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A Note on Homological Dimensions of Artinian Local Cohomology Modules

Kamal Bahmanpour

Abstract. Let $(R, \mathfrak{m})$ be a non-zero commutative Noetherian local ring (with identity) and let M be a non-zero finitely generated R-module. In this paper for any $\mathfrak{p} \in \operatorname{Spec}(R)$ we show that

$$\operatorname{injdim}_{R_\mathfrak{p}} H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \quad \text{and} \quad \operatorname{fd}_{R_\mathfrak{p}} H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})$$

are bounded from above by $\operatorname{injdim}_R H^i_\mathfrak{m}(M)$ and $\operatorname{fd}_R H^i_\mathfrak{m}(M)$ respectively, for all integers $i \geq \dim(R/\mathfrak{p})$.

1 Introduction

Let R be a commutative ring with identity and I be an ideal of R. For an arbitrary R-module M, the i-th local cohomology module of M with respect to I is defined as

$$H^i_I(M) = \varinjlim_{n \geq 1} \operatorname{Ext}^i_R(R/I^n, M).$$

We refer the reader to [2, 4] for more details. The module $H^i_I(M)$ has both algebraic and geometric aspects, but this is very difficult to treat. Hartshorne [5] introduced an interesting class of modules. He defined an R-module M to be I-cofinite if $\operatorname{Supp} M \subseteq V(I)$ and $\operatorname{Ext}^j_R(R/I, M)$ is finitely generated for all j.

Delfino and Marley [3, Theorem 1] and Yoshida [9, Theorem 1.1] have shown that for any ideal of dimension one of a Noetherian local ring $(R, \mathfrak{m})$ (*i.e.,* $\dim R/I = 1$), the modules $H^i_I(M)$ are I-cofinite for all i and all finitely generated modules M. Also, the author and Naghipour [1] have removed the local assumption on R.

In the sequel $(R, \mathfrak{m})$ denotes a non-zero commutative Noetherian local ring (with identity). In this paper we establish some results for finiteness of extension and torsion functors of Artinian local cohomology modules. Then, as an application, we get a couple of inequalities on homological dimensions of local cohomology modules. More precisely, this paper's main result is the following theorem.

Theorem 1.1 *Let M be a finitely generated R-module. Then for any $\mathfrak{p} \in \operatorname{Spec}(R)$ and all integers $i \geq \dim(R/\mathfrak{p})$, we have the following inequalities:*

(i) $\operatorname{injdim}_{R_\mathfrak{p}} H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \leq \operatorname{injdim}_R H^i_\mathfrak{m}(M)$;

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(ii) $\operatorname{fd}_{R_{\mathfrak{p}}} H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \operatorname{fd}_R H^i_{\mathfrak{m}}(M)$.

For any ideal $\mathfrak{a}$ of R, we denote the set $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq \mathfrak{a}\}$ by $V(\mathfrak{a})$. Also, *the radical of* $\mathfrak{a}$, denoted by $\operatorname{Rad}(\mathfrak{a})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{a}$ for some $n \in \mathbb{N}\}$. For any R-module M we denote injective and flat dimensions of M by $\operatorname{injdim}_R(M)$ and $\operatorname{fd}_R(M)$ respectively. For any unexplained notation and terminology, we refer the reader to [2, 6].

2 Homological Dimensions of Artinian Local Cohomology Modules

The following lemmas will be useful in the proof of Lemma 2.4.

Lemma 2.1 *Let R be a Noetherian (not necessary local) ring and let I be an ideal of R. Let M be an R-module with support in $V(I)$. Then the following statements are equivalent:*

(i) *the R-module $(0 :_M I)$ has finite length;*
(ii) *M is Artinian and I-cofinite.*

Proof See [7, Proposition 4.1]. ∎

Lemma 2.2 *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be a non-zero R-module. Then the following statements are equivalent:*

(i) *M is finitely generated and* $\operatorname{Supp}(M) = \{\mathfrak{m}\}$;
(ii) *M is finitely generated and Artinian;*
(iii) *M has finite length.*

Proof (i)⇒(ii) Since M is finitely generated, it follows that

$$\operatorname{Supp}_R(M) = V\big(\operatorname{Ann}_R(M)\big).$$

Hence, from the hypothesis we have $V(\operatorname{Ann}_R(M)) = \{\mathfrak{m}\}$. So the ideal $\operatorname{Ann}_R(M)$ is $\mathfrak{m}$-primary. In particular, the ring $R/\operatorname{Ann}_R(M)$ is Artinian. Now, since M can be viewed as a homomorphic image of a finitely generated free $R/\operatorname{Ann}_R(M)$-module, it follows that M is an Artinian $R/\operatorname{Ann}_R(M)$-module. But this implies that M is an Artinian R-module.

(ii)⇒(iii) and (iii)⇒(i) are clear. ∎

Lemma 2.3 *Let $(R, \mathfrak{m})$ be a Noetherian local ring, I an ideal of R, and M be a non-zero I-cofinite R-module. Then the R-module $\Gamma_{\mathfrak{m}}(M) = \cup_{n=1}^{\infty}(0 :_M \mathfrak{m}^n)$ is Artinian and I-cofinite.*

Proof We may assume that $\Gamma_{\mathfrak{m}}(M) \neq 0$. The exact sequence $0 \to \Gamma_{\mathfrak{m}}(M) \hookrightarrow M$ induces the exact sequence

$$0 \longrightarrow \operatorname{Hom}_R\big(R/I, \Gamma_{\mathfrak{m}}(M)\big) \longrightarrow \operatorname{Hom}_R(R/I, M).$$

By hypothesis M is I-cofinite, and so the R-module $\operatorname{Hom}_R(R/I, M)$ is finitely generated. Therefore, the R-module $(0 :_{\Gamma_{\mathfrak{m}}(M)} I) \cong \operatorname{Hom}_R(R/I, \Gamma_{\mathfrak{m}}(M))$ is also finitely

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generated. Since the R-module $\Gamma_{\mathfrak{m}}(M)$ is $\mathfrak{m}$-torsion, it follows that $\Gamma_{\mathfrak{m}}(M)$ is a non-zero I-torsion R-module. Hence, we have $(0 :_{\Gamma_{\mathfrak{m}}(M)} I) \neq 0$. So

$$\varnothing \neq \operatorname{Supp}\big((0 :_{\Gamma_{\mathfrak{m}}(M)} I)\big) \subseteq \operatorname{Supp}\big(\Gamma_{\mathfrak{m}}(M)\big) \subseteq \{\mathfrak{m}\},$$

which implies that $\operatorname{Supp}((0 :_{\Gamma_{\mathfrak{m}}(M)} I)) = \{\mathfrak{m}\}$. Now using Lemma 2.2 we deduce that the R-module $(0 :_{\Gamma_{\mathfrak{m}}(M)} I)$ has finite length. Therefore, as $\operatorname{Supp}(\Gamma_{\mathfrak{m}}(M)) = \{\mathfrak{m}\} \subseteq V(I)$, the assertion follows from Lemma 2.1. ∎

The following lemma is crucial for the proof of the main result.

Lemma 2.4 *Let $(R, \mathfrak{m})$ be a Noetherian local ring with $\dim R = d \geq 1$ and $\mathfrak{p}$ be a prime ideal of R such that $\dim R/\mathfrak{p} = 1$. Let M be a $\mathfrak{p}$-cofinite R-module such that $\operatorname{Supp} M = V(\mathfrak{p})$ and let $n \geq 0$ be an integer. Then the following hold:*

(i) *the R-module $\operatorname{Ext}^n_R(R/\mathfrak{p}, H^1_{\mathfrak{m}}(M))$ is finitely generated if and only if*

$$\operatorname{Ext}^n_R(R/\mathfrak{p}, M)_{\mathfrak{p}} = 0;$$

(ii) *the R-module $\operatorname{Tor}^R_n(R/\mathfrak{p}, H^1_{\mathfrak{m}}(M))$ is finitely generated if and only if*

$$\operatorname{Tor}^R_n(R/\mathfrak{p}, M)_{\mathfrak{p}} = 0.$$

Proof In view of Lemma 2.3, the R-module $\Gamma_{\mathfrak{m}}(M)$ is $\mathfrak{p}$-cofinite. Now, from the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{m}}(M) \longrightarrow M \longrightarrow M/\Gamma_{\mathfrak{m}}(M) \longrightarrow 0,$$

it follows that the R-module $M/\Gamma_{\mathfrak{m}}(M)$ also is $\mathfrak{p}$-cofinite. Moreover, there is an isomorphism of R-modules as $H^1_{\mathfrak{m}}(M) \cong H^1_{\mathfrak{m}}(M/\Gamma_{\mathfrak{m}}(M))$, and using the fact that $\operatorname{Supp}(\Gamma_{\mathfrak{m}}(M)) \subseteq \{\mathfrak{m}\}$, by [6, Exercise 7.7] we have the following isomorphisms:

$$\begin{aligned}\operatorname{Ext}^n_R\big(R/\mathfrak{p}, M/\Gamma_{\mathfrak{m}}(M)\big)_{\mathfrak{p}} &\cong \operatorname{Ext}^n_{R_{\mathfrak{p}}}\big(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, \big(M/\Gamma_{\mathfrak{m}}(M)\big)_{\mathfrak{p}}\big)\\ &\cong \operatorname{Ext}^n_{R_{\mathfrak{p}}}\big(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}}/\big(\Gamma_{\mathfrak{m}}(M)\big)_{\mathfrak{p}}\big)\\ &\cong \operatorname{Ext}^n_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}}) \cong \operatorname{Ext}^n_R(R/\mathfrak{p}, M)_{\mathfrak{p}}.\end{aligned}$$

Therefore, replacing M by $M/\Gamma_{\mathfrak{m}}(M)$), we may assume without loss of generality that $\Gamma_{\mathfrak{m}}(M) = 0$. Since $\dim R/\mathfrak{p} = 1$, it follows that there is an element $x \in \mathfrak{m} \setminus \mathfrak{p}$ such that $\dim R/(\mathfrak{p} + Rx) = 0$. In particular $\operatorname{Rad}(\mathfrak{p} + Rx) = \mathfrak{m}$. In view of [2, Theorem 2.2.4], there is an exact sequence:

$$0 \longrightarrow M \longrightarrow D_{\mathfrak{m}}(M) \longrightarrow H^1_{\mathfrak{m}}(M) \longrightarrow 0. \tag{2.1}$$

As $\operatorname{Ass}_R(M) \subseteq \operatorname{Supp}(M) = \{\mathfrak{p}, \mathfrak{m}\}$ and $\Gamma_{\mathfrak{m}}(M) = 0$, we have $\mathfrak{m} \notin \operatorname{Ass}_R(M)$ and so $\operatorname{Ass}_R(M) = \{\mathfrak{p}\}$. Since $x \notin \mathfrak{p}$, it follows that x is an M-regular element, and so the sequence $0 \to M \xrightarrow{x} M$, is exact. Therefore, for each $k \geq 1$, there is an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(\mathfrak{m}^k, M) \xrightarrow{x} \operatorname{Hom}_R(\mathfrak{m}^k, M).$$

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By definition we have

$$D_{\mathfrak{m}}(M) = \varinjlim_{k\geq 1} \operatorname{Hom}_R(\mathfrak{m}^k, M),$$

hence it follows that x is a $D_{\mathfrak{m}}(M)$-regular element. In particular $\Gamma_{Rx}(D_{\mathfrak{m}}(M)) = 0$. Now the exact sequence (2.1) induces an exact sequence

$$(2.2) \qquad 0 \longrightarrow H^1_{\mathfrak{m}}(M) \xrightarrow{f} H^1_{Rx}(M) \longrightarrow H^1_{Rx}\big(D_{\mathfrak{m}}(M)\big) \longrightarrow H^1_{Rx}\big(H^1_{\mathfrak{m}}(M)\big).$$

But, from the Grothendieck's Vanishing Theorem, [2, Theorem 6.1.2], we deduce that $H^1_{Rx}(H^1_{\mathfrak{m}}(M)) = 0$. Also from the facts that $\Gamma_{\mathfrak{p}}(M) = M$ and $\operatorname{Rad}(\mathfrak{p} + Rx) = \mathfrak{m}$, it follows that $H^1_{Rx}(M) = H^1_{\mathfrak{m}}(M)$. Consequently, the exact sequence (2.2) gives the exact sequence

$$(2.3) \qquad 0 \longrightarrow H^1_{\mathfrak{m}}(M) \xrightarrow{f} H^1_{\mathfrak{m}}(M) \longrightarrow H^1_{Rx}\big(D_{\mathfrak{m}}(M)\big) \longrightarrow 0.$$

By [1, Theorem 2.15], the R-module $H^1_{\mathfrak{m}}(M)$ is Artinian, therefore it follows from the exact sequence (2.3) that f is an epimorphism, and so $H^1_{Rx}(D_{\mathfrak{m}}(M)) = 0$. Using

$$\Gamma_{Rx}\big(D_{\mathfrak{m}}(M)\big) = 0 = H^1_{Rx}\big(D_{\mathfrak{m}}(M)\big),$$

we deduce that the map $D_{\mathfrak{m}}(M) \xrightarrow{x} D_{\mathfrak{m}}(M)$ is an isomorphism. On the other hand the exact sequence (2.1) induces the exact sequence

$$(2.4) \qquad \operatorname{Ext}^{n-1}_R\big(R/\mathfrak{p}, H^1_{\mathfrak{m}}(M)\big) \longrightarrow \operatorname{Ext}^n_R(R/\mathfrak{p}, M) \longrightarrow \operatorname{Ext}^n_R\big(R/\mathfrak{p}, D_{\mathfrak{m}}(M)\big) \longrightarrow \operatorname{Ext}^n_R\big(R/\mathfrak{p}, H^1_{\mathfrak{m}}(M)\big).$$

Since $\operatorname{Supp} H^1_{\mathfrak{m}}(M) \subseteq \{\mathfrak{m}\}$, it follows that

$$\operatorname{Ext}^n_R\big(R/\mathfrak{p}, H^1_{\mathfrak{m}}(M)\big)_{\mathfrak{p}} = 0 = \operatorname{Ext}^{n-1}_R\big(R/\mathfrak{p}, H^1_{\mathfrak{m}}(M)\big)_{\mathfrak{p}},$$

and therefore $\operatorname{Ext}^n_R(R/\mathfrak{p}, M)_{\mathfrak{p}} = 0$ if and only if $\operatorname{Ext}^n_R(R/\mathfrak{p}, D_{\mathfrak{m}}(M))_{\mathfrak{p}} = 0$ if and only if $\operatorname{Ext}^n_R(R/\mathfrak{p}, D_{\mathfrak{m}}(M)) = 0$. (In fact if $\operatorname{Ext}^n_R(R/\mathfrak{p}, D_{\mathfrak{m}}(M))_{\mathfrak{p}} = 0$, then $\operatorname{Ext}^n_R(R/\mathfrak{p}, D_{\mathfrak{m}}(M))$ is $\mathfrak{m}$-torsion and hence is Rx-torsion and therefore from the isomorphism $D_{\mathfrak{m}}(M) \xrightarrow{x} D_{\mathfrak{m}}(M)$, we deduce that $\operatorname{Ext}^n_R(R/\mathfrak{p}, D_{\mathfrak{m}}(M)) = 0$.) But using the NAK lemma and the isomorphism $D_{\mathfrak{m}}(M) \xrightarrow{x} D_{\mathfrak{m}}(M)$, we have $\operatorname{Ext}^n_R(R/\mathfrak{p}, D_{\mathfrak{m}}(M)) = 0$ if and only if the R-module $\operatorname{Ext}^n_R(R/\mathfrak{p}, D_{\mathfrak{m}}(M))$ is finitely generated. Now since M is $\mathfrak{p}$-cofinite, it follows from the exact sequence (2.4) that the R-module $\operatorname{Ext}^n_R(R/\mathfrak{p}, H^1_{\mathfrak{m}}(M))$ is finitely generated if and only if $\operatorname{Ext}^n_R(R/\mathfrak{p}, M)_{\mathfrak{p}} = 0$. This completes the proof of (i).

(ii) In view of method used in the proof of (i), the R-module $M/\Gamma_{\mathfrak{m}}(M)$ is $\mathfrak{p}$-cofinite. Also using [6, Exercise 7.7] we have the isomorphism

$$\operatorname{Tor}^R_n\big(R/\mathfrak{p}, M/\Gamma_{\mathfrak{m}}(M)\big)_{\mathfrak{p}} \cong \operatorname{Tor}^R_n(R/\mathfrak{p}, M)_{\mathfrak{p}}.$$

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Therefore, again as in the the proof of (i) we can assume $\Gamma_{\mathfrak{m}}(M) = 0$. Consequently, the map $D_{\mathfrak{m}}(M) \xrightarrow{x} D_{\mathfrak{m}}(M)$ is an isomorphism. The exact sequence (2.1) induces the following exact sequence:

$$\operatorname{Tor}_n^R\big(R/\mathfrak{p}, D_{\mathfrak{m}}(M)\big) \longrightarrow \operatorname{Tor}_n^R\big(R/\mathfrak{p}, H^1_{\mathfrak{m}}(M)\big) \longrightarrow \operatorname{Tor}_{n-1}^R(R/\mathfrak{p}, M)$$
$$\operatorname{Tor}_{n+1}^R\big(R/\mathfrak{p}, H^1_{\mathfrak{m}}(M)\big) \longrightarrow \operatorname{Tor}_n^R(R/\mathfrak{p}, M) \longrightarrow$$

Since $\operatorname{Supp} H^1_{\mathfrak{m}}(M) \subseteq \{\mathfrak{m}\}$, it follows that

$$\operatorname{Tor}_n^R\big(R/\mathfrak{p}, H^1_{\mathfrak{m}}(M)\big)_{\mathfrak{p}} = 0 = \operatorname{Tor}_{n+1}^R\big(R/\mathfrak{p}, H^1_{\mathfrak{m}}(M)\big)_{\mathfrak{p}}.$$

Also, since M is $\mathfrak{p}$-cofinite, it follows from [7, Theorem 2.1], that the R-modules $\operatorname{Tor}_i^R(R/\mathfrak{p}, M)$ are finitely generated for all integers $i \geq 0$. Now the remaining part of the proof follows from the method of the proof of (i), using the NAK lemma and the fact that, in this situation, we have $\operatorname{Tor}_n^R(R/\mathfrak{p}, D_{\mathfrak{m}}(M))_{\mathfrak{p}} = 0$ if and only if $\operatorname{Tor}_n^R(R/\mathfrak{p}, D_{\mathfrak{m}}(M)) = 0$. ■

Theorem 2.5 *Let $(R, \mathfrak{m})$ be a Noetherian local ring with $\dim R = d \geq 1$ and let M be a finitely generated R-module. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R/\mathfrak{p} = 1$. Then for all integers $n \geq 0$ and $i \geq 1$ the following conditions are equivalent:*

(i) *the R-module $\operatorname{Ext}^n_R(R/\mathfrak{p}, H^i_{\mathfrak{m}}(M))$ is finitely generated;*
(ii) $\operatorname{Ext}^n_R(R/\mathfrak{p}, H^{i-1}_{\mathfrak{p}}(M))_{\mathfrak{p}} = 0$.

Proof Since $\dim R/\mathfrak{p} = 1$, it follows that there is an element $x \in \mathfrak{m} \setminus \mathfrak{p}$ such that $\operatorname{Rad}(\mathfrak{p} + Rx) = \mathfrak{m}$. By [8, Corollary 3.5] there is the exact sequence

$$0 \longrightarrow H^1_{Rx}\big(H^{i-1}_{\mathfrak{p}}(M)\big) \longrightarrow H^i_{\mathfrak{p}+Rx}(M) \longrightarrow H^0_{Rx}\big(H^i_{\mathfrak{p}}(M)\big) \longrightarrow 0. \tag{2.5}$$

Since the R-module $H^i_{\mathfrak{p}}(M)$ is $\mathfrak{p}$-torsion and $\operatorname{Rad}(\mathfrak{p} + Rx) = \mathfrak{m}$, using Remark 1.2.3 of [2], it follows that

$$H^0_{Rx}\big(H^i_{\mathfrak{p}}(M)\big) = H^0_{\mathfrak{p}+Rx}\big(H^i_{\mathfrak{p}}(M)\big) = H^0_{\mathfrak{m}}\big(H^i_{\mathfrak{p}}(M)\big) \quad \text{and} \quad H^i_{\mathfrak{p}+Rx}(M) = H^i_{\mathfrak{m}}(M).$$

Also since R is Noetherian it follows from the definition that the ideal $\mathfrak{p}$ is finitely generated. Hence there are elements $a_1, \ldots, a_k \in \mathfrak{p}$ such that $\mathfrak{p} = (a_1, \ldots, a_k)$, for some integer $k \geq 0$. Now if $k \geq 1$, then in view of [8, Corollary 3.5], there exists an exact sequence as follows:

$$0 \longrightarrow H^1_{Ra_1}\Big(H^0_{Rx}\big(H^{i-1}_{\mathfrak{p}}(M)\big)\Big) \longrightarrow H^1_{Rx+Ra_1}\big(H^{i-1}_{\mathfrak{p}}(M)\big) \longrightarrow \tag{2.6}$$
$$H^0_{Ra_1}\Big(H^1_{Rx}\big(H^{i-1}_{\mathfrak{p}}(M)\big)\Big) \longrightarrow 0.$$

But since the R-module $H^{i-1}_{\mathfrak{p}}(M)$ is $\mathfrak{p}$-torsion and $a_1 \in \mathfrak{p}$, it follows that the R-module $H^{i-1}_{\mathfrak{p}}(M)$ is Ra_1-torsion. Consequently, the R-modules

$$H^0_{Rx}(H^{i-1}_{\mathfrak{p}}(M)) \quad \text{and} \quad H^1_{Rx}(H^{i-1}_{\mathfrak{p}}(M))$$

are also Ra_1-torsion. Hence,

$$H^1_{Ra_1}\big(H^0_{Rx}\big(H^{i-1}_{\mathfrak{p}}(M)\big)\big) = 0 \quad \text{and} \quad H^0_{Ra_1}\big(H^1_{Rx}\big(H^{i-1}_{\mathfrak{p}}(M)\big)\big) = H^1_{Rx}\big(H^{i-1}_{\mathfrak{p}}(M)\big).$$

So it follows from the exact sequence (2.6) that $H^1_{Rx+Ra_1}(H^{i-1}_{\mathfrak{p}}(M)) \cong H^1_{Rx}(H^{i-1}_{\mathfrak{p}}(M))$. Now, proceeding in the same way, we see that for each $1 \le i \le k$, there is an isomorphism of R-modules as $H^1_{Rx}(H^{i-1}_{\mathfrak{p}}(M)) = H^1_{Rx+(a_1,\dots,a_i)}(H^{i-1}_{\mathfrak{p}}(M))$. In particular,

$$H^1_{Rx}\big(H^{i-1}_{\mathfrak{p}}(M)\big) = H^1_{Rx+\mathfrak{p}}\big(H^{i-1}_{\mathfrak{p}}(M)\big).$$

But, since $\operatorname{Rad}(\mathfrak{p}+Rx) = \mathfrak{m}$, using Remark 1.2.3 of [2], we have $H^1_{Rx+\mathfrak{p}}(H^{i-1}_{\mathfrak{p}}(M)) = H^1_{\mathfrak{m}}(H^{i-1}_{\mathfrak{p}}(M))$. Therefore, $H^1_{Rx}(H^{i-1}_{\mathfrak{p}}(M)) \cong H^1_{\mathfrak{m}}(H^{i-1}_{\mathfrak{p}}(M))$. Hence from the exact sequence (2.5) we deduce the exact sequence

$$0 \longrightarrow H^1_{\mathfrak{m}}(H^{i-1}_{\mathfrak{p}}(M)) \longrightarrow H^i_{\mathfrak{m}}(M) \longrightarrow H^0_{\mathfrak{m}}(H^i_{\mathfrak{p}}(M)) \longrightarrow 0. \tag{2.7}$$

By [1, Corollary 2.7] the R-modules $H^i_{\mathfrak{p}}(M)$ and $H^{i-1}_{\mathfrak{p}}(M)$ are $\mathfrak{p}$-cofinite. Consequently, according to the Lemma 2.3 the R-module $\Gamma_{\mathfrak{m}}(H^i_{\mathfrak{p}}(M)) = H^0_{\mathfrak{m}}(H^i_{\mathfrak{p}}(M))$ is $\mathfrak{p}$-cofinite. The exact sequence (2.7) induces the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \operatorname{Hom}_R\big(R/\mathfrak{p}, H^1_{\mathfrak{m}}\big(H^{i-1}_{\mathfrak{p}}(M)\big)\big) \longrightarrow \operatorname{Hom}_R\big(R/\mathfrak{p}, H^i_{\mathfrak{m}}(M)\big) \\ &\longrightarrow \operatorname{Hom}_R\big(R/\mathfrak{p}, H^0_{\mathfrak{m}}\big(H^i_{\mathfrak{p}}(M)\big)\big) \longrightarrow \operatorname{Ext}^1_R\big(R/\mathfrak{p}, H^1_{\mathfrak{m}}\big(H^{i-1}_{\mathfrak{p}}(M)\big)\big) \\ &\longrightarrow \operatorname{Ext}^1_R\big(R/\mathfrak{p}, H^i_{\mathfrak{m}}(M)\big) \longrightarrow \operatorname{Ext}^1_R\big(R/\mathfrak{p}, H^0_{\mathfrak{m}}\big(H^i_{\mathfrak{p}}(M)\big)\big) \longrightarrow \cdots, \end{aligned}$$

which implies that for each $n \ge 0$ the R-module $\operatorname{Ext}^n_R(R/\mathfrak{p}, H^i_{\mathfrak{m}}(M))$ is finitely generated if and only if the R-module $\operatorname{Ext}^n_R(R/\mathfrak{p}, H^1_{\mathfrak{m}}(H^{i-1}_{\mathfrak{p}}(M)))$ is finitely generated. By Lemma 2.4 this is equivalent to $\operatorname{Ext}^n_R(R/\mathfrak{p}, H^{i-1}_{\mathfrak{p}}(M))_{\mathfrak{p}} = 0$. ■

Theorem 2.6 *Let $(R, \mathfrak{m})$ be a Noetherian local ring with $\dim R = d \ge 1$ and let M be a finitely generated R-module. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R/\mathfrak{p} = 1$. Then for all integers $n \ge 0$ and $i \ge 1$, the following conditions are equivalent:*

(i) *the R-module $\operatorname{Tor}^R_n(R/\mathfrak{p}, H^i_{\mathfrak{m}}(M))$ is finitely generated;*
(ii) $\operatorname{Tor}^R_n(R/\mathfrak{p}, H^{i-1}_{\mathfrak{p}}(M))_{\mathfrak{p}} = 0$.

Proof As in the proof of Theorem 2.5, there is an exact sequence

$$0 \longrightarrow H^1_{\mathfrak{m}}\big(H^{i-1}_{\mathfrak{p}}(M)\big) \longrightarrow H^i_{\mathfrak{m}}(M) \longrightarrow H^0_{\mathfrak{m}}\big(H^i_{\mathfrak{p}}(M)\big) \longrightarrow 0.$$

This exact sequence induces the long exact sequence

$$\begin{aligned} &R/\mathfrak{p} \otimes_R H^i_{\mathfrak{m}}(M) \longrightarrow R/\mathfrak{p} \otimes_R H^0_{\mathfrak{m}}\big(H^i_{\mathfrak{p}}(M)\big) \longrightarrow 0 \\ &\operatorname{Tor}^R_1\big(R/\mathfrak{p}, H^0_{\mathfrak{m}}\big(H^i_{\mathfrak{p}}(M)\big)\big) \longrightarrow R/\mathfrak{p} \otimes_R H^1_{\mathfrak{m}}\big(H^{i-1}_{\mathfrak{p}}(M)\big) \longrightarrow \\ &\cdots \longrightarrow \operatorname{Tor}^R_1\big(R/\mathfrak{p}, H^1_{\mathfrak{m}}\big(H^{i-1}_{\mathfrak{p}}(M)\big)\big) \longrightarrow \operatorname{Tor}^R_1\big(R/\mathfrak{p}, H^i_{\mathfrak{m}}(M)\big) \longrightarrow \end{aligned}$$

But in view of [1, Corollary 2.7] and Lemma 2.3, the R-module $H^0_{\mathfrak{m}}(H^i_{\mathfrak{p}}(M))$ is $\mathfrak{p}$-cofinite. Therefore, using [7, Theorem 2.1], it follows that the R-modules $\operatorname{Tor}^R_j(R/\mathfrak{p}, H^0_{\mathfrak{m}}(H^i_{\mathfrak{p}}(M)))$ are finitely generated for all integers $j \geq 0$. Now the above long exact sequence implies that for each $n \geq 0$ the R-module $\operatorname{Tor}^R_n(R/\mathfrak{p}, H^i_{\mathfrak{m}}(M))$ is finitely generated if and only if the R-module $\operatorname{Tor}^R_n(R/\mathfrak{p}, H^1_{\mathfrak{m}}(H^{i-1}_{\mathfrak{p}}(M)))$ is finitely generated. By Lemma 2.4 and [1, Corollary 2.7], this is equivalent to $\operatorname{Tor}^R_n(R/\mathfrak{p}, H^{i-1}_{\mathfrak{p}}(M))_{\mathfrak{p}} = 0$. ■

Before stating the next corollary, note that in this paper, for technical reasons, for each ring R, the injective and flat dimensions of the zero R-module are defined as follows: $\operatorname{fd}_R(0) = -1 = \operatorname{id}_R(0)$.

Corollary 2.7 *Let $(R, \mathfrak{m})$ be a Noetherian local ring with $\dim R = d \geq 1$ and let M be a non-zero finitely generated R-module. Let $i \geq 1$ be an integer. Then for each $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim(R/\mathfrak{p}) = 1$, the following assertions hold:*

(i) *the R-module $\operatorname{Ext}^k_R(R/\mathfrak{p}, H^i_{\mathfrak{m}}(M))$ is not finitely generated, for any $0 \leq k \leq \operatorname{injdim}_{R_{\mathfrak{p}}} H^{i-1}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$;*

(ii) *the R-module $\operatorname{Ext}^k_R(R/\mathfrak{p}, H^i_{\mathfrak{m}}(M))$ is finitely generated, for each*

$$k > \operatorname{injdim}_{R_{\mathfrak{p}}} H^{i-1}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}});$$

whenever $\operatorname{injdim}_{R_{\mathfrak{p}}} H^{i-1}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty$.

(iii) $\operatorname{injdim}_{R_{\mathfrak{p}}} H^{i-1}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \operatorname{injdim}_R H^i_{\mathfrak{m}}(M)$.

Proof (i) and (ii) follow immediately from Theorem 2.5, and (iii) follows from (i). ■

Lemma 2.8 *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let A be a non-zero Artinian R-module. Let $j \geq 0$ be an integer and N be a non-zero R-module such that $\operatorname{Tor}^R_j(N, A) \neq 0$. Then there exists an integer $n \geq j$ such that $\operatorname{Tor}^R_n(R/\mathfrak{m}, A) \neq 0$.*

Proof Since N can be viewed as the direct limit of its finitely generated submodules and the torsion functor $\operatorname{Tor}^R_j(-, A)$ commutes with direct limits, it follows from the hypothesis $\operatorname{Tor}^R_j(N, A) \neq 0$ that N has a finitely generated non-zero submodule M such that $\operatorname{Tor}^R_j(M, A) \neq 0$. Now using [6, Theorem 6.4] it follows that there exists a prime ideal $\mathfrak{p}$ of R such that $\operatorname{Tor}^R_j(R/\mathfrak{p}, A) \neq 0$. Next, let

$$\mathcal{S} := \left\{ \mathfrak{q} \in \operatorname{Spec}(R) : \operatorname{Tor}^R_n(R/\mathfrak{q}, A) \neq 0, \text{ for some integers } n \geq j \right\}.$$

Then as $\mathfrak{p} \in \mathcal{S}$, it follows that $\mathcal{S} \neq \varnothing$. Since R is Noetherian, $\mathcal{S}$ has a maximal element, say, $\mathfrak{q}$. By the definition of $\mathcal{S}$ there exists an integer $n \geq j$ such that $\operatorname{Tor}^R_n(R/\mathfrak{q}, A) \neq 0$. We must show that $\mathfrak{q} = \mathfrak{m}$. Suppose the contrary be true. Then there exists an element $x \in \mathfrak{m} \setminus \mathfrak{q}$. The exact sequence

$$0 \longrightarrow R/\mathfrak{q} \xrightarrow{x} R/\mathfrak{q} \longrightarrow R/(\mathfrak{q} + Rx) \longrightarrow 0,$$

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induces the exact sequence

$$\mathrm{Tor}_{n+1}^{R}(R/(\mathfrak{q}+Rx),A) \longrightarrow \mathrm{Tor}_{n}^{R}(R/\mathfrak{q},A) \xrightarrow{x} \mathrm{Tor}_{n}^{R}(R/\mathfrak{q},A). \tag{2.8}$$

Since

$$\varnothing \neq \mathrm{Supp}(\mathrm{Tor}_{n}^{R}(R/\mathfrak{q},A)) \subseteq \mathrm{Supp}(A)=\{\mathfrak{m}\},$$

it follows that $\mathrm{Supp}(\mathrm{Tor}_{n}^{R}(R/\mathfrak{q},A))=\{\mathfrak{m}\}$. Therefore, $(0:_{\mathrm{Tor}_{n}^{R}(R/\mathfrak{q},A)} x)\neq 0$. Whence, the exact sequence (2.8) implies that $\mathrm{Tor}_{n+1}^{R}(R/(\mathfrak{q}+Rx),A)\neq 0$. Now again using [6, Theorem 6.4] it follows that there exists a prime ideal $\mathfrak{q}_1 \in \mathrm{Supp}(R/(\mathfrak{q}+Rx))$ such that $\mathrm{Tor}_{n+1}^{R}(R/\mathfrak{q}_1,A)\neq 0$. So $\mathfrak{q}\subset\mathfrak{q}_1\in\mathcal{S}$, which is a desired contradiction. ■

The following consequence of Lemma 2.8 will be useful in the proof of the main theorem.

Corollary 2.9 *Let $(R,\mathfrak{m})$ be a Noetherian local ring and let A be a non-zero Artinian R-module. Then*

$$\mathrm{fd}_R(A)=\sup\{n\in\mathbb{N}_0:\mathrm{Tor}_{n}^{R}(R/\mathfrak{m},A)\neq 0\}.$$

Proof Using Lemma 2.8 we have

$$\begin{aligned}\mathrm{fd}_R(A)&=\sup\bigl\{\,n\in\mathbb{N}_0:\mathrm{Tor}_{n}^{R}(N,A)\neq 0,\ \text{for some } R\text{-module } N\neq 0\bigr\}\\ &\leq \sup\bigl\{\,n\in\mathbb{N}_0:\mathrm{Tor}_{n}^{R}(R/\mathfrak{m},A)\neq 0\bigr\}\ \leq \mathrm{fd}_R(A).\end{aligned}$$ ■

Corollary 2.10 *Let $\dim R=d\geq 1$ and let M be a non-zero finitely generated R-module. Let $i\geq 1$ be an integer. Then for each $\mathfrak{p}\in\mathrm{Spec}(R)$ with $\dim(R/\mathfrak{p})=1$ we have*

$$\mathrm{fd}_{R_\mathfrak{p}}\bigl(H^{i-1}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})\bigr)\ \leq \mathrm{fd}_R\bigl(H^{i}_{\mathfrak{m}}(M)\bigr).$$

Proof In the case where $H^{i-1}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})=0$, the assertion is clear. So we may assume that $H^{i-1}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})\neq 0$. Then since by [2, Theorem 7.1.3], $H^{i-1}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})$ is an Artinian $R_\mathfrak{p}$-module, it follows from Corollary 2.9 that

$$\mathrm{fd}_{R_\mathfrak{p}}\bigl(H^{i-1}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})\bigr)\ =\sup\bigl\{\,j\in\mathbb{N}_0:\mathrm{Tor}_{j}^{R_\mathfrak{p}}\bigl(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p},H^{i-1}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})\bigr)\ \neq 0\bigr\}.$$

By Theorem 2.6 we have

$$\begin{aligned}&\sup\bigl\{\,j\in\mathbb{N}_0:\mathrm{Tor}_{j}^{R_\mathfrak{p}}\bigl(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p},H^{i-1}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})\bigr)\ \neq 0\bigr\}\\ &\quad=\sup\bigl\{\,j\in\mathbb{N}_0:\mathrm{Tor}_{j}^{R}\bigl(R/\mathfrak{p},H^{i}_{\mathfrak{m}}(M)\bigr)\ \text{is not finitely generated}\bigr\}\\ &\quad\leq \mathrm{fd}_R\bigl(H^{i}_{\mathfrak{m}}(M)\bigr).\end{aligned}$$ ■

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Proof of Theorem 1.1 (i) Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and suppose that $\dim R/\mathfrak{p} = n$. Then it follows from the definition that there is a chain of prime ideals of R as:

$$\mathfrak{p} = \mathfrak{p}_n \subset \mathfrak{p}_{n-1} \subset \cdots \subset \mathfrak{p}_1 \subset \mathfrak{p}_0 = \mathfrak{m}$$

such that $\operatorname{height}(\mathfrak{p}_i / \mathfrak{p}_{i+1}) = 1$, for each $0 \leq i \leq n-1$. Then using Corollary 2.7(iii) we have

$$\begin{aligned} \operatorname{injdim}_{R_\mathfrak{p}} H^{i-n}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) &= \operatorname{injdim}_{R_{\mathfrak{p}_n}} H^{i-n}_{\mathfrak{p}_n R_{\mathfrak{p}_n}}(M_{\mathfrak{p}_n}) \\ &\leq \operatorname{injdim}_{R_{\mathfrak{p}_{n-1}}} H^{i-n+1}_{\mathfrak{p}_{n-1} R_{\mathfrak{p}_{n-1}}}(M_{\mathfrak{p}_{n-1}}) \\ &\quad \vdots \\ &\leq \operatorname{injdim}_{R_{\mathfrak{p}_1}} H^{i-1}_{\mathfrak{p}_1 R_{\mathfrak{p}_1}}(M_{\mathfrak{p}_1}) \leq \operatorname{injdim}_R H^i_\mathfrak{m}(M). \end{aligned}$$

This completes the proof of (i).

(ii) The assertion follows from Corollary 2.10, applying the method used in the proof of (i). ∎

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Department of Mathematics, Ardabil branch, Islamic Azad University, P.O. Box 5614633167, Ardabil, Iran.
e-mail: bahmanpour.k@gmail.com