LEMMA 8. *If $V(U)$ has a non-vertical critical subset, U has an optimal odd-vertex-pairing.*

Proof. Let X be a non-vertical critical subset of $V(U)$, and let $H = U_{\bar{X}}$, $K = U_X$. The definition of "critical" implies that X and $\bar{X}$ are non-empty; hence, since X is non-vertical, $|X| \geqslant 2$ and $|\bar{X}| \geqslant 2$. Therefore ord $H <$ ord U and ord $K <$ ord U. Therefore, by the inductive hypothesis, there exist optimal odd-vertex-pairings P, Q of H, K respectively. Since X is critical, $d(X)$ is even. Therefore $\bar{X}'$, X' are even in H, K respectively. Moreover, each element of $X, \bar{X}$ has clearly the same degree in U as in H, K respectively. Therefore $P \cup Q$ $(=R$, say) is an odd-vertex-pairing of U. We will show that R is optimal in U.

[4]This term is taken from **(1)**.

If $S \subset V(U)$, then, by Lemmas 2 (i) and 4,

$$\begin{aligned} d(S) + d(X) &\geqslant d(S \cap X) + d(\bar{S} \cap \bar{X}) \\ &\geqslant |P_{S \cap X}| + c(S \cap X)^* + |Q_{\bar{S} \cap \bar{X}}| + c(\bar{S} \cap \bar{X})^* \\ &= |R_S| + c(S \cap X)^* + c(\bar{S} \cap \bar{X})^*, \end{aligned}$$

whence

(3) $$d^R(S) + d(X) \geqslant c(S \cap X)^* + c(\bar{S} \cap \bar{X})^*.$$

If $T \subset V(U)$, application of (3) for $S = T$ and $S = \bar{T}$ gives

$$\begin{aligned} d^R(T) + d(X) &\geqslant \max\,(c(T \cap X)^* + c(\bar{T} \cap \bar{X})^*, c(\bar{T} \cap X)^* + c(T \cap \bar{X})^*) \\ &\geqslant c(T)^* + c(X)^* \end{aligned}$$

by Lemma 6. But $d(X) = c(X)^*$ by Lemma 7. Therefore $d^R(T) \geqslant c(T)^*$. Hence, by Corollary 3D, R is optimal; and Lemma 8 is proved.

LEMMA 9. *If $\lambda \in E(U)$, $Y \subset V(U)$ and λ belongs to no Y-critical cincture, then*

(4) $$c_{U-\lambda}(\xi, \eta)^* = c(\xi, \eta)^*$$

for each pair ξ, η of distinct elements of Y.

Proof. Let ξ, η be distinct elements of Y. It is clear that

$$c_{U-\lambda}(\xi, \eta) = c(\xi, \eta)$$

(which implies (4)) unless there is a subset X of $V(U)$ such that X separates ξ and η, $\lambda \in X\delta$ and

(5) $$d(X) = c(\xi, \eta).$$

If there is such an X, then clearly $c_{U-\lambda}(\xi, \eta) = c(\xi, \eta) - 1$, so that (4) still holds if $c(\xi, \eta)$ is odd. But $c(\xi, \eta)$ cannot now be even, since this, together with (5), would imply that X was $\xi\eta$-critical and hence Y-critical, so that $X\delta$ would be a Y-critical cincture including λ.

LEMMA 10. *If $Y \subset V(U)$ and some edge λ of U belongs to no Y-critical cincture, then U has a Y-optimal odd-vertex-pairing.*

Proof. Let α, β be the vertices joined by λ. By the inductive hypothesis, we can select an optimal odd-vertex-pairing P of $U - \lambda$. Since α, β each have different parities in $U - \lambda$ and U, and every other vertex has the same parity in each graph, an odd-vertex-pairing R of U may be defined as follows:

(i) if α, β are both odd in U, let $R = P \cup \{\{\alpha, \beta\}\}$;

(ii) if α is even and β odd in U, let $R = (P - \{\{\alpha, \sigma\}\}) \cup \{\{\beta, \sigma\}\}$, where σ is the vertex paired with α by P;

(iii) if α is odd and β even in U, let $R = (P - \{\{\beta, \tau\}\}) \cup \{\{\alpha, \tau\}\}$, where τ is paired with β by P;

(iv) if α, β are both even in U and $\{\alpha, \beta\} \in P$, let $R = P - \{\{\alpha, \beta\}\}$;

(v) if α, β are both even in U and $\{\alpha, \beta\} \notin P$, let

$$R = (P - \{\{\alpha, \sigma\}, \{\beta, \tau\}\}) \cup \{\{\sigma, \tau\}\},$$

where σ, τ are paired with α, β respectively by P.

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LEMMA 10A[5]. *If* $S \subset V(U)$, $|R_S| \leqslant |P_S| + |\{\{\alpha, \beta\}\}_S|$.

Proof. In Cases (i) and (iv), the result is clear. In Case (ii),

$$|R_S| = |P_S| - |\{\{\alpha, \sigma\}\}_S| + |\{\{\beta, \sigma\}\}_S| \leqslant |P_S| + |\{\{\alpha, \beta\}\}_S|$$

by Lemma 1, and the discussion of Case (iii) is similar. In Case (v),

$$|R_S| = |P_S| - |\{\{\sigma, \alpha\}\}_S| - |\{\{\beta, \tau\}\}_S| + |\{\{\sigma, \tau\}\}_S|,$$

which yields the required result since, by two applications of Lemma 1,

$$|\{\{\sigma, \alpha\}\}_S| + |\{\{\alpha, \beta\}\}_S| + |\{\{\beta, \tau\}\}_S| \geqslant |\{\{\sigma, \tau\}\}_S|.$$

If $S \subset V(U)$, $\xi \in S \cap Y$ and $\eta \in \bar{S} \cap Y$, then, since P is optimal in $U - \lambda$,

(6) $$c_{U-\lambda}(\xi, \eta)^* = c_{U-\lambda}^P(\xi, \eta) \leqslant d_{U-\lambda}^P(S) = d_{U-\lambda}(S) - |P_S| .$$

By Lemmas 9 and 10A and (6),

$$\begin{aligned} c(\xi, \eta)^* + |R_S| &\leqslant c_{U-\lambda}(\xi, \eta)^* + |P_S| + |\{\{\alpha, \beta\}\}_S| \\ &\leqslant d_{U-\lambda}(S) + |\{\{\alpha, \beta\}\}_S| = d(S). \end{aligned}$$

Hence, by Corollary 3C, R is Y-optimal; and Lemma 10 is proved.

LEMMA 11. *If* $Y \subset V(U)$ and $\bar{Y} \circ \bar{Y} \neq \Lambda$, *then U has a Y-optimal odd-vertex-pairing.*

Proof. Let $\lambda \in \bar{Y} \circ \bar{Y}$. If λ belongs to no Y-critical cincture, Lemma 10 gives the required result; we may therefore assume that $\lambda \in X\delta$ for some Y-critical subset X of $V(U)$. If X were vertical, it would be of the form $\{\omega\}$ or $V(U) - \{\omega\}$ for some vertex ω. But then λ would be incident with ω since $\lambda \in X\delta$, and ω would belong to Y since the Y-criticality of X requires X to separate two elements of Y; these conclusions contradict the assumption that $\lambda \in \bar{Y} \circ \bar{Y}$. Hence X must be non-vertical. Therefore, by Lemma 8, U has an optimal, and therefore Y-optimal, odd-vertex-pairing.

LEMMA 12. *If $Y \subset V(U)$ and X is a Y-minimal subset of $V(U)$, then*
(i) $c(\xi, \eta) \geqslant d(X)$ *for each pair ξ, η of distinct elements of Y;*
(ii) $c(S) \geqslant d(X)$ *for every subset S of $V(U)$ which divides Y.*

Proof. Since X is Y-minimal, $d(T) \geqslant d(X)$ for every subset T of $V(U)$ which divides Y. This fact implies (i), and (i) implies (ii).

LEMMA 13. *If $Y \subset V(U)$, $\bar{Y} \circ \bar{Y} = \Lambda$ and $V(U)$ has a non-vertical Y-minimal subset, then U has a Y-optimal odd-vertex-pairing.*

Proof. Let X be a non-vertical Y-minimal subset of $V(U)$. Then X divides Y, so that we can select two vertices $\sigma \in X \cap Y$, $\tau \in \bar{X} \cap Y$. Then $c(\sigma, \tau) \geqslant d(X)$ by Lemma 12 (i), and $c(\sigma, \tau) \leqslant d(X)$ since X separates σ, τ;

[5]We give the names Lemma nA, Lemma nB to lemmas which themselves form part of the proof of Lemma n.

hence $c(\sigma, \tau) = d(X)$. It follows that, if $d(X)$ is even, then X is critical, so that U has an optimal and therefore Y-optimal odd-vertex-pairing by Lemma 8. We shall therefore assume that $d(X)$ is odd.

Write $U_{\bar{X}} = H$, $U_X = K$. Since X divides Y, $x \neq \Lambda$ and $\bar{X} \neq \Lambda$; therefore, since X is non-vertical, $|X| \geqslant 2$ and $|\bar{X}| \geqslant 2$. Therefore ord $H <$ ord U and ord $K <$ ord U; hence H, K have, by the inductive hypothesis, optimal odd-vertex-pairings P, Q respectively. Since $d(X)$ is odd, $\bar{X}'$, X' are odd vertices of H, K respectively; let $\bar{X}'$, X' be paired with θ, ϕ by P, Q respectively. Then, since each element of X, $\bar{X}$ has the same degree in U as in H, K respectively,

$$R = (P - \{\{\theta, \bar{X}'\}\}) \cup (Q - \{\{\phi, X'\}\}) \cup \{\{\theta, \phi\}\}$$

is an odd-vertex-pairing of U.

LEMMA 13A. *If either $Z \subset X$ or $Z \subset \bar{X}$, then $d^R(Z) \geqslant c(Z)^*$.*

Proof. This follows from Lemma 4 and the obvious fact that $|R_Z| = |P_Z|$ or $|Q_Z|$ if $Z \subset X$ or $\bar{X}$ respectively.

To prove that R is Y-optimal (which will establish Lemma 13), it suffices, by Corollary 3C, to prove

LEMMA 13B. *If $S \subset V(U)$, $\xi \in S \cap Y$ and $\eta \in \bar{S} \cap Y$, then*

$$d^R(S) \geqslant c(\xi, \eta)^*. \tag{7}$$

Proof. We shall consider separately the cases (I) $S \cap X$, $S \cap \bar{X}$, $\bar{S} \cap X$, $\bar{S} \cap \bar{X}$ all meet Y; (II) $S \cap X \subset \bar{Y}$; (III) $S \cap \bar{X} \subset \bar{Y}$; (IV) $\bar{S} \cap X \subset \bar{Y}$; (V) $\bar{S} \cap \bar{X} \subset \bar{Y}$. (It suffices that these cases are jointly exhaustive; that not all pairs of them are mutually exclusive does not matter.)

Proof of (7) *in Case* I. Let Z_1 be whichever of $S \cap X$, $S \cap \bar{X}$ includes ξ and Z_2 be the other. Let Z_3 be whichever of $\bar{S} \cap X$, $\bar{S} \cap \bar{X}$ includes η and Z_4 be the other. Then

$$c(Z_1) \geqslant c(\xi, \eta),\ c(Z_3) \geqslant c(\xi, \eta) \tag{8}$$

since $\xi \in Z_1 \subset \bar{Z}_3$ and $\eta \in Z_3 \subset \bar{Z}_1$. Moreover, since the Z_i all meet Y by the hypothesis of Case I, they all divide Y; therefore

$$c(Z_i) \geqslant d(X)\ \ (i = 1, 2, 3, 4) \tag{9}$$

by Lemma 12 (ii). Using Lemma 2 (ii), Lemma 13A and (8), and using (9) for $i = 2, 4$, we obtain

$$d^R(S) + d(X) \geqslant \frac{1}{2}\sum_{i=1}^{4} d^R(Z_i) \geqslant \frac{1}{2}\sum_{i=1}^{4} c(Z_i)^* \geqslant c(\xi, \eta)^* + d(X)^*,$$

which implies (7) since $c(\xi, \eta)^*$ and (by Corollary 3A) $d^R(S)$ are even.

Proof of (7) *in Case* II. To prove (7) by *reductio ad absurdum*, let us suppose that (7) is false. Since $c(\xi, \eta)^*$ and, by Corollary 3A, $d^R(S)$ are even, the falsity of (7) implies that

$$(10) \qquad d^R(S) \leqslant c(\xi, \eta)^* - 2.$$

Since $S \cap X \subset \bar{Y}$ by the hypothesis of Case II and $\xi \in S \cap Y$, it follows that $\xi \in S \cap \bar{X}$. But $\eta \in \bar{S} \cap Y \subset V(U) - (S \cap \bar{X})$. Therefore $c(S \cap \bar{X}) \geqslant c(\xi, \eta)$, and so, by Lemma 13A,

$$(11) \qquad d^R(S \cap \bar{X}) \geqslant c(\xi, \eta)^*.$$

Since X is Y-minimal, it divides Y; therefore, since $S \cap X \subset \bar{Y}$ by the hypothesis of Case II, $\bar{S} \cap X$ divides Y. Therefore $c(\bar{S} \cap X) \geqslant d(X)$ by Lemma 12 (ii), and so

$$(12) \qquad d^R(\bar{S} \cap X) \geqslant d(X)^*$$

by Lemma 13A. By (10), Lemma 2 (iii), (11) and (12),

$$\begin{aligned} d(X) + c(\xi, \eta)^* - 2 &\geqslant d^R(S) + d(X) \\ (13) \qquad &\geqslant d^R(S \cap \bar{X}) + d^R(\bar{S} \cap X) - 1 \\ (14) \qquad &\geqslant d(X)^* + c(\xi, \eta)^* - 1 \\ &= d(X) + c(\xi, \eta)^* - 2 \end{aligned}$$

since we are assuming $d(X)$ to be odd. Hence each inequality in the above sequence must in fact be an equality. Equality in (13) implies that

$$(15) \qquad \theta \in S \cap X$$

(and $\phi \in \bar{S} \cap \bar{X}$) by Lemma 2 (iii); and equality in (14) implies equality in (11) and (12), which, in the case of (12), gives

$$(16) \qquad d^R(\bar{S} \cap X) = d(X)^*.$$

Since $\theta \in S \cap X$, $S \cap X \subset \bar{Y}$ and $\bar{Y} \circ \bar{Y} = \Lambda$ by (15), the hypothesis of Case II and a hypothesis of Lemma 13 respectively, it follows that

$$(17) \qquad \{\theta\} \circ ((S \cap X) - \{\theta\}) = \Lambda.$$

Since X, being Y-minimal, divides Y and $\theta \in \bar{Y}$ by (15) and the hypothesis of Case II, it follows that $X - \{\theta\}$ divides Y. Therefore $d(X - \{\theta\}) \geqslant d(X)$, since X is Y-minimal. But, by (15) and Lemma 2 (i),

$$\begin{aligned} d((\bar{S} \cap X) \cup \{\theta\}) + d(X - \{\theta\}) - (d(\bar{S} \cap X) + d(X)) \\ = 2|\{\theta\} \circ ((S \cap X) - \{\theta\})|, \end{aligned}$$

which vanishes by (17). It follows from the last two sentences that

$$(18) \qquad d((\bar{S} \cap X) \cup \{\theta\}) \leqslant d(\bar{S} \cap X).$$

By (15) and the facts that $\{\theta, \phi\} \in R$ and $\phi \in \bar{X}$,

$$(19) \qquad |R_{(\bar{S} \cap X) \cup \{\theta\}}| = |R_{\bar{S} \cap X}| + 1.$$

Since X, being Y-minimal, divides Y, $S \cap X \subset \bar{Y}$ by the hypothesis of Case II and $\theta \in X$, it follows that $(\bar{S} \cap X) \cup \{\theta\}$ divides Y. Therefore, by Lemma 12 (ii),

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$$c((\bar{S} \cap X) \cup \{\theta\}) \geqslant d(X). \tag{20}$$

By subtracting (19) from (18), and using Lemma 13A and (20),

$$d^R(\bar{S} \cap X) - 1 \geqslant d^R((\bar{S} \cap X) \cup \{\theta\}) \geqslant c((\bar{S} \cap X) \cup \{\theta\})^* \geqslant d(X)^*,$$

which contradicts (16). This contradiction proves by *reductio ad absurdum* the truth of (7) in Case II.

The truth of (7) in Cases III–V can be proved by arguments similar to that given for Case II. (There is no real asymmetry between X and $\bar{X}$ in our discussion, since a subset of $V(U)$ is non-vertical [Y-minimal] if and only if its complement is non-vertical [Y-minimal].) We have therefore now completed the proof of Lemma 13B and hence also that of Lemma 13.

LEMMA 14. *If $Y \subset V(U)$ and U has a Z-optimal odd-vertex-pairing for every proper subset Z of Y, then U has a Y-optimal odd-vertex-pairing.*

Proof. If $|Y| = 0$ or 1, any odd-vertex-pairing of U is vacuously Y-optimal; we may therefore assume that $|Y| \geqslant 2$. We may also, by Lemmas 11 and 13, assume that

$$\bar{Y} \circ \bar{Y} = \Lambda \tag{21}$$

and that $V(U)$ has no non-vertical Y-minimal subset. But $V(U)$ has a Y-minimal subset since $|Y| \geqslant 2$; hence it must have a vertical one. This clearly implies that $\{\omega\}$ is Y-minimal for some $\omega \in V(U)$. Since $\{\omega\}$ is Y-minimal, it divides Y; therefore $\omega \in Y$. Therefore, by the data of Lemma 14, U has a $(Y - \{\omega\})$-optimal odd-vertex-pairing P.

LEMMA 14A. *If $S \subset V(U)$, $S \cap Y = \{\omega\}$ and $\xi \in \bar{S}$, then $d^P(S) \geqslant c(\omega, \xi)^*$.*

Proof. Let $\tau \in S - \{\omega\}$, and let $A_\tau = \{\tau\} \circ (V(U) - \{\omega, \tau\})$, $B_\tau = \{\tau\} \circ \{\omega\}$. Since $S \cap Y = \{\omega\}$, $\tau \in \bar{Y}$. But $\{\omega\}$ divides Y since it is Y-minimal. Therefore $\{\omega, \tau\}$ divides Y. Therefore, since $\{\omega\}$ is Y-minimal, $d(\{\omega, \tau\}) \geqslant d(\omega)$, which implies that $|A_\tau| \geqslant |B_\tau|$. Moreover, if this last inequality becomes an equality, τ must be even, since $d(\tau) = |A_\tau| + |B\tau|$. Therefore

$$|A\tau| \geqslant |B_\tau| + \epsilon(\tau). \tag{22}$$

Furthermore, since $S \cap Y = \{\omega\}$ and $\tau \in S - \{\omega\}$, it follows from (21) that

$$A_\tau = \{\tau\} \circ \bar{S}. \tag{23}$$

Since

$$d(S) = |\{\omega\} \circ \bar{S}| + \sum_{\tau \in S - \{\omega\}} |\{\tau\} \circ \bar{S}|, \; d(\omega) = |\{\omega\} \circ \bar{S}| + \sum_{\tau \in S - \{\omega\}} |B_\tau|,$$

it follows from (22) and (23) that

$$d(S) \geqslant d(\omega) + \sum_{\tau \in S - \{\omega\}} \epsilon(\tau) = d(\omega) + o(S - \{\omega\}) = d(\omega)^* + o(S). \tag{24}$$

But obviously $o(S) \geqslant |P_S|$; and, since $\{\omega\}$ separates ω, ξ, we have $d(\omega) \geqslant c(\omega, \xi)$ and therefore $d(\omega)^* \geqslant c(\omega, \xi)^*$. Therefore, by (24), $d^P(S) \geqslant c(\omega, \xi)^*$.

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Let θ be any element of $Y - \{\omega\}$. If

(25) $$S \subset V(U),\, \omega \in S,\, \theta \in \bar{S},$$

then either $S \cap Y = \{\omega\}$, in which case $d^P(S) \geqslant c(\omega, \theta)^*$ by Lemma 14A, or S includes an element ψ of $Y - \{\omega\}$, in which case $d^P(S) \geqslant c^P(\psi, \theta) = c(\psi, \theta)^*$ since S separates ψ, θ and P is $(Y - \{\omega\})$-optimal, $c(\psi, \theta) \geqslant d(\omega)$ by Lemma 12 (i) and $d(\omega) \geqslant c(\omega, \theta)$ since $\{\omega\}$ separates ω, θ. Hence (25) implies that $d^P(S) \geqslant c(\omega, \theta)^*$. Therefore $c^P(\omega, \theta) \geqslant c(\omega, \theta)^*$, which must of course reduce to equality by Corollary 3B. Since this holds for any $\theta \in Y - \{\omega\}$, and since P is already $(Y - \{\omega\})$-optimal, it follows that P is Y-optimal. Lemma 14 is therefore proved.

Using induction on $|Y|$, we infer from Lemma 14 that U has an optimal odd-vertex-pairing. This, in turn, completes the inductive step in our proof of Theorem 2 by induction on the order of the graph.

3. Proof of Theorem 1. U is *Eulerian* if its vertices are all even. N is *quasi-symmetrical* if $x(\{\xi\}) = e(\{\xi\})$ for every $\xi \in V(N)$. If $S, T \subset V(N)$, $S \to T$ will denote the number of edges λ of N such that $\lambda t \in S, \lambda h \in T$. If H is a subgraph of U, an orientation L of U will be said to *induce* the orientation M of H such that $\lambda t_M = \lambda t_L$ and $\lambda h_M = \lambda h_L$ for every $\lambda \in E(H)$.

LEMMA 15. *If N is quasi-symmetrical and $S \subset V(N)$, then $x(S) = \frac{1}{2}d(S)$.*

Proof. Since N is quasi-symmetrical, we have

$$S \to V(N) = \sum_{\xi \in S} x(\{\xi\}) = \sum_{\xi \in S} e(\{\xi\}) = V(N) \to S.$$

Subtracting $S \to S$ from each side gives $S \to \bar{S} = \bar{S} \to S$, which clearly implies the required result.

Given any unoriented graph U, the following argument now shows that it has an admissible orientation. By Theorem 2, we can select an optimal odd-vertex-pairing P of U. Construct an unoriented graph H such that (i) $V(H) = V(U)$, (ii) U is a subgraph of H, and (iii) two distinct vertices ξ, η are joined in H by exactly one element of $E(H) - E(U)$ if $\{\xi, \eta\} \in P$ and by no such element otherwise. Since P is an odd-vertex-pairing of U, H is Eulerian and therefore (**3**, p. 30, ll. 4–9) has a quasi-symmetrical orientation Q. Let N, M be the induced orientations of U, F respectively, where F is the subgraph of H defined by $V(F) = V(H)$ $(= V(U))$, $E(F) = E(H) - E(U)$. If $S \subset V(U)$, then $x_Q(S) = \frac{1}{2}d_H(S)$ by Lemma 15 and $x_M(S) \leqslant d_F(S)$ obviously. Therefore

$$x_N(S) = x_Q(S) - x_M(S) \geqslant \tfrac{1}{2}d_H(S) - d_F(S) = \tfrac{1}{2}(d(S) - d_F(S)) = \tfrac{1}{2}d^P(S).$$

Hence, if ξ, η are distinct vertices of U and S runs through all subsets of $V(U)$ which include ξ but not η, we have

$$a_N(\xi, \eta) = \min x_N(S) \geqslant \min \tfrac{1}{2}d^P(S) = \tfrac{1}{2}c^P(\xi, \eta) = \tfrac{1}{2}c(\xi, \eta)^*,$$

since P is optimal. Since $\frac{1}{2}c(\xi, \eta)^* = [\frac{1}{2}c(\xi, \eta)]$, this proves that N is admissible.

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