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ON DIFFERENT TYPES OF CLEAVABILITY OF TOPOLOGICAL SPACES

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Abstract

The notion of pointwise cleavability is introduced. We clarify those results concerning cleavability which can be or can not be generalized to the case of pointwise cleavability.

The importance of compactness in this theory is shown. Among other things we prove that t, t_S, π_χ, the property to be Fréchet-Urysohn, radiality, biradiality, bisequentiality and so on are preserved by pointwise cleavability on the class of compact Hausdorff spaces.

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0. Introduction

In 1985 Arhangel'skii [4, 5, 6], introduced the notion of cleavability (originally called splittability) of a topological space as follows. Let $\mathscr{P}$ be a class of topological spaces and let $\mathscr{M}$ be a class of mappings. We say that a topological space X is *$(\mathscr{M}, \mathscr{P})$-cleavable* or *$\mathscr{M}$-cleavable over $\mathscr{P}$* if for every $A \subset X$ there exists $f \in \mathscr{M}$, $f : X \to Y$, such that $Y \in \mathscr{P}$ and $A = f^{-1}f(A)$.

If $\mathscr{M}$ is the class of all continuous mappings (open, closed, perfect and so on) we shall use the term *cleavable* (*open-cleavable, closed cleavable, perfect-cleavable,...*) *over $\mathscr{P}$*.

If $\mathscr{P} = \{Y\}$ with Y a fixed topological space we shall use the term *cleavable* (*open-cleavable, closed-cleavable, perfect-cleavable,...*) *over Y*.

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The cases $Y = \mathbb{R}^n$ with $n \in \mathbb{N}$ and $Y = \mathbb{R}^\omega$ are of particular interest.

The cleavability over $\mathbb{R}^\omega$ is equivalent to the cleavability over the class of all separable and metrizable spaces; hence in what follows cleavability over $\mathbb{R}^\omega$ will be the same as cleavability over separable and metrizable spaces.

Recently many papers concerning the notion of cleavability were published ([2, 1, 7, 9, 10, 11, 19, 18]).

Of basic importance is the paper [8] in which a characterization of cleavability in terms of the countable functional approximation property is given.

In the present paper the notion of pointwise cleavability is introduced, in different versions, corresponding to different classes of mappings. It is clarified which results concerning cleavability can be or can not be generalized to the case of pointwise cleavability.

The role of compactness in this theory is emphasized. Among other things we prove that the cardinal functions t, t_S, π_χ, the property of being Fréchet-Urysohn, radiality, biradiality, bisequentiality and so on are preserved by pointwise cleavability on the class of Hausdorff compact spaces.

Our topological notations are standard, they are the same as in [13, 14]. In particular, $\mathbb{N}$ is the set of all natural numbers and $\mathbb{R}$ is the space of real numbers with the usual topology.

1. Pointwise cleavability

Let $\mathscr{M}$ be a class of mappings and $\mathscr{P}$ a class of spaces.

DEFINITION 1.1. A topological space X is said to be *$\mathscr{M}$-pointwise cleavable on $\mathscr{P}$* if for every point $x \in X$ there exists $f \in \mathscr{M}$, $f : X \to Y$, where $Y \in \mathscr{P}$, such that $\{x\} = f^{-1}f(x)$.

REMARK 1. Now we can speak about pointwise cleavability (open, closed, perfect,...), and more specifically of pointwise cleavability over $\mathbb{R}$ or over $\mathbb{R}^\omega$.

From now on when we speak of cleavability or pointwise cleavability without mentioning the class $\mathscr{P}$ of spaces we mean that $\mathscr{P} = \{\mathbb{R}^\omega\}$ or equivalently that $\mathscr{P}$ consists of all separable metrizable spaces. Where we do not mention explicitly the class $\mathscr{M}$ of all mappings, we mean that $\mathscr{M}$ consists of all continuous mappings. The next proposition is obvious.

PROPOSITION 1.1. *The following assertions hold:*

(1) *every subspace of a space is cleavable over this space;*

(2) *if X is cleavable over Y then X is cleavable over any $Z \supset Y$;*

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(3) *if X is cleavable over Y then any $Z \subset X$ is cleavable over Y;*
(4) *if X is cleavable over Y then X is cleavable over any class $\mathscr{P}$ containing Y.*

For example the unit segment $I = [0, 1] \subset \mathbb{R}$ is a subspace of $\mathbb{R}$ and $\mathbb{R}$ is homeomorphic to $(0, 1) \subset I$; therefore cleavability over $\mathbb{R}$ is equivalent to cleavability over I. If X is (pointwise) cleavable over $\mathbb{R}^p$ then X is (pointwise) cleavable over $\mathbb{R}^n$ for any $n \geq p$.

PROPOSITION 1.2. *If X is cleavable (pointwise-cleavable) over Y and Y is cleavable (pointwise-cleavable) over Z then X is cleavable (pointwise-cleavable) over Z.*

PROOF. By the hypothesis for every $A \subset X$ there exists a continuous mapping $f : X \to Y$ such that $A = f^{-1}f(A)$. Also for $B = f(A)$ there exists a continuous mapping $g : Y \to Z$ such that $B = g^{-1}g(B)$. The mapping $h = g \circ f : X \to Z$ obviously cleaves X along A.

By a similar argument the following assertion is proved.

PROPOSITION 1.3. *If X is cleavable (pointwise-cleavable) over a class $\mathscr{P}$ of spaces such that every $Y \in \mathscr{P}$ is cleavable (pointwise-cleavable) over a class $\mathscr{Q}$ of spaces then X is cleavable (pointwise-cleavable) over $\mathscr{Q}$.*

EXAMPLE 1. Every space is cleavable over the space $\{0, 1\}$ with the anti-discrete topology.

PROOF. Let X be a space; for any $A \subset X$ we define a mapping $f_A : X \to \{0, 1\}$ as follows: $f(A) = \{0\}$ and $f(X \backslash A) = \{1\}$. Clearly f_A cleaves X along A and is continuous.

THEOREM 1.1. *Let X be a topological space. Then the following conditions are equivalent.*

(i) *X is discrete;*
(ii) *X is cleavable over the double point space D($\{0, 1\}$ with discrete topology);*
(iii) *X is closed-cleavable over D;*
(iv) *X is open-cleavable over D;*
(v) *X is pointwise-cleavable over D.*

PROOF. (i) implies all other conditions. It is clear that the condition (v) is the weakest one. We show then that (v) implies (i). Take any $x \in X$. There exists a continuous mapping $f : X \to D$ such that $f(x) = 0$ and $f(y) = 1$ for any $y \in X \backslash \{x\}$. Since $\{0\}$ is open in D it follows that $\{x\} = f^{-1}(0)$ is open in X, hence X is discrete.

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If Y is a T_1-space with $|Y| > 1$ then every discrete space is closed-cleavable over Y. For example this is true when $Y = I = [0, 1] \subset \mathbb{R}$ with the usual topology. It is clear that I is cleavable over itself but it is not cleavable over D because it is not discrete.

EXAMPLE 2. It is not true that all separable metrizable spaces are cleavable over $\mathbb{R}$. In fact $\mathbb{R}^2$ is separable and metrizable but it is not cleavable over $\mathbb{R}$ (see [1]). On the other hand, every separable metrizable space is cleavable over $\mathbb{R}^\omega$ but this is not true for all metrizable spaces. Indeed, let $I = [0, 1]$ and let Y be a discrete space with $|Y| > 2^{\aleph_0}$; the product space $X = I \times Y$ is metrizable but it is not cleavable over $\mathbb{R}^\omega$ (see [8]).

It is important to know the behaviour of separation axioms with respect to the pointwise cleavability and cleavability. Now we prove only two elementary assertions which will be helpful later.

PROPOSITION 1.4. *If X is pointwise-cleavable over the class of Hausdorff (T_1) spaces then X is Hausdorff (T_1).*

PROOF. We consider only the case of Hausdorff spaces. Let $x \in X$; then there exists a Hausdorff space Y and a continuous mapping $f : X \to Y$ such that $\{x\} = f^{-1}f(x)$. This implies that for every $y \in Y$ with $x \neq y$ we have $f(x) \neq f(y)$. Since Y is Hausdorff, there exist two open neighbourhoods $U_{f(x)}$ and $V_{f(y)}$ in Y such that $U_{f(x)} \cap V_{f(y)} = \emptyset$. Then $f^{-1}(U_{f(x)})$ and $f^{-1}(V_{f(y)})$ are two disjoint open neighbourhoods of x and y respectively.

THEOREM 1.2. *A Tychonoff space X is pointwise-cleavable over $I = [0, 1]$ (over $\mathbb{R}$) if and only if every point in X is G_δ (in other words if and only if $\psi(X) \le \aleph_0$).*

PROOF. Let $x \in X$. There exists a continuous mapping $f : X \to Y$ such that $\{x\} = f^{-1}f(x)$. Choose a countable base of neighbourhoods $\mathscr{A} = \{A_n\}_{n\in\mathbb{N}}$ of $f(x) \in I$. Then $\{f^{-1}(A_n)\}_{n\in\mathbb{N}}$ is a countable family of open sets and since $\{x\} = f^{-1}f(x)$, we have $\{x\} = \bigcap_{n\in\mathbb{N}} f^{-1}(A_n)$, so that x is G_δ in X. Conversely, let $x \in X$. Since X is a Tychonoff space and x is G_δ in X there exists a continuous mapping $f : X \to \mathbb{R}$ such that $f(x) = 0$ and $f(y) \neq 0$ for every $y \in X\backslash\{x\}$. Thus $\{x\} = f^{-1}f(x)$.

PROPOSITION 1.5. *If X is pointwise-cleavable over the class of spaces with countable pseudocharacter then X has countable pseudocharacter.*

PROOF. The proof is similar to the preceding argument.

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DEFINITION 1.2. A mapping $f : X \to Y$ is said to be *closed in* $x \in X$ if for every $A \subset X$ with $x \notin \overline{A}$ we have $f(x) \notin \overline{f(A)}$.

DEFINITION 1.3. A topological space X is said to be *c-cleavable in* $x \in X$ *over a topological space* Y if there exists a continuous mapping $f : X \to Y$ such that:

(1) f is closed in x, and
(2) $\{x\} = f^{-1}f(x)$.

DEFINITION 1.4. Let $\mathscr{P}$ be a class of topological spaces. We say that a topological space X is *pointwise c-cleavable over* $\mathscr{P}$ if for every $x \in X$ there exist $Y \in \mathscr{P}$ and a continuous mapping $f : X \to Y$ such that X is c-cleavable in x over Y by f in the sense of Definition 1.3.

Obviously a mapping $f : X \to Y$ is closed if and only if it is closed in every $x \in X$.

LEMMA 1.1. *Let $f : X \to Y$ be a continuous mapping closed in $x \in X$ and such that $\{x\} = f^{-1}f(x)$. Then if $\mathscr{U}_{f(x)} = \{U_i\}_{i\in J}$ is a base (π-base) of $f(x)$ in Y then the family $\{f^{-1}(U_i)\}_{i\in J}$ is a base (π-base) of x in X.*

PROOF. We only consider the case when $\mathscr{U}_{f(x)}$ is a base of x in X. Let O_x be an open neighbourhood of x in X. The set $X\backslash O_x$ is closed in X and $x \notin X\backslash O_x$. Hence $f(x) \notin \overline{f(X\backslash O_x)}$. Obviously the set $Y\backslash\overline{f(X\backslash O_x)}$ is an open neighbourhood of $f(x)$ and there exists $U_k \in \mathscr{U}_{f(x)}$ such that $f(x) \in U_k \subset Y\backslash\overline{f(X\backslash O_x)}$. Since f is continuous, $f^{-1}(U_k)$ is an open set in X such that $x \in f^{-1}(U_k)$. So we have: $x \in f^{-1}(U_k) \subset f^{-1}[Y\backslash\overline{f(X\backslash O_x)}] \subset O_x$. The proof is complete.

THEOREM 1.3. *If X is pointwise c-cleavable over the class of first countable spaces then X is first countable.*

PROOF. Let $x \in X$. There exists a first countable space Y and a continuous mapping $f : X \to Y$ closed in x such that $\{x\} = f^{-1}f(x)$. By hypothesis there exists a countable open base $\{V_n\}_{n\in\mathbb{N}}$ of $f(x)$ in Y. By Lemma 1.1, $\{f^{-1}(V_n)\}_{n\in\mathbb{N}}$ is a countable open base of x in X.

REMARK 2. It is true that every continuous mapping from a countably compact space in I or $\mathbb{R}$ is closed. It is also known that in a Tychonoff countable compact space every G_δ point is a point of first countability. Then by Theorem 1.2 we have the following

COROLLARY 1.3. *Let X be a Tychonoff countably compact space. Then the following conditions are equivalent:*

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(i) *X is first countable;*
(ii) *X is pointwise cleavable over I;*
(iii) *X is pointwise c-cleavable over I.*

COROLLARY 1.4. *Every metrizable space is pointwise cleavable over $\mathbb{R}$: it is even cleavable over $\mathbb{R}$ along all closed subsets.*

Now we are able to exhibit some examples of pointwise cleavable spaces which are not cleavable.

EXAMPLE 3. Let $X = \sum_{\alpha\in A} I_\alpha$ the free topological sum where I_α, for every $\alpha \in A$, is a copy of the segment $I = [0, 1]$ with the usual topology. In [8] it is shown that if $|A| > 2^{\aleph_0}$ then X is not cleavable over $\mathbb{R}^\omega$. Since X is metrizable, X is pointwise-cleavable over $\mathbb{R}$ and hence over $\mathbb{R}^\omega$.

EXAMPLE 4. Let X be a compact first countable non-metrizable space. Then X is pointwise-cleavable over $\mathbb{R}$ but is not cleavable over $\mathbb{R}^\omega$ since otherwise it would have been metrizable (by a theorem in [8]: if X is a compact space cleavable over $\mathbb{R}^\omega$ then X is metrizable). For such a space we can take the Alexandroff duplicate of I.

2. Pointwise-cleavability and cleavability

Since the principal aim of the first part of our paper is to clarify the differences between cleavability and pointwise-cleavability, we wish to check which results on cleavability can be extended to pointwise cleavability.

Observe that not every closed-cleavable space over the class of Hausdorff compact spaces is compact, since every discrete space is closed-cleavable over the class of Hausdorff compact spaces. On the other hand, Arhangel'skii [1] has proved the following assertion:

THEOREM 2.1. *If X is a countably compact space cleavable over the class of Hausdorff compact sequential spaces, then X is compact.*

Theorem 2.1 cannot be extended to pointwise cleavability over the class of Hausdorff compact spaces since every Tychonoff countably compact first countable space is pointwise-cleavable over the segment I. Such a space need not be compact. (Consider the space $T(\omega_1)$ of all ordinals smaller then the first uncountable ordinal ω_1.)

The following result was established in [4]:

THEOREM 2.2. *If X is compact and cleavable over the class of Hausdorff zero-dimensional spaces then X is a Hausdorff zero-dimensional space.*

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We can generalize this result to pointwise cleavability.

THEOREM 2.3. *If X is pointwise c-cleavable over the class of Hausdorff zero-dimensional spaces then X is a Hausdorff zero-dimensional space.*

PROOF. Let $x \in X$. There exists a Hausdorff zero-dimensional space Y and a continuous mapping $f : X \to Y$ such that f is closed in x and $\{x\} = f^{-1}f(x)$. By hypothesis there exists a clopen neighbourhood base $\{A_i\}_{i \in J}$ of $f(x)$ in Y. Put $\mathscr{A}' = \{f^{-1}(A_i)\}_{i \in J}$. Since f is continuous, $\mathscr{A}'$ is a family of clopen sets in X, and by Lemma 1.1 $\mathscr{A}'$ is a clopen base of x, so that the space X is zero-dimensional. The space X is also Hausdorff by Proposition 1.4.

COROLLARY 2.3. *If X is compact and pointwise-cleavable over the class of Hausdorff zero-dimensional spaces then X is Hausdorff and zero-dimensional.*

PROOF. This follows from Theorem 2.3 since every continuous mapping from compact spaces into Hausdorff spaces is closed.

REMARK 3. For the case $\dim Y = n \neq 0$ we have the following example. By Corollary 1.3 every compact Hausdorff first countable space is pointwise-cleavable over $I = [0, 1]$. Thus $X = I^{\aleph_0}$ satisfies all the assumptions of Corollary 1.3; X has infinite dimension, while $\dim Y = 1$. This shows that the only case for the preservation of dimension under pointwise cleavability is the one of dimension zero.

3. Pointwise cleavability and cardinal invariants

We have already proved a couple of results in this direction: if X is pointwise c-cleavable over the class of first countable spaces then X is first countable, and if X is pointwise-cleavable over the class of spaces of countable pseudocharacter, then the pseudocharacter of X is also countable.

Now we consider more systematically the behaviour of various cardinal invariants with respect to pointwise c-cleavability.

We recall the following

DEFINITION 3.1. The *tightness* of X, denoted by $t(X)$, is the smallest infinite cardinal τ such that for each non-closed set $A \subset X$ and for every point $x \in \overline{A} \backslash A$ there exists a set $B \subset A$ satisfying the conditions: $|B| \leq \tau$ and $x \in \overline{B}$.

DEFINITION 3.2. The *π-character* of a point $x \in X$, denoted by $\pi\chi(x, X)$, is the smallest (infinite) cardinality of a local π-base of X in x.

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By a *local π-base* of X in x we mean a collection $\mathscr{V}$ of non-empty open sets in X such that for each open neighbourhood R of x, one has $V \subset R$ for some $V \in \mathscr{V}$.

PROPOSITION 3.1. *If X is pointwise c-cleavable over the class of spaces Y with $\pi\chi(Y) \leq \tau$, then $\pi\chi(X) \leq \tau$.*

PROOF. Let $x \in X$; by the hypothesis, there exists a topological space Y with $\pi\chi(Y) \leq \tau$, and a continuous mapping $f : X \to Y$ closed in x such that $\{x\} = f^{-1}f(x)$. Let η be a π-base of $f(x)$ in Y such that $|\eta| \leq \tau$. Then $\eta' = \eta_{/f(X)} = \{M \cap f(X) : M \in \eta\}$ is a π-base of $f(x)$ in $f(X)$ and $|\eta'| \leq \tau$. It follows that $f^{-1}(\eta')$ is a π-base in x, by Lemma 1.1, satisfying the condition: $|f^{-1}(\eta')| \leq \tau$.

COROLLARY 3.1. *If X is a compact space pointwise-cleavable over the class of Hausdorff spaces Y such that $\pi\chi(Y) \leq \tau$, then X is Hausdorff and $\pi\chi(X) \leq \tau$.*

PROOF. This follows by Proposition 3.1 and Proposition 1.4, taking into account that every continuous mapping of a compact space into a Hausdorff space is closed.

THEOREM 3.1. *If X is pointwise c-cleavable over the class of spaces Y such that $t(Y) \leq \tau$ then $t(X) \leq \tau$.*

PROOF. Let $A \subset X$ be a non-closed subset of X and let $x \in \overline{A}\backslash A$; then there exists a topological space Y such that $t(Y) \leq \tau$ and a continuous mapping $f : X \to Y$ closed in x and such that $\{x\} = f^{-1}f(x)$. Put $y = f(x)$; by continuity of f we have: $y \in f(\overline{A}) \subset \overline{f(A)}$ so that $y \in \overline{f(A)}\backslash f(A)$. Hence there exists $B \subset f(A)$ such that $|B| \leq \tau$ and $y \in \overline{B}$. For every $z \in B$ fix a point $x_z \in f^{-1}(z) \cap A$. Put $C = \{x_z \in f^{-1}(z) \cap A,\ \forall z \in B\}$. Then $C \subset A$ and $|C| = |B| \leq \tau$. Since $\{x\} = f^{-1}f(x)$ and f is closed in x, we have $x \in C$. The proof is complete.

Theorem 3.1 improves a similar result from [9], dealing with cleavability.

COROLLARY 3.2. *If X is a compact space, pointwise cleavable over the class of Hausdorff spaces Y such that $t(Y) \leq \tau$, then $t(X) \leq \tau$ and X is Hausdorff.*

PROOF. We apply Theorem 3.1 and Proposition 1.4 taking into account that every continuous mapping of a compact space into a Hausdorff space is closed.

Recall now that a space X is said to be *Fréchet-Urysohn* if for every $A \subset X$ and any $x \in \overline{A}$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in A converging to x.

THEOREM 3.2. *If X is pointwise c-cleavable over the class of Fréchet-Urysohn spaces then X is Fréchet-Urysohn.*

PROOF. Let $A \subset X$ and $x \in \overline{A}$. Then there exists a Fréchet-Urysohn space Y and a continuous mapping $f : X \to Y$ such that $\{x\} = f^{-1}f(x)$ and f is closed in x. Put $y = f(x)$. Then $y \in f(\overline{A}) \subset \overline{f(A)}$. Since Y is Fréchet-Urysohn, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ in $f(A)$ converging to y. We consider the sequence of fibers $\{f^{-1}(y_n)\}_{n\in\mathbb{N}}$. Since $y_n \in f(A)$ for each $n \in \mathbb{N}$, we have: $f^{-1}(y_n) \cap A \neq \emptyset$. Choosing in every $f^{-1}(y_n) \cap A$ a point x_n, we obtain a sequence $\{x_n\}_{n\in\mathbb{N}}$ in A. Since $\{x\} = f^{-1}f(x)$ and f is closed in x, the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x.

Theorem 3.2 improves a similar result of Kočinac [18], involving cleavability. The next result is proved in the same way as Corollaries 3.1 and 3.2.

COROLLARY 3.3. *If X is compact and pointwise cleavable over the class of Hausdorff Fréchet-Urysohn spaces then X is a Hausdorff Fréchet-Urysohn space.*

DEFINITION 3.3. A mapping $f : X \to Y$ is said to be *pseudo-open* if for every $y \in Y$ and for every neighbourhood U of $f^{-1}(y)$, $f(U)$ is a neighbourhood (not necessarily open) of y in Y, that is, $y \in \operatorname{Int} f(U)$.

All open mappings and all closed mappings are pseudo-open.

It is known that $f : X \to Y$ is pseudo-open if and only if for every $B \subset Y$ and for every $y \in \overline{B}\setminus B$ there exists $x \in X$ such that $f(x) = y$ and $x \in \overline{f^{-1}(B)}$, that is $x \in f^{-1}(y) \cap \overline{f^{-1}(B)}$.

LEMMA 3.1. *Let $f : X \to Y$ be a pseudo-open (in particular, open or closed) surjection, such that $\{x\} = f^{-1}f(x)$ for some $x \in X$. Then $t(f(x), Y) \leq t(x, X)$.*

PROOF. Let $x \in X$ and $t(x, X) \leq \tau$. We assume that $A \subset Y$ is a non-closed subset of Y such that $f(x) \in \overline{A}\setminus A$. Then $f^{-1}f(x) \cap \overline{f^{-1}(A)} \neq \emptyset$. Since $\{x\} = f^{-1}f(x)$, we have $x \in \overline{f^{-1}(A)}$, which implies that there exists $B \subset f^{-1}(A)$ such that $x \in \overline{B}$ and $|B| \leq \tau$. Obviously $f(B) \subset A$ and $|f(B)| \leq \tau$. By continuity of f, $f(x) \in \overline{f(B)}$ and, hence, $t(f(x), Y) \leq \tau$.

PROPOSITION 3.2. *Suppose that X is pointwise c-cleavable by continuous surjections over the class of spaces Y such that $\pi\chi(y, Y) \leq t(y, Y)$, for any $y \in Y$. Then $\pi\chi(x, X) \leq t(x, X)$ for any $x \in X$.*

PROOF. Let $x \in X$; then there exists a topological space Y and a continuous surjection $f : X \to Y$ such that $\pi\chi(f(x), Y) \leq t(f(x), Y)$, f is closed in x and $\{x\} = f^{-1}f(x)$. Put $t(x, X) = \tau$. To complete the argument, it remains to prove that $\pi\chi(x, X) \leq \tau$. By Lemma 3.1 we have $t(f(x), Y) \leq \tau$. Hence $\pi\chi(f(x), Y) \leq t(f(x), Y) \leq \tau$, so that by Proposition 3.1 we get: $\pi\chi(f^{-1}f(x), X) \leq \tau$, that is $\pi\chi(x, X) \leq \tau$.

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COROLLARY 3.4. *If a space X is pointwise c-cleavable by continuous surjections over the class of Hausdorff compact spaces, then $\pi\chi(x, X) \leq t(x, X)$ for any $x \in X$.*

PROOF. This follows from Proposition 3.2, since in compact spaces Y it is true that $\pi\chi(y, Y) \leq t(y, Y)$ for any $y \in Y$ (see Juhasz and Shelah [15]).

It is clear that the class of spaces which are pointwise c-cleavable by continuous surjections over the class of compact Hausdorff spaces is much larger than the class of compact Hausdorff spaces. For instance, it contains all discrete spaces. At the moment, it is not clear how to describe in intrinsic terms the spaces in this class and for this reason we introduce the following:

DEFINITION 3.4. A topological space X is said to be *Sicilian* provided that it is pointwise c-cleavable by continuous surjections over the class of Hausdorff compact spaces.

DEFINITION 3.5. A topological space X is said to be *Messinese* provided that it is closed-cleavable over the class of Hausdorff compact spaces.

It is obvious that every Messinese space is Sicilian. In this language Corollary 3.4 can be written in the following way: For every Sicilian space X the inequality $\pi\chi(x, X) \leq t(x, X)$ holds for each $x \in X$.

COROLLARY 3.5. *If a topological group G is pointwise c-cleavable by continuous surjections over the class of compact Hausdorff spaces Y such that $t(Y) \leq \aleph_0$, then G is metrizable.*

PROOF. By Theorem 3.1 we have $t(G) \leq \aleph_0$. Since G is a Sicilian space, $\pi\chi(G) \leq \aleph_0$. Thus (see [3]) the topological group G is first countable, and hence, by Kakutani's [17] result, G is metrizable.

We present now some results on closed cleavability. It is appropriate to recall that a mapping $f : X \to Y$ is closed if and only if [13] for every $y \in Y$ and for any neighbourhood U of $f^{-1}(y)$ in X there exists a neighbourhood V of y in Y such that $f^{-1}(V) \subset U$.

THEOREM 3.3. *Let X be a first countable normal space without isolated points and let X be closed-cleavable over the class of countably compact Hausdorff spaces. Then X itself is Hausdorff and countably compact.*

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PROOF. Let $A \subset X$, then there exists a countably compact Hausdorff space Y and a closed continuous mapping $f : X \to Y$ such that $A = f^{-1}f(A)$. Since f is closed and Y is countably compact and Hausdorff, for every $y \in Y$ we have: $\mathrm{Fr}(f^{-1}(y)) = f^{-1}(y)\setminus\mathrm{Int}(f^{-1}(y))$ is countably compact. We know also [13] that if $f : X \to Y$ is a closed continuous mapping such that $f^{-1}(y)$ is countably compact for each $y \in Y$ and Y is countably compact, then X is countably compact. Hence it is sufficient to prove we can choose f in a such way that $\mathrm{Int}(f^{-1}(y)) = \emptyset$ for every $y \in Y$. It is possible to decompose the space X into the union of two disjoint dense subspaces [1]. Clearly if $X = X_1 \cup X_2$, where $\overline{X_1} = \overline{X_2} = X$ and $X_1 \cap X_2 = \emptyset$ then the interiors of X_1 and X_2 are empty. We can choose f to be a closed continuous mapping of X into a countably compact Hausdorff space Y such that $X_1 = f^{-1}f(X_1)$. Then for every $y \in f(X_1)$, $\mathrm{Int}(f^{-1}(y)) = \emptyset$, since $f^{-1}(y) \subset X_1$ and $\mathrm{Int}(X_1) = \emptyset$.

COROLLARY 3.6. *If X is a first countable weakly paracompact (that is, metacompact) space without isolated points and X is closed-cleavable over the class of countably compact Hausdorff spaces then X is compact and Hausdorff.*

PROOF. By Theorem 3.3 the space X is Hausdorff and countably compact; since X is metacompact, it follows that X is compact.

4. Pointwise cleavability, double cleavability and set-tightness

We recall that *set tightness* of X, denoted by $t_S(X)$, is the smallest infinite cardinal number τ such that for any subset B of X and any point $x \in \overline{B}\setminus B$ there exists a family $\Gamma \subset P(B)$ such that $|\Gamma| \le \tau$, $x \in \overline{\cup\Gamma}$ and $x \notin \cup\overline{\Gamma}$.

THEOREM 4.1. *Let a space X be open-cleavable over the class of spaces Y such that $t_S(Y) \le \tau$. Then $t_S(X) \le \tau$.*

PROOF. Let $A \subset X$ be a non-closed subset of X and $x \in \overline{A}\setminus A$. There exist a topological space Y such that $t_S(Y) \le \tau$ and an open continuous mapping $f : X \to Y$ satisfying the condition: $A = f^{-1}f(A)$. By continuity of f, we have $y = f(x) \in \overline{f(A)}\setminus f(A)$. There exists a family $\{B_i : i \in \lambda\}$ of subsets of $f(A)$ such that $|\lambda| \le \tau$ and $f(x) \in \overline{\cup\{B_i : i \in \lambda\}}$ while $f(x) \notin \overline{B_i}$ for every $i \in \lambda$. Put $C_i = f^{-1}(B_i)$, for each $i \in \lambda$. Then $C_i \subseteq A = f^{-1}f(A)$. Let us show that $x \in \overline{\cup\{C_i : i \in \lambda\}}$ and $x \notin \overline{C_i}$ for every $i \in \lambda$. Take an open neighbourhood U_x of x in X. Then $f(U_x)$ is open in Y and $y \in f(U_x)$. Since $y \in \overline{\cup B_i}$, we have: $f(U_x) \cap (\cup B_i) \ne \emptyset$, so that there exists $i_0 \in \lambda$ such that $f(U_x) \cap B_{i_0} \ne \emptyset$. Hence there exist $z \in f(U_x) \cap B_{i_0}$ and $x_0 \in f^{-1}(z)$ such that $x_0 \in U_x$. Then $x_0 \in C_{i_0}$ and $U_x \cap C_{i_0} \ne \emptyset$ which implies that

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$U_x \cap (\cup_{i\in\lambda} C_i) \neq \emptyset$ and $x \in \overline{\cup_{i\in\lambda} C_i}$. Now $x \notin f^{-1}(\overline{B_i})$, and it follows that $x \notin \overline{C_i}$ for every $i \in \lambda$. Hence $t_S(X) \leq \tau$. The proof is complete.

REMARK 4. One can prove a result similar to Theorem 4.1 for closed-cleavability, as was shown in [9].

DEFINITION 4.1. A topological space X is said to be *double-cleavable* over a class $\mathcal{P}$ of topological spaces provided that for every pair A, B of subsets of X there exist $Y \in \mathcal{P}$ and a continuous mapping $f : X \to Y$ such that $A = f^{-1}f(A)$ and $B = f^{-1}f(B)$.

REMARK 5. In a similar way to pointwise cleavability we can define double pointwise cleavability by the following condition: for every $x, y \in X$ such that $x \neq y$ there exist $Y \in \mathcal{P}$ and a continuous mapping $f : X \to Y$ such that $\{x\} = f^{-1}f(x)$ and $\{y\} = f^{-1}f(y)$.

THEOREM 4.2. *If a space X is pseudo-open double-cleavable over the class of spaces Y such that $t_S(Y) \leq \tau$ then $t_S(X) \leq \tau$.*

PROOF. Let $A \subset X$ be a non-closed set and $x \in \overline{A}\setminus A$. There exist a space Y and a pseudo-open continuous mapping $f : X \to Y$ such that $\{x\} = f^{-1}f(x)$, $t_S(Y) \leq \tau$ and $A = f^{-1}f(A)$. Clearly $f(x) \in \overline{f(A)}\setminus f(A)$. Since $t_S(Y) \leq \tau$, there exists a family $\mathcal{B}$ of subsets of $f(A)$ such that $|\mathcal{B}| \leq \tau$, $f(x) \notin \overline{B}$ for every $B \in \mathcal{B}$ and $f(x) \in \overline{\cup\mathcal{B}}$. For every $B \in \mathcal{B}$ we have $f^{-1}(B) \subset A$ and $x \notin f^{-1}(\overline{B})$ which implies that $x \notin \overline{f^{-1}(B)}$. Put $\mathcal{B}' = \{f^{-1}(B) : B \in \mathcal{B}\}$. Obviously $|\mathcal{B}'| \leq \tau$. Since $x \notin \overline{f^{-1}(B)}$ for $B \in \mathcal{B}$, to complete the proof it is sufficient to show that $x \in \overline{\cup\mathcal{B}'}$. Since f is pseudo-open, for every neighbourhood U_x of $f^{-1}f(x) = \{x\}$ in X we have $f(x) \in \operatorname{Int}(f(U_x)) = U'_{f(x)}$. Since $f(x) \in \overline{\cup\mathcal{B}}$ we have $U'_{f(x)} \cap (\cup\mathcal{B}) \neq \emptyset$ which implies that $f(y) \in U'_{f(x)} \cap B$ for some $y \in U_x$ and some $B \in \mathcal{B}$. Then $y \in U_x \cap f^{-1}(B)$ and hence $U_x \cap (\cup\mathcal{B}') \neq \emptyset$, that is, $x \in \overline{\cup\mathcal{B}'}$. The proof is complete.

COROLLARY 4.2. *If X is open (closed)-double-cleavable over the class of spaces Y such that $t_S(Y) \leq \tau$, then $t_S(X) \leq \tau$.*

PROOF. This follows from Theorem 4.2 and Corollary 3.4.

If $\mathcal{M}_\tau$ denote the class of open continuous mappings with density of every fibre $\leq \tau$. We have the following

THEOREM 4.3. *If X is $\mathscr{M}_\tau$-cleavable over the class of spaces Y such that $t(Y) \leq \tau$, then $t(X) \leq \tau$.*

PROOF. Let $A \subset X$ and $x \in \overline{A} \setminus A$. There exist a space Y and a mapping $f \in \mathscr{M}_\tau$, $f : X \to Y$, such that $t(Y) \leq \tau$ and $A = f^{-1}f(A)$. Obviously $f(x) = y \in \overline{f(A)} \setminus f(A)$. Since $t(Y) \leq \tau$, there exists a subset $B \subset f(A)$ such that $|B| \leq \tau$ and $y = f(x) \in \overline{B}$. Then $f^{-1}(B) \subset A$, and for every $z \in B$ there exists a subset $C_z \subset f^{-1}(z)$ such that $|C_z| \leq \tau$ and $\overline{C_z} = f^{-1}(z)$. We have $C = \bigcup_{z \in B} C_z \subseteq \bigcup_{z \in B} f^{-1}(z) = f^{-1}(B)$ and $|C| \leq \tau$. Clearly $f^{-1}(B) \subset \overline{C}$, which implies that $\overline{f^{-1}(B)} \subset \overline{C}$. It remains to prove that $x \in \overline{C}$, and for that it is enough to show that $x \in \overline{f^{-1}(B)}$. Assume the contrary. Then the set $U = X \setminus \overline{f^{-1}(B)}$ is open in X and $x \in U$. Since f is open, the set $f(U)$ is open. Clearly, $f(x) \in f(U)$ and $f(U) \cap B = \emptyset$. But this is in contradiction with $f(x) \in \overline{B}$. The proof is complete.

For the class $\mathscr{M}_{\aleph_0}$ of open continuous mappings with every fibre separable we have the following.

COROLLARY 4.3. *If X is $\mathscr{M}_{\aleph_0}$-cleavable over the class of spaces with countable tightness then X has countable tightness.*

5. Closed pointwise cleavability over some special classes of spaces

In this section we consider the spaces which are pointwise cleavable over the class of all *Lots*, that is, over the class of all linearly ordered topological spaces.

We exhibit some necessary conditions for this kind of cleavability which are unfortunately not sufficient.

Let us recall the following

DEFINITION 5.1. A topological space X is said to be

(i) *pseudo-radial* or *chain-net*, provided that for any non-closed set $A \subset X$ there exists a transfinite sequence $\{x_\alpha\}_{\alpha \in \lambda}$ in A which converges to some point $x \in \overline{A} \setminus A$; (if in the above definition the sequence can be chosen in such a way that λ is regular and $x \notin \overline{\{x_\alpha : \alpha \in \beta\}}$ for any $\beta \in \lambda$ then the space is said to be almost radial);

(ii) *radial*, provided that for any set $A \subset X$ and any point $x \in \overline{A}$ there exists a transfinite sequence in A converging to x;

(iii) *weakly radial*, provided that for any closed set $F \subset X$ and for any non-isolated point $x \in F$ there exists a transfinite sequence in $F \setminus \{x\}$ converging to x;

(iv) *sequential*, provided that a set $A \subset X$ is closed if and only if for every sequence $\{x_n : n \in \mathbb{N}^+\}] \subset A$ converging to a point $x \in X$ the point x belongs to A;

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(v) *weakly sequential*, provided that for any closed set $F \subset X$ and for any non-isolated point $x \in F$ there exists a sequence in $F \backslash \{x\}$, converging to x.

THEOREM 5.1. *If a space X is closed-pointwise cleavable over the class of radial spaces, then X is radial.*

PROOF. This is demonstrated by the same argument as Theorem 3.2.

COROLLARY 5.1. *If X is a compact space pointwise-cleavable over the class of Hausdorff radial spaces then X is a Hausdorff radial space.*

We do not know the answer to the following question:

QUESTION 1. Let X be a compact space pointwise-cleavable over the class of pseudo-radial Hausdorff spaces. Is then X pseudo-radial?

THEOREM 5.2. *If a space X is closed-pointwise cleavable over the class of weakly radial spaces then X is weakly radial.*

PROOF. Let F be a closed subset of X and let x be an accumulation point of F. There exist a weakly radial space Y and a closed continuous mapping $f : X \to Y$ such that $\{x\} = f^{-1}f(x)$. Since f is closed, $f(F)$ is a closed subset of Y such that $f(x) \in f(F)$. Obviously, $f(x)$ is not isolated in $f(F)$. There exists a transfinite sequence $\{y_\alpha : \alpha \in \lambda\}$ in $f(F)\backslash\{f(x)\}$ converging to $f(x)$. For every $\alpha \in \lambda$ pick a point $x_\alpha \in F \cap f^{-1}(y_\alpha)$. In this way we obtain a transfinite sequence $\{x_\alpha : \alpha \in \lambda\}$ in $F\backslash f^{-1}f(x) = F\backslash\{x\}$ converging to x, since the mapping f is closed. The proof is complete.

THEOREM 5.3. *If X is closed pointwise-cleavable over the class of weakly sequential spaces then X is weakly sequential.*

PROOF. This is demonstrated by the same argument as Theorem 5.2.

COROLLARY 5.2. *If X is a compact space pointwise cleavable over the class of Hausdorff weakly sequential spaces then X is a Hausdorff weakly sequential space.*

PROOF. This follows from Theorem 5.3 and Proposition 1.4.

Making use of the the result [12] 'Every Hausdorff compact weakly sequential space is sequential', and applying Theorem 5.3 we conclude that it is compatible with ZFC that every Hausdorff compact space which is pointwise-cleavable over the class of Hausdorff sequential spaces is sequential. Now it is natural to ask the following question.

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QUESTION 2. Is every Hausdorff compact space X, which is pointwise-cleavable over the class of Hausdorff sequential spaces, sequential (in ZFC)?

DEFINITION 5.2. Let X be a topological space and let ξ be a filter base on X, We say that

(1) ξ is *converging to a point* $x \in X$ provided that for any neighbourhood O_x of x in X there exists $P \in \xi$ such that $P \subseteq O_x$;
(2) $x \in X$ is an *adherence* point of ξ, that is, $x \in \overline{\xi}$, provided that $x \in \cap\{\overline{P} : P \in \xi\}$;
(3) ξ is a *chain*, provided that for every $A, B \in \xi$ either $A \subset B$ or $B \subset A$.

Two filter bases ξ and η are said to be *synchronous* (we write $\xi©\eta$) if $A \cap B \neq \emptyset$ for every $A \in \xi$ and every $B \in \eta$.

DEFINITION 5.3. A topological space X is said to be *bisequential* provided that for any filter base ξ in X and any point $x \in X$ such that $x \in \overline{\xi}$ there exists a filter base η in X such that

(i) η is converging to x,
(ii) η is countable,
(iii) $\eta©\xi$ – that is, η and ξ are synchronous.

REMARK 6. All first countable spaces are bisequential and all bisequential spaces are Fréchet-Urysohn. An interesting property of bisequential spaces is that they form a countably productive class of spaces.

DEFINITION 5.4. A topological space X is said to be biradial provided that for any filter base ξ in X and for any $x \in \overline{\xi}$, there exists a chain η such that

(i) η is converging to x,
(ii) $\eta©\xi$.

REMARK 7. Every Lots and every subspace of a Lots is a biradial space.

THEOREM 5.4. *If a space X is pointwise c-cleavable over the class of bisequential spaces, then X is bisequential.*

PROOF. Let $x \in X$. There exist a bisequential space Y and a continuous mapping $f : X \to Y$ closed in x such that $\{x\} = f^{-1}f(x)$. Let ξ be a filter base on X such that $x \in \overline{\xi}$. Put $f(\xi) = \{f(A) : A \in \xi\}$; $f(\xi)$ is a filter base on Y and $f(x) \in \overline{f(\xi)}$. There exists a countable filter base η on X converging to $f(x)$ and synchronous with $f(\xi)$. Put $\eta' = f^{-1}(\eta) = \{f^{-1}(B) : B \in \eta\}$. Then η' is a countable filter base converging to x since $\{x\} = f^{-1}f(x)$ and f is closed in x. It is obvious that $\eta'©\xi$. The proof is complete.

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COROLLARY 5.3. *If a space X is compact and pointwise-cleavable over the class of Hausdorff bisequential spaces then X is a Hausdorff bisequential space.*

PROOF. This follows from Theorem 5.4 and Proposition 1.4.

THEOREM 5.5. *If X is closed pointwise-cleavable over the class of biradial spaces, then X is biradial.*

PROOF. Let $x \in X$. There exists a biradial space Y and a closed continuous mapping $f : X \to Y$ such that $\{x\} = f^{-1}f(x)$. Let ξ be a filter base on X such that $x \in \overline{\xi}$. Then $f(\xi)$ is a filter base on Y and $f(x) \in \overline{f(\xi)}$. There exists a chain η on Y which is converging to $f(x)$ and is synchronous with $f(\xi)$. Put $\eta' = f^{-1}(\eta) = \{f^{-1}(B) : B \in \eta\}$. As in the proof of Theorem 5.4, η' is converging to x and η'©ξ. Clearly η' is a chain.

COROLLARY 5.4. *If X is a compact space pointwise-cleavable over the class of Hausdorff biradial spaces, then X is a Hausdorff biradial space.*

PROOF. This follows from Theorem 5.5 and Proposition 1.4.

REMARK 8. Since every Lots is a biradial space, we have the following result.

COROLLARY 5.5. *If a space X is closed pointwise-cleavable over the class of all Lots then X is biradial.*

Corollary 5.5 gives a necessary condition for cleavability over the class of all Lots which is not sufficient.

We now present a much stronger necessary condition for the closed pointwise-cleavability over Lots.

THEOREM 5.6. *If a space X is closed pointwise-cleavable over the class of all Lots, then for every $x \in X$ there exist two chains η_1 and η_2 on X such that*

(1) *η_1 and η_2 are converging to x,*
(2) *for every filter base ξ on X such that $x \in \overline{\xi}$, either η_1©ξ or η_2©ξ (equivalently, for any $A \in \eta_1$ and for any $B \in \eta_2$, $A \cup B$ is a neighbourhood of x (not necessarily open)).*

PROOF. Let $x \in X$. There exist a Lots Y and a closed continuous mapping $f : X \to Y$ such that $\{x\} = f^{-1}f(x)$. Since Y is a Lots, there are two chains μ_1 and μ_2 of subsets of Y, converging to $f(x)$ such that for every filter base γ on Y, satisfying the condition $f(x) \in \overline{\gamma}$ either μ_1 or μ_2 is synchronous with γ. It is easy to

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see that the chains $\eta_1 = \{f^{-1}(P) : P \in \mu_1\}$ and $\eta_2 = \{f^{-1}(P) : P \in \mu_2\}$ satisfy the conditions (1) and (2).

QUESTION 3. Is the converse to Theorem 5.6 true in the class of Hausdorff (or regular) spaces?

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