Canad. Math. Bull. Vol. **49** (2), 2006 pp. 265–269

Endomorphisms That Are the Sum of a Unit and a Root of a Fixed Polynomial

W. K. Nicholson and Y. Zhou

Abstract. If $C = C(R)$ denotes the center of a ring R and $g(x)$ is a polynomial in $C[x]$, Camillo and Simón called a ring $g(x)$-clean if every element is the sum of a unit and a root of $g(x)$. If V is a vector space of countable dimension over a division ring D, they showed that $\operatorname{end}_D V$ is $g(x)$-clean provided that $g(x)$ has two roots in $C(D)$. If $g(x) = x - x^2$ this shows that $\operatorname{end}_D V$ is clean, a result of Nicholson and Varadarajan. In this paper we remove the countable condition, and in fact prove that $\operatorname{end}_R M$ is $g(x)$-clean for any semisimple module M over an arbitrary ring R provided that $g(x) \in (x - a)(x - b)C[x]$ where $a, b \in C$ and both b and $b - a$ are units in R.

An element in a ring R is called *clean* in R if it is the sum of a unit and an idempotent, and the ring itself is called a *clean ring* if every element is clean. Camillo and Yu [2] showed that all semiperfect rings and unit-regular rings are clean, and Nicholson and Varadarajan [3] showed that $\operatorname{end}_D V$ is clean for any vector space V of countable dimension over a division ring D.

More generally, if $C = C(R)$ denotes the center of the ring R, and if $g(x)$ is a polynomial in $C[x]$, Camillo and Simón [1] called the ring $g(x)$-*clean* if each element $r \in R$ has the form $r = u + s$ where u is a unit and $g(s) = 0$. They went on to show that $\operatorname{end}_D V$ is $g(x)$-clean for any vector space V of countable dimension over a division ring D. The main result of the present paper is to simultaneously remove the countable condition on the dimension of V, and to extend the result to the case when V is any semisimple module. In fact, we prove the following theorem.

Theorem 1 *Let R be a ring, let ${}_RM$ be a semisimple module over R, and write $C = C(R)$. If $g(x) \in (x - a)(x - b)C[x]$ where $a, b \in C$ are such that b and $b - a$ are both units in R, then* $\operatorname{end}_R M$ *is $g(x)$-clean.*

The following corollary extends a theorem of Camillo and Simón [1] who obtained the countable-dimensional case.

Corollary 2 *Let ${}_DV$ be a vector space over a division ring D. If $g(x)$ is a polynomial in $C(D)[x]$ with at least two roots in $C(D)$, then* $\operatorname{end}_D V$ *is $g(x)$-clean.*

Corollary 3 *If ${}_RM$ is a semisimple module over a ring R, then* $\operatorname{end}_R M$ *is a clean ring.*

Received by the editors June 1, 2004; revised January 10, 2005.
The authors were supported by NSERC Grant A8075 and Grant OGP0194196 respectively. The second author is grateful for the hospitality provided by the University of Calgary.
AMS subject classification: Primary: 16S50; secondary: 16E50.
Keywords: Clean rings, linear transformations, endomorphism rings.
©Canadian Mathematical Society 2006.

https://doi.org/10.4153/CMB-2006-027-6 Published online by Cambridge University Press

Proof of Theorem 1 Write $g(x) = c_0 + c_1x + \cdots + c_nx^n$ where each $c_i \in C$. If $c \in C$ we identify $c = c1_M \in \text{end}\,_RM$, and note that $c \in C(\text{end}\,_RM)$. Thus $g(\alpha) = c_0 + c_1\alpha + \cdots + c_n\alpha^n \in \text{end}\,_RM$ for any $\alpha \in \text{end}\,_RM$, and we must show that there exists an element $\beta \in \text{end}\,_RM$ such that $g(\beta) = 0$ and $\alpha - \beta$ is a unit in $\text{end}\,_RM$. To this end, let $\mathcal{S}$ denote the set of ordered pairs (W, β) such that

- $W \subseteq {}_RM$ is α-invariant,
- $\beta \in \text{end}\,_RW$ satisfies $g(\beta) = 0$, and
- $\alpha_{|W} - \beta$ is a unit in $\text{end}\,_RW$.

Then $(0, a) \in \mathcal{S}$ because $g(a) = 0$, so $\mathcal{S}$ is nonempty. Define a partial ordering on $\mathcal{S}$ by setting $(W, \beta) \leq (W', \beta')$ if $W \subseteq W'$ and $\beta'_{|W} = \beta$. If $\{(W_i, \beta_i) : i \in I\}$ is a chain in $\mathcal{S}$, define $\beta \in \text{end}\,(\bigcup_i W_i)$ by $\beta(w) = \beta_i(w)$ whenever $w \in W_i$. It is easy to see that $(\bigcup_i W_i, \beta) \in \mathcal{S}$ and $(W_i, \beta_i) \leq (\bigcup_i W_i, \beta)$ for each $i \in I$. Hence Zorn's lemma provides a maximal element (W, ρ) in $\mathcal{S}$; we complete the proof by showing that $W = M$.

Claim 4 *If $0 \neq m \in M$ and $W \cap Rm = 0$ then $\alpha(m) \notin W$ and $(\alpha - a)(m) \notin W$.*

Proof Suppose on the contrary that $(\alpha - c)(m) \in W$ where c denotes either 0 or a (and so $c \in C$). Then $\alpha(m) \in W \oplus Rm$ so $W \oplus Rm$ is α-invariant. Extend ρ to $W \oplus Rm$ by setting $\rho(rm) = brm$ for all $r \in R$. Then $\rho \in \text{end}\,_R(W \oplus Rm)$ because $b \in C$, and we obtain the desired contradiction by showing that $(W \oplus Rm, \rho) \in \mathcal{S}$. We have $g(\rho) = 0$ on $W \oplus Rm$ because $g(\rho) = 0$ on W and $g(\rho) = g(b) = 0$ on Rm. Hence it remains to show that $\alpha_{|W\oplus Rm} - \rho$ is a unit in $\text{end}\,_R(W \oplus Rm)$.

To see that $\alpha_{|W\oplus Rm} - \rho$ is monic, let $(\alpha - \rho)(w + rm) = 0$ where $w \in W$ and $r \in R$. Since $(\alpha - c)(m) \in W$ this gives $[(\alpha - \rho)(w) + (\alpha - c)rm] + (c - b)rm = 0$ in $W \oplus Rm$. It follows that $(c - b)rm = 0$, so $rm = 0$ because $c - b$ is a unit in R. Thus $(\alpha - \rho)(w) = 0$, so $w = 0$ because $\alpha_{|W} - \rho$ is a unit in $\text{end}\,_RW$. It follows that $\alpha_{|W\oplus Rm} - \rho$ is monic. Finally, $(\alpha - \rho)(W) = W$ because $(W, \rho) \in \mathcal{S}$. Moreover, $(\alpha - \rho)(m) = (\alpha - c)(m) + (c - b)(m) \in W \oplus Rm$, so it follows that $(c - b)(m) \in im(\alpha_{|W\oplus Rm} - \rho)$. Hence $Rm \subseteq im(\alpha_{|W\oplus Rm} - \rho)$ because $c - b$ is a unit in R. This implies that $\alpha_{|W\oplus Rm} - \rho$ is epic in $\text{end}\,_R(W \oplus Rm)$, and Claim 4 is proved.

Now suppose that $M \neq W$; we show that this leads to a contradiction. Since M is semisimple, choose $0 \neq z$ such that Rz is a simple module and $W \cap Rz = 0$. We separate the proof into two cases.

Case 1 *There exists an integer $l \geq 0$ such that there exists a linear combination*

$$d_0z + d_1(\alpha - a)(z) + \cdots + d_l(\alpha - a)^l(z) \in W, \quad d_i \in R,$$

for which at least one of the terms $d_i(\alpha - a)^i(z)$ is nonzero .

Choose l to be the smallest integer satisfying this condition. Note that $l > 0$ because $W \cap Rz = 0$, and $d_l(\alpha - a)^l(z) \neq 0$ by the choice of l. Define

$$V = Rz + R(\alpha - a)(z) + \cdots + R(\alpha - a)^{l-1}(z).$$

https://doi.org/10.4153/CMB-2006-027-6 Published online by Cambridge University Press

Then $V \neq 0$; indeed $d_1 z + d_2(\alpha - a)(z) + \cdots + d_l(\alpha - a)^{l-1}(z) \neq 0$. (Otherwise $d_l(\alpha - a)^{l-1}(z) = 0$ by the choice of l, so $d_l(\alpha - a)^l(z) = (\alpha - a)[d_l(\alpha - a)^{l-1}(z)] = 0$, a contradiction.) Moreover, $W \cap V = 0$ by the choice of l, and $d_l(\alpha - a)^l(z) \in W \oplus V$. Since $d_l(\alpha - a)^l(z) \neq 0$ and Rz is simple, we obtain $Rd_l(\alpha - a)^l(z) = (\alpha - a)^l(Rd_l z) = (\alpha - a)^l(Rz) = R(\alpha - a)^l(z)$. It follows that $(\alpha - a)^l(z) \in W \oplus V$ and so, since $a \in C$, that $W \oplus V$ is $(\alpha - a)$-invariant. Thus $W \oplus V$ is α-invariant (as $\alpha = (\alpha - a) + a$). Now extend ρ to $W \oplus V$ by setting $\rho(v) = av$ for all $v \in V$. Then $\rho \in \operatorname{end}_R(W \oplus V)$ because $a \in C$, and $g(\rho) = 0$ on $W \oplus V$ because $g(\rho) = 0$ on W and $g(\rho) = g(a) = 0$ on V. Hence we contradict the maximality of (W, ρ) by showing that $\alpha_{|W \oplus V} - \rho$ is a unit of $\operatorname{end}_R(W \oplus V)$.

Note first that $d_0 z \neq 0$. (Otherwise $(\alpha - a)(m) \in W$, where $m = d_1 z + d_2(\alpha - a)(z) + \cdots + d_l(\alpha - a)^{l-1}(z)$. But $m \neq 0$ as verified above, so $W \cap Rm \neq 0$ by Claim 4. Thus $0 \neq rm \in W$ for some $r \in R$, contradicting the choice of l.) Now observe that

$$
\begin{aligned}
(\alpha - \rho)(W \oplus V) &= (\alpha - \rho)(W) + (\alpha - \rho)(V) \\
&= W + (\alpha - a)(V) = W + \Big[\sum_{i=1}^{l} R(\alpha - a)^i(z)\Big].
\end{aligned}
$$

Since $d_0 z + d_1(\alpha - a)(z) + \cdots + d_l(\alpha - a)^l(z) \in W$, it follows that

$$d_0 z \in (\alpha - \rho)(W \oplus V).$$

But $Rz = Rd_0 z$ because $d_0 z \neq 0$ and Rz is simple, and it follows that $z \in (\alpha - \rho)(W \oplus V)$. So $\alpha - \rho \colon W \oplus V \to W \oplus V$ is epic. Let $(\alpha - \rho)(w + v) = 0$ where $w \in W$ and $v \in V$. Then $(\alpha - a)(v) = (\alpha - \rho)(v) = -(\alpha - \rho)(w) \in W$, so $v = 0$ by Claim 4. It follows that $(\alpha - \rho)(w) = 0$, and so $w = 0$ because $\alpha|_W - \rho$ is a unit in $\operatorname{end}_R W$. So $\alpha - \rho \colon W \oplus V \to W \oplus V$ is monic, as required.

Case 2 *For any $l \geq 0$, a linear combination $d_0 z + d_1(\alpha - a)(z) + \cdots + d_l(\alpha - a)^l(z)$, $d_i \in R$, lies in W if and only if $d_i(\alpha - a)^i(z) = 0$ for each $i = 0, 1, \dots, l$.*

In this case we have a direct sum $U = \bigoplus_{i=0}^{\infty} R(\alpha - a)^i(z)$ of R-modules. Clearly $W \cap U = 0$, $U \neq 0$, and U is α-invariant (it is clearly $(\alpha - a)$-invariant). Moreover, $\sum_{i=0}^{n} R(\alpha - a)^i(z) = \sum_{i=0}^{n} R\alpha^i(z)$ for each $n \geq 0$ as is easily verified, and it follows that

$$U = Rz \oplus R\alpha(z) \oplus R\alpha^2(z) \oplus \cdots .$$

We begin by using this representation to construct $\theta \in \operatorname{end}_R U$ such that $g(\theta) = 0$ on U and $\alpha_{|U} - \theta$ is a unit in $\operatorname{end}_R U$. For each $n \geq 0$, define

$$
\begin{aligned}
&\theta_{2n} \colon R\alpha^{2n}(z) \to U, \quad r\alpha^{2n}(z) \mapsto br\alpha^{2n}(z), \\
&\theta_{2n+1} \colon R\alpha^{2n+1}(z) \to U, \quad r\alpha^{2n+1}(z) \mapsto (b - ba)r\alpha^{2n}(z) + ar\alpha^{2n+1}(z) + r\alpha^{2n+2}(z).
\end{aligned}
$$

https://doi.org/10.4153/CMB-2006-027-6 Published online by Cambridge University Press

To see that θ_{2n+1} is well defined, let $r\alpha^{2n+1}(z) = 0$. Then $r\alpha^{2n+2}(z) = 0 = ar\alpha^{2n+1}(z)$, and $r\alpha^{2n}(z) = 0$ by Claim 4 because $\alpha(r\alpha^{2n}(z)) = 0 \in W$. So θ_{2n+1} is well defined and we obtain the map $\theta \in \text{end}\,_R U$ given by $\theta = \bigoplus_{n\geq 0} \theta_n$. Hence

$$\theta[r\alpha^k(z)] = \theta_k[r\alpha^k(z)] \quad \text{for all } r \in R \text{ and } k \geq 0.$$

For each $n \geq 0$ we compute:

$$\begin{aligned}
(\theta - a)&(\theta - b)(\alpha^{2n+1}(z)) \\
&= (\theta - a)\big[(b - ba)\alpha^{2n}(z) + (a - b)\alpha^{2n+1}(z) + \alpha^{2n+2}(z)\big] \\
&= (b - ba)b\alpha^{2n}(z) + (a - b)\big[(b - ba)\alpha^{2n}(z) + a\alpha^{2n+1}(z) + \alpha^{2n+2}(z)\big] \\
&\qquad + b\alpha^{2n+2}(z) - (b - ba)a\alpha^{2n}(z) - (a - b)a\alpha^{2n+1}(z) - a\alpha^{2n+2}(z) \\
&= 0.
\end{aligned}$$

So $(\theta - a)(\theta - b) = 0$ on $R\alpha^{2n+1}(z)$ for all $n \geq 0$. By hypothesis, $g(x) = (x - a)(x - b)f(x)$ where $f(x) \in C[x]$. So $g(\theta) = f(\theta)(\theta - a)(\theta - b)$, and it follows that $g(\theta) = 0$ on $R\alpha^{2n+1}(z)$ for all $n \geq 0$. It is clear that $g(\theta) = g(b) = 0$ on $R\alpha^{2n}(z)$ for all $n \geq 0$. Therefore, $g(\theta) = 0$ on U.

To see that $\alpha_{|U} - \theta : U \to U$ is monic, suppose on the contrary that $(\alpha - \theta)(u) = 0$ where $0 \neq u = s\alpha^{2n}(z) + t\alpha^{2n+1}(z) + \cdots \in U$, where $s, t, \ldots$ are in R, and where either $s\alpha^{2n}(z) \neq 0$ or $t\alpha^{2n+1}(z) \neq 0$. Thus,

$$\begin{aligned}
0 &= \alpha(u) - \theta(u) \\
&= [s\alpha^{2n+1}(z) + t\alpha^{2n+2}(z) + \cdots] \\
&\qquad - [sb\alpha^{2n}(z) + t(b - ba)\alpha^{2n}(z) + ta\alpha^{2n+1}(z) + t\alpha^{2n+2}(z) + \cdots]
\end{aligned}$$

It follows that $b[(s - at) + t]\alpha^{2n}(z) = 0$ and $(s - at)\alpha^{2n+1}(z) = 0$. Applying α to the first of these (and using the second) gives $bt\alpha^{2n+1}(z) = 0$, so $t\alpha^{2n+1}(z) = 0$ because b is a unit, whence $s\alpha^{2n+1}(z) = 0$. Thus $\alpha[s\alpha^{2n}(z)] = 0 \in W$ so $s\alpha^{2n}(z) = 0$ by Claim 4. This contradiction shows that $\alpha_{|U} - \theta$ is monic.

Finally, note that $(\alpha - \theta)(a\alpha^{2n}(z) + \alpha^{2n+1}(z)) = -b\alpha^{2n}(z)$ and $(\alpha - \theta)(\alpha^{2n}(z)) = \alpha^{2n+1}(z) - b\alpha^{2n}(z)$. So $\alpha^{2n}(z)$ and $\alpha^{2n+1}(z)$ are in $im(\alpha_{|U} - \theta)$ for all $n \geq 0$. This shows that $\alpha_{|U} - \theta : U \to U$ is epic. Therefore, $\alpha_{|U} - \theta$ is a unit in end$\,_R U$.

Since $g(\rho) = 0$ on W and $g(\theta) = 0$ on U, $g(\rho \oplus \theta) = 0$ on $W \oplus U$. Moreover $\alpha_{|W} - \rho$ is a unit in end$\,_R W$ and $\alpha_{|U} - \theta$ is a unit in end$\,_R U$, so $\alpha_{|W\oplus U} - (\rho \oplus \theta) = (\alpha_{|W} - \rho) \oplus (\alpha_{|U} - \theta)$ is a unit in end$\,_R(W \oplus U)$. Thus $(W \oplus U, \rho \oplus \theta) \in \mathcal{S}$, once again contradicting the maximality of (W, ρ) in $\mathcal{S}$. Hence $W = M$ and the proof of Theorem 1 is complete. ■

References

[1] V. P. Camillo and J. J. Simón, *The Nicholson-Varadarajan theorem on clean linear transformations.* Glasg. Math. J. **44** (2002), 365–369.

https://doi.org/10.4153/CMB-2006-027-6 Published online by Cambridge University Press

[2] V. P. Camillo and H. P. Yu, *Exchange rings, units and idempotents.* Comm. Algebra **22**(1994), no. 12, 4737–4749.

[3] W. K. Nicholson and K. Varadarajan, *Countable linear transformations are clean.* Proc. Amer. Math. Soc. **126** (1998), no. 1, 61–64.

Department of Mathematics
University of Calgary
Calgary T2N 1N4, Canada
e-mail: wknichol@ucalgary.ca

Department of Mathematics
Memorial University of Newfoundland
St. John's A1C 5S7, Canada
e-mail: zhou@math.mun.ca

https://doi.org/10.4153/CMB-2006-027-6 Published online by Cambridge University Press