Rates of convergence for renewal sequences in the null-recurrent case

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Abstract

Motivated by work of Garsia and Lamperti we consider null-recurrent renewal sequences with a regularly varying tail and seek information about their rate of convergence to zero. The main result shows that such sequences subject to a monotonicity condition obey a limit law whatever the value of the exponent \( \alpha \) is, \( 0 < \alpha < 1 \). This monotonicity property is seen to hold for a large class of renewal sequences, the so-called Kaluza sequences. This class includes moment sequences, and therefore includes the sequences generated by reversible Markov chains. Several subsidiary results are proved.


1. Introduction

Let \( \{ f_n \}, n = 1, 2, \ldots, \) be a sequence of real numbers with

\[
(1.1) \quad f_n \geq 0, \quad \sum_{n=1}^{\infty} f_n = 1, \quad \text{g.c.d.}\{n: f_n > 0\} = 1.
\]

Define another sequence \( \{ u_n \}, n = 0, 1, 2, \ldots, \) by

\[
(1.2) \quad u_0 = 1, \quad u_n = \sum_{k=1}^{n} f_k u_{n-k}, \quad n \geq 1.
\]

It can be seen that \( 0 \leq u_n \leq 1 \). The sequences \( \{ f_n \} \) and \( \{ u_n \} \) are related to Markov chain theory as follows: consider a recurrent aperiodic Markov chain...
\( \{X_n, n \geq 0\} \) with state space the integers and \( P(X_0 = 0) = 1 \). Let \( T \) be the time of first return to the origin. If we put

\[
P(T = n) = f_n, \quad n \geq 1,
\]

then (1.1) is satisfied (the second condition there is equivalent to recurrence of the process and the third to its aperiodicity). Let

\[
P(X_n = 0 \mid X_0 = 0) = u_n.
\]

Then \( \{u_n\} \) satisfies (1.2).

The classical renewal theorem [2] states

\[
\lim_{n \to \infty} u_n = \frac{1}{\sum_{k=1}^{\infty} kf_k}
\]

where the right side is taken to be zero when the denominator diverges. In Markov chain terminology the denominator diverges when the chain is null-recurrent, and this is the case of interest in this paper.

Garsia and Lamperti [5] studied the rate of convergence to zero in (1.5) in the null-recurrent case when \( T \) is in the domain of attraction of a stable law of index \( \alpha \), \( 0 < \alpha < 1 \). Their main result (Theorem 1.1) states that if

\[
\sum_{k=n+1}^{\infty} f_k = n^{-\alpha} L(n), \quad 0 < \alpha < 1,
\]

where \( L(n) \) is a slowly varying function, then

\[
\liminf_{n \to \infty} n^{1-\alpha} L(n) u_n = \frac{\sin \pi \alpha}{\pi}
\]

and if \( \frac{1}{2} < \alpha < 1 \) then (1.7) can be sharpened to

\[
\lim_{n \to \infty} n^{1-\alpha} L(n) u_n = \frac{\sin \pi \alpha}{\pi}.
\]

The principal result of this note (Theorem 3.1) is the observation that if the renewal sequence \( \{u_n\} \) satisfies the monotonicity property (3.2), then (1.6) is sufficient to imply (1.8) without regard to the value of \( \alpha \), \( 0 < \alpha < 1 \). In particular it follows that any renewal sequence \( \{u_n\} \) such that \( \{u_n\} \) is a Kaluza or moment sequence for some fixed \( k \geq 1 \) (see Section 4) satisfies (1.8) when (1.6) is true; this includes the case of reversible Markov chains (Corollary 4.1).

Section 2 presents the mostly well-known tools on rates of growth needed for the rest of the article. Finally, Proposition 3.1 gives some information on the boundary cases \( \alpha = 0 \) and \( \alpha = 1 \), including Erickson’s renewal theorem (3.14) when \( \alpha = 1 \).
2. Preliminary results on rates of growth

DEFINITION. A positive function $L$ defined on the positive real axis is slowly varying (at infinity) if, for each $\lambda > 0$

\[
\lim_{t \to \infty} \frac{L(\lambda t)}{L(t)} = 1.
\]

$U$ is regularly varying with exponent $\rho$ if

\[ U(x) = x^\rho L(x) \]

with $-\infty < \rho < \infty$ and $L$ slowly varying. A basic reference on slow and regular variation is [11]. We require the following results.

**Lemma 2.1.** Let $0 < \alpha < 1$, and let $L(x)$ be slowly varying. Then

\[
\sum_{k=1}^{n} \frac{1}{k^\alpha} L(k) \sim \frac{1}{1 - \alpha} n^{1-\alpha} L(n).
\]

**Lemma 2.2.** Let $L(x)$ be slowly varying with

\[
\sum_{k=1}^{n} \frac{1}{k} L(k) \uparrow \infty.
\]

Then the function

\[
\int_{1}^{x} \frac{1}{y} L(y) \, dy = L_1(x)
\]

is slowly varying, and

\[
\sum_{k=1}^{\infty} \frac{1}{k} L(k) \sim L_1(n).
\]

**Lemma 2.3.** Let $\sum_{k=1}^{n} p_k \sim n^\alpha L(n)$, $0 < \alpha \leq 1$, where $L(n)$ is slowly varying and $p_n$ is monotone non-increasing. Then

\[
p_n \sim \alpha n^{\alpha-1} L(n).
\]

**Lemma 2.4.** Let $\sum_{k=1}^{n} p_k \sim L(n)$ where $L(n)$ is slowly varying and $p_n$ is monotone non-increasing. Then

\[
\lim_{n \to \infty} \frac{n p_n}{L(n)} = 0.
\]

From (2.8) one gets

\[
\lim_{n \to \infty} n^{1-\delta} p(n) = 0, \quad \text{for all} \; \delta > 0.
\]
The above results are either all well known or easily accessible. Observe that from [11, 4°, pages 19–21], integral test comparisons on [11, Theorem 2.1] yield Lemma 2.1, and a similar argument using [11, Exercise 2.2] proves Lemma 2.2 (see also [4, Theorem 8.9.1]). Lemma 2.3 is part of [4, Theorem 13.5.4] or [11, Exercise 2.8], and the latter reference yields (2.8). Then (2.9) follows from (2.8) and

$$\lim_{x \to -\infty} x^\delta (L(x))^{-1} = \infty \quad \text{for} \; \delta > 0.$$  

(See, for example, [11, 1° and 3°, page 18].)

3. Principal results

Recall the definitions of the sequences \( \{f_n\} \) and \( \{u_n\} \) and of the random variable \( T \) given in Section 1. Let

$$r_n = \sum_{k=n+1}^{\infty} f_k = P(T > n).$$

Throughout this section it will be assumed that (1.1) holds and that \( ET = \infty \) (or equivalently, \( \sum r_k \) diverges).

**Theorem 3.1.** Let \( T \) be in the domain of attraction of a stable law of index \( \alpha, 0 < \alpha < 1 \); more precisely, suppose

(3.1) \quad r_n \sim n^{-\alpha} L(n)

for \( L(n) \) slowly varying. If

(3.2) \quad \text{there exists a fixed integer} \; k \geq 1 \; \text{such that the sequence} \; \{u_{nk}\} \; \text{is monotone non-increasing, then}

(3.3) \quad u_n \sim \frac{\sin \pi \alpha}{\pi} \frac{n^{\alpha-1}}{L(n)}.

Conversely, suppose (3.3) is true for some \( \alpha, 0 < \alpha < 1 \), and \( L(n) \) slowly varying. Then (3.1) holds.

**Proof.** The sum

(3.4) \quad \sum_{j=0}^{(n+1)k-1} u_j

may be decomposed into the \( k \) sums

$$\sum_{j=0}^{n} u_{j+k-i} = U_i(n), \quad 0 \leq i \leq k - 1.$$
The monotonicity of \( \{u_{nk}\} \) implies that the sequence \( \{u_n\} \) possesses the strong ratio limit property (SRLP) (see [10]) so that

\[
u_{nk+i} \sim u_{nk+j}
\]

for fixed \( i, j, 0 \leq i, j \leq k - 1 \). Now (3.4) diverges, in fact, by [5, Lemma 2.3.1] we know

\[
\sum_{j=0}^{n} u_j \sim \frac{\sin \pi \alpha}{\pi \alpha} \frac{n^\alpha}{L(n)}.
\]

Thus at least one \( U_i(n) \) diverges, and (3.5) easily implies

\[
U_i(n) \sim U_j(n), \quad 0 \leq i, j \leq k - 1.
\]

By (3.6) and properties of slowly varying functions we obtain

\[
\sum_{j=0}^{k-1} U_j(n) = \sum_{j=0}^{(n+1)k-1} u_j \sim \frac{C((n+1)(k-1)^\alpha}{L((n+1)(k-1))} \sim \frac{C(nk)^\alpha}{L(nk)},
\]

\[
C = (\pi \alpha)^{-1} \sin \pi \alpha.
\]

From (3.7) we conclude that

\[
U_i(n) \sim \frac{C(nk)^\alpha}{kL(nk)} = Ck^{\alpha-1} \frac{n^\alpha}{L(nk)}
\]

for each \( i \). The terms of \( U_0(n) \) are monotone non-increasing and so, by Lemma 2.3

\[
u_{nk} \sim \alpha C k^{\alpha-1} \frac{n^{\alpha-1}}{L(nk)} = \alpha C \frac{(nk)^{\alpha-1}}{L(nk)}.
\]

Using the SRLP

\[
u_{nk+i} \sim \alpha C \frac{(nk+i)^{\alpha-1}}{L(nk+i)}, \quad 0 \leq i \leq k - 1,
\]

proving (3.3).

To prove the converse assertion, it will be sufficient to show that if

\[
u_{n} \sim C n^{\alpha-1} \frac{L(n)}{L(n)}
\]

for some constant \( C \), then \( r_n \sim C_1 n^{-\alpha} L(n) \) for some constant \( C_1 \). Below \( C \) denotes a constant, not necessarily the same one in different relations. Since the reciprocal of a slowly varying function is also slowly varying, (3.8) can be written as \( u_n \sim C n^{\alpha-1} L_1(n) \). Lemma 2.1 then gives

\[
\sum_{j=1}^{n} u_j \sim C n^\alpha L_1(n).
\]
An Abelian theorem [4, page 423] shows that the generating function \( U(s) \) of the sequence \( \{u_n\} \) satisfies

\[
U(s) \sim C(1 - s)^{-\alpha}L_1 \left( \frac{1}{1 - s} \right), \quad s \to 1^-.
\]

Use a standard renewal Tauberian argument (for example, see [5, Lemma 2.3.1], and reverse the steps) to obtain

\[
\sum_{j=0}^{n} r_j \sim Cn^{1-\alpha}L(n).
\]

Monotonicity of \( \{r_n\} \) and Lemma 2.3 allow us to deduce (3.1).

Suppose we now relax the condition on \( T \) in Theorem 3.1: let us assume that \( T \) only has a regularly varying tail. This means that (3.1) now holds where \( L \) is slowly varying and \( \alpha \) is some real number. Since we are interested in the null-recurrent case, \( \sum r_k \) diverges and hence \( 0 \leq \alpha \leq 1 \). So there are two extreme cases, \( \alpha = 0 \) and \( \alpha = 1 \), not covered by Theorem 3.1. Erickson obtained the result (3.14) for \( \alpha = 1 \) [3]. We have the following

**Proposition 3.1.** (a) Let (3.1) hold with \( \alpha = 0 \). Then

\[
\sum_{j=0}^{n} u_j \sim (L(n))^{-1}.
\]

If the monotonicity condition (3.2) also holds, then

\[
\lim_{n \to \infty} \frac{n u_n}{L(n)} = 0 \quad \text{and} \quad \lim_{n \to \infty} n^{1-\delta} u_n = 0 \quad \text{for all} \quad \delta > 0.
\]

(b) Let (3.1) hold with \( \alpha = 1 \) and let

\[
\int_1^{x} \frac{1}{y} L(y) \, dy = L_1(x).
\]

Then

\[
\sum_{k=1}^{n} u_k \sim \frac{n}{L_1(n)}
\]

and

\[
u_n \sim \frac{1}{L_1(n)}.
\]

**Proof.** Under (a), Lemma 2.1 and the Tauberian argument of [5, Lemma 2.3.1] cited previously prove (3.10). An argument similar to that in the proof of Theorem 3.1 coupled with Lemma 2.4 proves (3.11).

Under (b), divergence of \( \sum r_k \) implies divergence of (3.12) so that by Lemma 2.2, \( L_1(x) \) is slowly varying and (2.6) is true. Again, the Tauberian argument...
easily gives (3.13). Note that (3.14) does not follow immediately from (3.13), for we have not assumed monotonicity here; we refer the reader to Erickson’s proof [3, page 266].

Remark 1. The failure of Lemma 2.3 for the case $\alpha = 0$ means that we are not able to obtain the exact rate of convergence of $\{u_n\}$ in this case. Lemma 2.4 gives us (3.11), but this is unsatisfactory. The case of simple random walk in the plane suggests improvement on (3.11) may be possible; there one has

$$\tau_n \sim \frac{\pi}{\log n}, \quad \sum_{j=0}^{n} u_j \sim \frac{\log n}{\pi} \quad \text{and} \quad u_{2n} \sim \frac{1}{\pi n}.$$  

Remark 2. It is perhaps not surprising that the case $\alpha = 1$ can be added to the Garsia-Lamperti range $\frac{1}{2} < \alpha < 1$ of values of $\alpha$ where renewal theorems hold automatically without further conditions. Thus there is a kind of continuity at $\alpha = 1$ of the good behavior at $\alpha = 1^-$, although (3.3) and (3.14) are different. Whether such continuity also holds at $\alpha = \frac{1}{2}$ is an open question (see [5, page 230], the discussion following (3.4.9)).

4. Applications

Throughout this section the renewal sequence $\{u_n\}$ is associated with the sequence $\{f_n\}$ where (1.1) is assumed to be valid, and $\sum \tau_k$ diverges.

The sequence $\{u_n\}$ is called a Kaluza sequence if

$$(4.1) \quad u_n^2 \leq u_{n-1} \cdot u_{n+1}, \quad n \geq 1,$$

and it is called a moment sequence if there exists a probability measure $\nu$ on $[0,1]$ with $u_n = \int_0^1 x^n \nu(dx)$, $n \geq 0$. Every moment sequence is a Kaluza sequence. The most interesting property of Kaluza sequences in the present discussion is that they are non-increasing. Moreover, many renewal sequences turn out to have the Kaluza or moment properties. Perhaps the most famous case is $u_n = (2^n)^2 - 2^n$ where $\{u_n\}$ is associated with simple random walk on the line. We refer the reader to [8] (also see [7] and [9]) for further discussion of Kaluza sequences.

A class of moment sequences arises by considering reversible Markov chains. A chain is reversible if $\pi(i)p(i,j) = \pi(j)p(j,i)$ for all $i, j$, where $\pi$ is the invariant measure of the chain, and $p(\cdot, \cdot)$ is its transition probability (see for example [10, page 83]). Under our assumptions, the chain is recurrent and aperiodic and has a non-trivial $\sigma$-finite invariant measure. A result of Kendall ([6], also [10, page 83]) shows that for reversible chains $u_{2n}$ is a moment sequence. The monotonicity property of Kaluza sequences enables us to apply Theorem 3.1 or Proposition 3.1(a). We summarize this in the following corollary.
COROLLARY 4.1. Let \( \{u_n\} \) be a renewal sequence such that \( \{u_{nk}\} \) is a Kaluza sequence for a fixed integer \( k \geq 1 \).

(a) If (3.1) is valid for some \( \alpha, \, 0 < \alpha < 1 \), then (3.3) holds.

(b) If (3.1) is valid for \( \alpha = 0 \) then (3.11) holds.

In particular, if \( \{u_n\} \) is derived from a reversible Markov chain, then \( \{u_{2n}\} \) is a moment (hence Kaluza) sequence, so that if \( T \) has a regularly varying tail, (3.3), (3.11) or (3.14) holds, depending upon the value of \( \alpha \).

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References


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