J. Plasma Phys. (2026), *vol.* 92, E53
 doi:10.1017/S0022377826101469

Is there an exact magnetic moment for charged particle motion in a time-dependent, homogeneous magnetic field?

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(Received 4 November 2025; revision received 9 January 2026; accepted 12 February 2026)

The non-perturbative guiding-centre model provides an exact alternative to full-orbit simulations of charged particle dynamics in situations where traditional guiding-centre theory may fail. We demonstrate that the charged particle motion in a homogeneous, time-varying magnetic field is a solvable example of the non-perturbative guiding-centre model. This entails showing that the exact magnetic moment of Qin and Davidson can be constructed to be asymptotic to the adiabatic invariant series of Kruskal. In contrast to the perturbative invariant, the exact invariant contains information about parametric resonances. These resonances destroy the conservation of the usual magnetic moment over very long times. This refutes some previous claims about the all-time invariance of the magnetic moment.

Key words: plasma dynamics, plasma confinement

1. Introduction

Traditional models of particle motion in strong magnetic fields rely on Kruskal's asymptotic theory of nearly periodic systems (Kruskal 1962) to formally separate the fast gyration of charged particles from slow drift motions. Examples include the guiding-centre (Kruskal 1958; Littlejohn 1981, 1984; Cary & Brizard 2009) and gyrocentre (Brizard & Hahm 2007) models. For Hamiltonian theories, such as the aforementioned, Kruskal's theory implies the existence of a formally conserved adiabatic invariant series $\mu_\epsilon = \mu_0 + \epsilon\mu_1 + \epsilon^2\mu_2 + \cdots$, where ϵ is the small parameter of the theory. Consequently, traditional models of single-particle motion are accurate when μ_ϵ is well conserved, but suffer when the particle Larmor radius (cyclotron period) is not particularly small compared with the length (time) scale of magnetic field variation. These models may therefore lead to imprecise estimates of particle confinement, particularly in device regions where the magnetic field varies sharply. This is a primary concern for the confinement of energetic α-particles. The α-particle gyro-radii are only marginally smaller than the length scale of magnetic field variation in typical stellarators. As recounted in Burby *et al.* (2025), significant deviations in the predicted α-particle loss fraction between guiding-centre and full-orbit simulations may therefore occur in certain magnetic geometries (de Assunção 2023; Rodrigues *et al.* 2024; Carbajal *et al.* 2025).

These concerns motivated Burby *et al.* (2025) to develop a non-perturbative guiding-centre model. The non-perturbative guiding-centre model postulates the existence of an exact invariant $\mathcal{J}_\epsilon$, dubbed the non-perturbative adiabatic invariant, which both generates a $U(1)$ symmetry, and agrees with μ_ϵ to at all orders in ϵ. Assuming such a $\mathcal{J}_\epsilon$ can be found, the non-perturbative guiding-centre theory gives exact equations for charged particle motion free of the fast cyclotron time scale. Further, the non-perturbative guiding-centre model recovers the usual guiding-centre theory to all orders in ϵ. No examples of non-perturbative adiabatic invariants were given in Burby *et al.* (2025). Rather, Burby *et al.* demonstrated that approximate non-perturbative adiabatic invariants can be constructed from data. The particle dynamics recovered from these learned invariants outperformed the traditional guiding-centre model. This result motivates the search for true non-perturbative adiabatic invariants.

Recently, Hollas *et al.* (2025) were able to find exact formulas for non-perturbative adiabatic invariants in some symmetric, time-independent magnetic fields. In this work, we continue the program of Hollas *et al.* by showing that there is a non-perturbative adiabatic invariant for particle motion in a homogeneous, time-dependent magnetic field. This system is already known to be integrable. Qin & Davidson (2006) demonstrated that charged particle motion in a magnetic field of the form $\boldsymbol{B} = B(t)\hat{z}$ possesses a non-trivial, time-dependent invariant $\mathcal{M}_\epsilon$. The construction of Qin and Davidson follows closely the construction of H.R. Lewis (Lewis 1968), who showed that the time-dependent simple harmonic oscillator, henceforth referred to as Hill's equation,

$$\epsilon^2 y'' + \varOmega^2(t) y = 0, \tag{1.1}$$

possesses a time-dependent invariant of motion for any positive function $\varOmega(t)$. Parameterising either invariant is a solution branch $w(t;\epsilon)$ of the singularly perturbed Ermakov–Pinney equation

$$\epsilon^2 w''(t;\epsilon) + \varOmega^2(t) w(t;\epsilon) = \frac{1}{w(t;\epsilon)^3}. \tag{1.2}$$

Taking ϵ to be zero, Qin and Davidson showed the solution $w(t;0) = \varOmega^{-1/2}(t)$ reduces $\mathcal{M}_0 = (v_\perp^2/2B(t))$ to the usual magnetic moment. For $\mathcal{M}_\epsilon$ to comprise a non-perturbative adiabatic invariant, it must be shown that $\mathcal{M}_\epsilon$ agrees with μ_ϵ to all orders in ϵ, and that the Hamiltonian flow of $\mathcal{M}_\epsilon$ generates a $U(1)$ action. This was not previously shown by Qin and Davidson.

A complication to showing all orders agreement between $\mathcal{M}_\epsilon$ and μ_ϵ is that there is generally no solution to (1.2) taking the form of a convergent power series $w(t;\epsilon) = w_0(t) + \epsilon^2 w_2(t) + \epsilon^4 w_4(t) + \cdots$. It is not even clear if $w(t;\epsilon)$ can be constructed to admit an asymptotic power series. For time-periodic magnetic fields, for example, the formal solution $w_0(t) + \epsilon^2 w_2(t) + \cdots$ is periodic to all orders. However, we will show that there are generally arbitrarily small ϵ values where no periodic solution to (1.2) exists. At these ϵ values, particle trajectories grow exponentially unbounded. This behaviour was not explored in Qin and Davidson,

https://doi.org/10.1017/S0022377826101469 Published online by Cambridge University Press

since no careful exploration of the singularly perturbed Ermakov–Pinney equation was conducted. This complication led Qin and Davidson to erroneously conclude that the existence of an exact invariant $\mathcal{M}_\epsilon$ implies the usual magnetic moment is conserved up to $O(\epsilon)$ error for *all* times.

Despite this complication, we will show that the invariant $\mathcal{M}_\epsilon$ can be constructed to be a non-perturbative adiabatic invariant. This result follows after proving two facts. The first is that $\mathcal{M}_\epsilon$ generates a $U(1)$ action. The second is that there exists a branch of solutions $w(t;\epsilon)$ to the singularly perturbed Ermakov–Pinney equation such that both $w(t;\epsilon)$ and $w'(t;\epsilon)$ admit asymptotic power series valid over, at least, $O(1)$ times. After proving $\mathcal{M}_\epsilon$ is a non-perturbative adiabatic invariant, we explore how the non-perturbative dynamics predicted by the invariance of $\mathcal{M}_\epsilon$ differs from the approximate dynamics predicted by the invariance of μ_ϵ. As we will explore, the phenomenon of parametric resonance can exist for any $\epsilon > 0$. This presents an unexplored, strictly non-perturbative phenomenon which can be inferred from the invariance of $\mathcal{M}_\epsilon$, but not from the approximate conservation of μ_ϵ to any order in ϵ.

One of the distinguishing features of this work is that the guiding-centre scaling is not used to place the equations of motion in a form amenable to Kruskal's theory of nearly periodic systems. No assumption is made about the smallness of the gyroradius. Rather, as seen by our equations of motion, gyrocentre scaling is used. By appropriately choosing a fast time scale to work on, our equations limit to charged particle motion in a time-independent magnetic field as $\epsilon \to 0^+$. This difference is interesting but ultimately of little importance since the non-perturbative guiding-centre theory of Burby *et al.* can be adopted to any nearly periodic Hamiltonian system. The more important difference between this work and that of Hollas *et al.* (2025) is that the Arnold–Liouville theorem cannot be leveraged in proving $\mathcal{M}_\epsilon$ is a non-perturbative adiabatic invariant. The analysis in this work is therefore qualitatively different from previous works.

In § 2, we briefly review Kruskal's theory of adiabatic invariants in nearly periodic Hamiltonian systems and define precisely the notion of a non-perturbative adiabatic invariant. We then show that the magnetic system of Qin and Davidson can be given the form of a nearly periodic Hamiltonian system. We conclude by showing that $\mathcal{M}_\epsilon$ generates a $U(1)$ action for any choice of solution to (1.2). In § 3, we show that the Ermakov–Pinney equation is controlled by Hill's equation. We then use the Bremmer series to construct a smooth choice of $\mathcal{M}_\epsilon$ admitting an asymptotic series. In § 4, we focus on time-periodic magnetic fields and use Floquet theory to explore the non-perturbative dynamics of the charged particle motion. In particular, we show that long-term particle confinement cannot be predicted from perturbative theory alone. In § 5, we give an example and explore the validity of perturbative theory numerically. Finally, we conclude with some remarks and avenues of further work in § 6.

2. The exact adiabatic invariant

Consider the non-relativistic motion of a particle of mass m and charge $q = |q|\sigma$ in a uniform, time-dependent magnetic field of the form $\boldsymbol{B}(t) = B_0 B(t/T)\hat{z}$. Here, $B(t/T)$ is a dimensionless function, $\sigma = \pm 1$ is the sign of the particle charge, B_0 is the typical magnetic field strength and T is the characteristic time scale of magnetic field variation. Since we are only interested in non-vanishing magnetic fields, we

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assume without loss of generality that B is strictly positive. We define the dimensionless ordering parameter $\epsilon^{-1} = \omega_c T$, where $\omega_c = |q|B_0/m$ is the typical cyclotron frequency; ϵ is assumed to be a small parameter representing the ratio between the typical gyro-period and the time scale of magnetic field variation. We adopt units of time such that $T = 1$. The single-particle equations of motion on the fast time scale $\tau = \epsilon^{-1} t$ are

$$\frac{\mathrm{d}^2 \boldsymbol{x}}{\mathrm{d}\tau^2} = \sigma \left[B(\epsilon\tau) \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} \times \hat{z} + \epsilon \boldsymbol{E}(\boldsymbol{x}, \epsilon\tau) \right]. \tag{2.1}$$

We assume $\boldsymbol{B}$ is generated by a rotationally symmetric configuration, so the induced electric field is given in cylindrical coordinates by $\boldsymbol{E}(\boldsymbol{x}, t) = -1/2 r B'(t)\hat{\theta}$. Since $z'' = 0$, we ignore the trivial motion in the z direction and focus only on the transverse particle motion.

To study invariants of this time-dependent system, we must place the equations of motion in autonomous Hamiltonian form. Let (τ, p_τ) be a canonically conjugate position–momentum pair obeying $\dot{\tau} = 1$. Using polar coordinates, we also define the canonical momenta $p_r = \mathrm{d}r/\mathrm{d}\tau$ and $p_\theta = r^2(\mathrm{d}\theta/\mathrm{d}\tau) + (r^2 \sigma B(\epsilon\tau)/2)$. The Hamiltonian in the extended phase space can be taken to be

$$\tilde{H}_\epsilon = \frac{\epsilon}{2} \left(p_r^2 + \left(\frac{p_\theta}{r} - r\sigma\Omega(t) \right)^2 \right) + \epsilon p_\tau. \tag{2.2}$$

The symplectic form is $-\mathrm{d}\vartheta_\epsilon$ with the Liouville one-form ϑ_ϵ given by

$$\vartheta_\epsilon = \epsilon p_r \mathrm{d}r + \epsilon p_\theta \mathrm{d}\theta + p_\tau \mathrm{d}t. \tag{2.3}$$

The equations of motion are

$$\begin{aligned} \dot{p}_\theta &= 0, \\ \dot{\theta} &= \left(\frac{p_\theta}{r^2} - \sigma\Omega(t) \right), \\ \dot{p}_\tau &= \epsilon\sigma\Omega'(t) \left(p_\theta - r^2\sigma\Omega(t) \right), \\ \dot{t} &= \epsilon, \\ \dot{p}_r &= \frac{p_\theta^2}{r^3} - \Omega^2(t) r, \\ \dot{r} &= p_r. \end{aligned} \tag{2.4}$$

In the variables $(r, \theta, t, p_r, p_\theta, p_\tau)$, (2.4) takes the desired form of a nearly periodic Hamiltonian system. As shown in the seminal work of Kruskal (1962), nearly periodic Hamiltonian systems encompass a broad class of systems for which a perturbative adiabatic invariant $\mu_\epsilon = \mu_0 + \epsilon\mu_1 + \cdots$ can be constructed, and therefore provide a general context for defining the notion of a non-perturbative adiabatic invariant. For the sake of completeness, we recall the generalised definition of a nearly periodic system as defined by Burby and others in Burby & Squire (2020), Burby & Hirvijoki (2021) and Burby *et al.* (2023).

DEFINITION 2.1. *Nearly periodic Hamiltonian system. A nearly periodic system is an ordinary differential equation (ODE) of the form* $\dot{z} = X_\epsilon(z)$ *where*

(i) X_ϵ *is a smooth vector field depending smoothly on* ϵ *for* $0 \leqslant \epsilon \ll 1$.

(ii) *There exists a smooth, positive function* ω_0 *and a vector field* R_0 *such that* $X_0 = \omega_0 R_0$.

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(iii) R_0 *generates a* 2π*-periodic* $U(1)$ *action* i.e. *the streamlines of* R_0 *are* 2π*-periodic curves.*

(iv) $\mathcal{L}_{R_0}\omega_0 = 0$ i.e. ω_0 *is constant along the streamlines of* R_0.

If, in addition, there exists a smoothly parameterised family of functions H_ϵ *and one-forms* ϑ_ϵ *such that*

(i) $i_{X_\epsilon}\mathrm{d}\vartheta_\epsilon = -dH_\epsilon$ i.e. X_ϵ *is a Hamiltonian flow.*

Then we say that $(X_\epsilon, \vartheta_\epsilon, H_\epsilon)$ *is a nearly periodic Hamiltonian system.*

Critical to the construction of an adiabatic invariant μ_ϵ in nearly periodic Hamiltonian systems is the roto-rate vector $R_\epsilon = R_0 + \epsilon R_1 + \cdots$ which is the unique formal vector field satisfying, in the sense of formal power series in ϵ.

(i) $\lim_{\epsilon\to 0} R_\epsilon = R_0$.

(ii) $\exp(2\pi\mathcal{L}_{R_\epsilon}) = \mathrm{Id}$.

(iii) $[X_\epsilon, R_\epsilon] = 0$.

It is not hard to show that the conditions imposed on the roto-rate vector ensure R_ϵ is a formal Noether symmetry. Essentially, R_ϵ generates a formally ignorable angle-like coordinate, often the gyrophase-angle in the context of guiding-centre theory. The asymptotic invariant $\mu_\epsilon = \mu_0 + \epsilon\mu_1 + \cdots$ is then the associated formally conserved quantity obeying $i_{R_\epsilon}\mathrm{d}\vartheta_\epsilon = -\mathrm{d}\mu_\epsilon$. As argued in Gjaja & Bhattacharjee (1992), even under ideal conditions, the series for μ_ϵ may not converge for any $\epsilon > 0$. In general, there is no coordinate change separating, say, the fast motion of gyrating particles from the slow drift motion. The results of Neishtadt (1984), Ramis & Schäfke (1996) and Simó (1994) show that such a split may only be possible up to an exponentially small coupling between the fast and slow variables.

While it is too much to ask that the formal series for R_ϵ converges, it may nevertheless be true that a $U(1)$ symmetry persists for $\epsilon > 0$. In such a case, we say the symmetry generator $\mathcal{R}_\epsilon$ is a non-perturbative roto-rate vector if $\mathcal{R}_\epsilon$ depends continuously on $0 \leqslant \epsilon \ll 1$ and admits an asymptotic power series as $\epsilon \to 0^+$. Uniqueness of the roto rate then implies $\mathcal{R}_\epsilon$ is asymptotic at all orders to R_ϵ. When a non-perturbative roto-rate vector exists in a nearly periodic Hamiltonian system, the Hamiltonian $\mathcal{J}_\epsilon$ generating $\mathcal{R}_\epsilon$ is a non-perturbative adiabatic invariant. Indeed, since $\mathcal{R}_\epsilon$ is asymptotic to R_ϵ, it follows that $\mathcal{J}_\epsilon$ is asymptotic to Kruskal's adiabatic invariant μ_ϵ. This leads to the following definition.

DEFINITION 2.2. *Non-perturbative adiabatic invariant. Given a nearly periodic Hamiltonian system* $(X_\epsilon, \vartheta_\epsilon, H_\epsilon)$, *we say that an* ϵ*-parameterised family of functions* $\mathcal{J}_\epsilon$ *generating a Hamiltonian flow* $\mathcal{R}_\epsilon$ *is a non-perturbative adiabatic invariant if*

(i) $\mathcal{L}_{X_\epsilon}\mathcal{J}_\epsilon = 0$, i.e. $\mathcal{J}_\epsilon$ *is invariant to the flow of* X_ϵ.

(ii) $\mathcal{J}_\epsilon$ *is continuous in* ϵ *for* $0 \leqslant \epsilon \ll 1$.

(iii) $\mathcal{J}_\epsilon$ *admits an asymptotic power series of the form* $\mathcal{J}_\epsilon \sim J_0 + \epsilon J_1 + \cdots$ *as* $\epsilon \to 0^+$.

(iv) $\lim_{\epsilon\to 0}\mathcal{R}_\epsilon = R_0$.

(v) $\mathcal{R}_\epsilon$ *generates a* 2π*-periodic* $U(1)$ *action for all* $0 \leqslant \epsilon \ll 1$.

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In the present case, $\tilde{H}_\epsilon$ and ϑ_ϵ are given by (2.2) and (2.3) respectively, and X_ϵ is the Hamiltonian vector field (2.4). Taking the formal limit $\epsilon \to 0^+$, the limiting dynamics X_0 consists of periodic motion at an angular frequency of $2\Omega(t)$. We therefore have a nearly periodic Hamiltonian system with $\omega_0(t) = 2\Omega(t)$ and $R_0 = (X_0/2\Omega(t))$.

In Qin & Davidson (2006), it was shown that

$$2\epsilon^{-1}\mathcal{M}_\epsilon = \frac{p_\theta^2}{2}\left(\frac{w(t;\epsilon)^2}{r^2}\right) + \frac{1}{2}\left(\frac{r^2}{w(t;\epsilon)^2}\right) + \frac{1}{2}(p_r w(t;\epsilon) - \epsilon w'(t;\epsilon) r)^2 - p_\theta\sigma, \tag{2.5}$$

is an exact invariant of the dynamics when $w(t;\epsilon)$ solves the singularly perturbed Ermakov–Pinney equation

$$\epsilon^2 w''(t;\epsilon) + \Omega^2(t) w(t;\epsilon) = \frac{1}{w^3(t;\epsilon)}. \tag{2.6}$$

Taking ϵ to be zero, one computes that $w(t;0) = \Omega(t)^{-1/2}$. Thus, $\epsilon^{-1}\mathcal{M}_\epsilon$ formally limits to magnetic moment $\epsilon^{-1}\mathcal{M}_0 = (1/4\Omega)(r^2\dot{\theta}^2 + p_r^2) = \mu_1$, and the Hamiltonian vector field of $\mathcal{M}_\epsilon$ limits to R_0. To prove $\mathcal{M}_\epsilon$ is a non-perturbative adiabatic invariant, it must be shown that $\mathcal{M}_\epsilon$ generates a 2π-periodic $U(1)$ action, and that there exists a branch of solutions $w(t;\epsilon)$ to the Ermakov–Pinney equation such that both $w(t;\epsilon)$ and $w'(t;\epsilon)$ admit asymptotic power series in ϵ. Once both of these statements have been shown, one verifies all the conditions of Definition 2.2 are met.

We first assume $w(t;\epsilon)$ is given and show that $\mathcal{M}_\epsilon$ generates a $U(1)$ action. In the next section, we show a method by which $w(t;\epsilon)$ may be smoothly constructed. The Hamiltonian vector field of $2\mathcal{M}_\epsilon$ generates the flow

$$\begin{aligned}
\frac{\mathrm{d}t}{\mathrm{d}\xi} &= 0,\\
\frac{\mathrm{d}p_\tau}{\mathrm{d}\xi} &= \epsilon\left[-ww'\left(\frac{p_\theta^2}{r^2} + p_r^2\right) + r^2 w'\left(\frac{1}{w^3} - \epsilon^2 w''\right) + r p_r \epsilon\left(w'' w + w'^2\right)\right],\\
\frac{\mathrm{d}r}{\mathrm{d}\xi} &= p_r w^2 - \epsilon w w' r,\\
\frac{\mathrm{d}p_r}{\mathrm{d}\xi} &= \frac{p_\theta^2 w^2}{r^3} - \frac{r}{w^2} + \epsilon w'\left(p_r w - \epsilon w' r\right),\\
\frac{\mathrm{d}\theta}{\mathrm{d}\xi} &= \frac{p_\theta w^2}{r^2} - \sigma,\\
\frac{\mathrm{d}p_\theta}{\mathrm{d}\xi} &= 0.
\end{aligned} \tag{2.7}$$

The equation for the canonical pair (r, p_r) closes to give

$$r'' + r = \frac{p_\theta^2 w^4}{r^3}, \tag{2.8}$$

which has the general solution $r^2(\xi; E, \phi) = p_\theta w^2[E + \sqrt{E^2 - 1}\sin(2\xi + \phi)]$. Given that $p_r(\xi) = 1/w^2(\mathrm{d}r/\mathrm{d}\xi(\xi) + \epsilon w w' r(\xi))$, $p_r(\xi)$ is a π-periodic function for any constants $E \geqslant 1, \phi$. One computes that

$$\begin{aligned}
&\int_0^\pi \frac{\mathrm{d}\xi}{r^2(\xi)} = \frac{\pi}{p_\theta w^2},\\
&\int_0^\pi r^2(\xi)\mathrm{d}\xi = (\pi p_\theta E)w^2,\\
&\int_0^\pi \left[\frac{p_\theta^2}{r^2(\xi)} + (p_r(\xi))^2\right]\mathrm{d}\xi = (\pi p_\theta E)\left(\frac{1}{w^2} + \epsilon^2 w'^2\right),\\
&\int_0^\pi r(\xi)p_r(\xi)\mathrm{d}\xi = \epsilon(\pi p_\theta E)ww'.
\end{aligned} \tag{2.9}$$

It follows that $\epsilon^{-1}\mathcal{M}_\epsilon$ generates a 2π periodic $U(1)$ action since

$$\begin{aligned}
\Delta\theta &\equiv \int_0^\pi \frac{\mathrm{d}\theta}{\mathrm{d}\xi}(\xi)\mathrm{d}\xi = \pi - \sigma\pi \equiv 0\ (\mathrm{mod}\ 2\pi),\\
\Delta p_\tau &\equiv \int_0^\pi \frac{\mathrm{d}p_\tau}{\mathrm{d}\xi}(\xi)\mathrm{d}\xi = 0.
\end{aligned} \tag{2.10}$$

We have thus shown that the non-perturbative invariant $\mathcal{M}_\epsilon$ generates an angle-like, time-dependent ignorable coordinate. The physical meaning of $\mathcal{M}_\epsilon$ is somewhat obscure for $\epsilon \neq 0$ beyond the fact that $\mathcal{M}_\epsilon$ is linear combination of p_θ and the approximate constants of motion used in accelerator physics, as shown Qin & Davidson (2006).

3. Construction of the auxiliary function

We now construct a smooth function $w(t;\epsilon)$ satisfying the singularly perturbed Ermakov–Pinney equation (2.6) admitting an asymptotic power series in ϵ limiting to $\Omega(t)^{-1/2}$. We show that $w'(t;\epsilon)$ also admits an asymptotic power series obtained by term-by-term differentiation of the series for $w(t;\epsilon)$. Provided this choice of $w(t;\epsilon)$ is used in the construction of the exact invariant, $\mathcal{M}_\epsilon$ admits an asymptotic series and is hence a non-perturbative adiabatic invariant.

It is known by the general results of Sibuya (2000, 2003), and Canalis-Durand *et al.* (2000) that solutions to singularly perturbed analytic ODEs asymptotic to their formal series solution exist for short times. To obtain all-time results for merely smooth $\Omega(t)$, use the well-known fact that the Ermakov–Pinney equation is controlled by the singularly perturbed Hill's equation (Pinney 1950).

$$\epsilon^2\psi''(t;\epsilon) + \Omega^2(t)\psi(t;\epsilon) = 0. \tag{3.1}$$

Indeed, let $u(t)$, $v(t)$ be any two linearly independent solutions to Hill's equation. Then there exists smooth, real functions $R(t) > 0$ and $\phi(t) > 0$ such that

$$\psi \equiv u(t) + iv(t) = R(t)\exp\left(\mp\frac{i}{\epsilon}\int_0^t \phi(t')\mathrm{d}t'\right), \tag{3.2}$$

is a never-vanishing solution to Hill's equation; $R(t)$ can never vanish since that would imply the Wronskian matrix formed from $u(t)$ and $v(t)$ is singular at some point in time, contradicting the linear independence of u and v. The choice of the $\mp$ sign is determined by the condition that $\phi > 0$. R and ϕ define a solution to Hill's equation if and only if

$$\begin{aligned}
\epsilon^2 R'' + \Omega^2(t)R &= \phi^2 R,\\
2\phi R' + \phi' R &= 0.
\end{aligned} \tag{3.3}$$

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The latter equation implies $R = C\phi^{-1/2}$, justifying that ϕ is also non-vanishing. Provided the initial conditions are such that $C = 1$, which can always be ensured by rescaling u and v, we have that

$$\epsilon^2 R'' + \Omega^2(t)R = \frac{1}{R^3}, \tag{3.4}$$

which is the desired result. With this basic fact in mind, it is clear that the formal expansion of $w(t;\epsilon)$ into powers of ϵ is obtained from the usual WKB series of (3.1). Indeed, the first-order WKB approximation yields that $w(t;\epsilon \ll 1) \approx \Omega^{-1/2}$. Results about the accuracy of WKB (Olver 1961, 1974; Taylor 1982; Costin 2008) therefore justify that the formal solutions $w^{(n)}(t) = w_0(t) + \epsilon^2 w_2(t) + \cdots + \epsilon^n w_n(t)$ to (2.6) approximate some true solution for any fixed n.

To construct an exact solution $\psi(t;\epsilon)$ to Hill's equation asymptotic to the WKB series at all orders, we use the Bremmer series. Constructed independently in Bremmer (1951), Schelkunoff (1951) and Landauer (1951) and studied further in Bellman & Kalaba (1959), Atkinson (1960) and Kay (1961), the Bremmer series organises the solution of (3.1) into a convergent sum of waves undergoing a particular number of reflections. Importantly, the Bremmer series agrees with WKB at leading order. In Winitzki (2005), it was shown that the Bremmer-series solutions admit an asymptotic ϵ-series provided there exists a point in time such that all derivatives of $\Omega(t)$ vanish. By temporarily introducing a cutoff to $\Omega(t)$ in the far past, we show that this assumption can be dropped. We repeat parts of Winitzski for the sake of a self-contained construction, but refer the reader Winitzki (2005) for an interesting discussion on the precision of optimally truncated WKB. We first write $\Omega(t) = \Omega_0 + \phi(t)$ where $\phi(t)$ is smooth and the strict inequality $|\phi(t)| < \Omega_0$ holds uniformly. Let $0 \leqslant H(t) \leqslant 1$ be a smooth bump function supported on a domain of the form $[-K, \infty)$ such that $H(t \geqslant 0) \equiv 1$. We define the deformed magnetic field $\tilde{\Omega}(t) = \Omega_0 + H(t)\phi(t)$. Clearly, for $t \geqslant 0$, we have not changed $\Omega(t)$ so any linearly independent solutions to the deformed equation

$$\epsilon^2 \tilde{\psi}''(t;\epsilon) + \tilde{\Omega}^2(t)\tilde{\psi}(t;\epsilon) = 0, \tag{3.5}$$

will yield a solution to the undeformed Hill's equation, hence the undeformed Ermakov–Pinney equation, for $t \geqslant 0$. By deforming the magnetic field, we are able to assume $\tilde{\psi}$ is a positive-frequency WKB solution in the far past, i.e.

$$\tilde{\psi}(t \to -\infty;\epsilon) = \tilde{\Omega}^{-1/2}(t) \exp\left(-\frac{i}{\epsilon}\int_0^t \tilde{\Omega}(t')\,\mathrm{d}t'\right). \tag{3.6}$$

Making this notion more precise, we define the WKB solutions

$$X_\pm(t;\epsilon) = \tilde{\Omega}^{-1/2}(t) \exp\left(\mp\frac{i}{\epsilon}\int_0^t \tilde{\Omega}(t')\,\mathrm{d}t'\right). \tag{3.7}$$

Then there exists complex-valued functions $p(t;\epsilon)$ and $q(t;\epsilon)$ such that

$$\tilde{\psi}(t;\epsilon) = p(t;\epsilon)X_+(t;\epsilon) + q(t;\epsilon)X_-(t;\epsilon). \tag{3.8}$$

The condition that $\tilde{\psi}$ is a WKB solution in the far past implies initial conditions $p(-\infty;\epsilon) = 1$ and $q(-\infty;\epsilon) = 0$. Since $p(t;\epsilon)$ and $q(t;\epsilon)$ define four degrees of freedom, we may remove two of these degrees of freedom by demanding that

$$\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t}(t;\epsilon) = -\frac{i\Omega(t)}{\epsilon}\big[p(t;\epsilon)X_+(t;\epsilon) - q(t;\epsilon)X_-(t;\epsilon)\big]. \tag{3.9}$$

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Here, $\tilde{\psi}$ satisfies the deformed Hill's equation and has the desired form in the far past if and only if

$$
\begin{aligned}
p(t;\epsilon) &= 1 + \int_{-\infty}^{t} \frac{q(t';\epsilon)}{2\tilde{\Omega}(t')}\frac{\mathrm{d}\tilde{\Omega}}{\mathrm{d}t}(t') \exp\left(\frac{2i}{\epsilon}\int_0^{t'} \tilde{\Omega}(t'')\mathrm{d}t''\right)\mathrm{d}t', \\
q(t;\epsilon) &= \int_{-\infty}^{t} \frac{p(t';\epsilon)}{2\tilde{\Omega}(t')}\frac{\mathrm{d}\tilde{\Omega}}{\mathrm{d}t}(t') \exp\left(-\frac{2i}{\epsilon}\int_0^{t'} \tilde{\Omega}(t'')\mathrm{d}t''\right)\mathrm{d}t'.
\end{aligned} \tag{3.10}
$$

Equation (3.10) yields a convergent Picard iteration scheme when the initial guess $(p_0, q_0) = (1, 0)$ is assumed. Equivalent to the Piccard iteration, we may write $p(t;\epsilon) = \sum_{n=0}^{\infty} p_n(t;\epsilon)$ and $q(t;\epsilon) = \sum_{n=1}^{\infty} q_n(t;\epsilon)$ with $p_0 = 1$ and with

$$
\begin{aligned}
q_{n+1}(t;\epsilon) &= \int_{-\infty}^{t} \frac{p_n(t';\epsilon)}{2\tilde{\Omega}(t')}\frac{\mathrm{d}\tilde{\Omega}}{\mathrm{d}t}(t') \exp\left(-\frac{2i}{\epsilon}\int_0^{t'} \tilde{\Omega}(t'')\mathrm{d}t''\right)\mathrm{d}t', \\
p_{n+1}(t;\epsilon) &= \int_{-\infty}^{t} \frac{q_{n+1}(t';\epsilon)}{2\tilde{\Omega}(t')}\frac{\mathrm{d}\tilde{\Omega}}{\mathrm{d}t}(t') \exp\left(\frac{2i}{\epsilon}\int_0^{t'} \tilde{\Omega}(t'')\mathrm{d}t''\right)\mathrm{d}t'.
\end{aligned} \tag{3.11}
$$

The ϵ-independent bound

$$
\sum_{n=0}^{\infty} |p_n(t;\epsilon)| + \sum_{n=0}^{\infty} |q_n(t;\epsilon)| \leqslant \exp\left(\int_{-\infty}^{t} \left|\frac{1}{2\tilde{\Omega}(t')}\frac{\mathrm{d}\tilde{\Omega}}{\mathrm{d}t}(t')\right| \mathrm{d}t'\right) < \infty, \tag{3.12}
$$

proves that the series for p and q converges uniformly on time intervals of the form $(-\infty, T]$. To show that that $\tilde{\psi}$ admits an asymptotic series of the form $\tilde{\psi} \sim (1 + \epsilon\tilde{\psi}_1(t) + \cdots)X_+$, we define the variable $\zeta(t) = \int_0^t \tilde{\Omega}(t')\mathrm{d}t'$ and the function $\alpha(\zeta) = (1/2\tilde{\Omega}^2)(\mathrm{d}\tilde{\Omega}/\mathrm{d}t)(\zeta)$. Then

$$
\begin{aligned}
q_{n+1}(\zeta;\epsilon) &= \int_{-\infty}^{\zeta} p_n(\zeta';\epsilon)\alpha(\zeta') \exp\left(-\frac{2i}{\epsilon}\zeta'\right)\mathrm{d}\zeta', \\
p_{n+1}(\zeta;\epsilon) &= \int_{-\infty}^{\zeta} q_{n+1}(\zeta';\epsilon)\alpha(\zeta') \exp\left(\frac{2i}{\epsilon}\zeta'\right)\mathrm{d}\zeta'.
\end{aligned} \tag{3.13}
$$

Equation (3.13) is the advertised Bremmer series, as seen by integrating equations (9*a*) and (9*b*) in Bremmer (1951). Repeatedly integrating by parts, the standard result that $\lim_{\epsilon\to 0}\int_D f(\zeta')\exp(\pm i\zeta'/\epsilon)\mathrm{d}\zeta' = 0$ for any $f \in L^1(D \subset \mathbb{R})$ shows that

$$
q_1(\zeta;\epsilon) = \int_{-\infty}^{\zeta} \alpha(\zeta') \exp\left(-\frac{2i}{\epsilon}\zeta'\right)\mathrm{d}\zeta' \sim -\frac{\epsilon}{2i}\exp\left(-\frac{2i}{\epsilon}\zeta\right)\sum_{n=0}^{\infty}\frac{\epsilon^n}{(2i)^n}\frac{\mathrm{d}^n\alpha}{\mathrm{d}\zeta^n}(\zeta). \tag{3.14}
$$

Since term-by-term integration of an asymptotic series yields an asymptotic series, we have that

$$
p_1(\zeta;\epsilon) = \int_{-\infty}^{\zeta} q_1(\zeta';\epsilon)\alpha(\zeta') \exp\left(\frac{2i}{\epsilon}\zeta'\right)\mathrm{d}\zeta' \sim -\frac{\epsilon}{2i}\sum_{n=0}^{\infty}\frac{\epsilon^n}{(2i)^n}\int_{-\infty}^{\zeta}\alpha(\zeta')\left[\frac{\mathrm{d}^n\alpha}{\mathrm{d}\zeta^n}(\zeta')\right]\mathrm{d}\zeta'. \tag{3.15}
$$

Continuing in this fashion, its not hard to see that q_n admits an asymptotic series of the form $q_n \sim \epsilon^n \exp(-(2i/\epsilon)\zeta)(q_n^0 + \epsilon q_n^1 + \cdots)$ and p_n admits an asymptotic powers series of the form $p_n \sim \epsilon^n (p_n^0 + \epsilon p_n^1 + \cdots)$. For any N, the finite sums $\sum_{n=0}^{N} p_n$ and $\exp((2i/\epsilon)\zeta)\sum_{n=1}^{N} q_n$ admit asymptotic power series. To see that the limiting functions $p = \sum_{n=0}^{N} p_n + \sum_{n>N} p_n$ and $\exp((2i/\epsilon)\zeta)q = \exp((2i/\epsilon)\zeta)\sum_{n=1}^{N} q_n + \exp((2i/\epsilon)\zeta)\sum_{n>N} q_n$ admit asymptotic power series as well, we write $|q_{N+1}(t;\epsilon)| = \epsilon^{N+1} C_{N+1}(\zeta,\epsilon)$. One verifies that $C_{N+1}(t,\epsilon)$ is bounded on any domain of the sort $(\zeta,\epsilon) \in (-\infty, \Theta] \times (0,\infty) \equiv \mathcal{D}(\Theta)$. This yields the uniform bound

$$\frac{1}{\epsilon^{N+1}}\left(\sum_{n>N} |p_n(\zeta;\epsilon)| + \sum_{n>N} |q_n(\zeta;\epsilon)|\right) \leqslant \exp\left(||C||_{L^\infty(\mathcal{D}(\zeta))} \int_{-\infty}^{\zeta} |\alpha(\zeta')| \mathrm{d}\zeta'\right) < \infty. \tag{3.16}$$

Hence, $\sum_{n>N} p_n, \sum_{n>N} q_n \in O(\epsilon^{N+1})$. Summing the asymptotic expression for p_n and q_n therefore shows that p and $\exp((2i/\epsilon)\zeta)q$ admit asymptotic power series. Using the form of $\tilde{\psi}$ in the far past, we verify the real and imaginary parts of $\tilde{\psi}$ are linearly independent and normalised so that $\tilde{w}(t;\epsilon) = |\tilde{\psi}(t;\epsilon)|$ solves the deformed Ermakov–Pinney equation. Defining $w(t;\epsilon) = \tilde{w}(t;\epsilon)$ for $t \geqslant 0$ and uniquely extending $w(t;\epsilon)$ to $t < 0$, we have a solution to the Ermakov–Pinney equation (2.6) admitting an asymptotic power series of the form $w(t;\epsilon) \sim \Omega^{-1/2}(t) + \epsilon^2 w_2(t) + \cdots$ when $t \geqslant 0$. To show that $w(t;\epsilon)$ admits the same asymptotic series for $t < 0$, one can again use the Bremmer series with $\tilde{\Omega}(t)$ replaced by $\Omega(t)$ and with $p(0;\epsilon)$ and $q(0;\epsilon)$ giving the initial conditions at $t = 0$. The proof essentially proceeds as above. We remark that the cutoff function H is only used to find suitable initial conditions for $w(t;\epsilon)$.

To complete the proof that $w(t;\epsilon)$ is the desired function, we now show that $w'(t;\epsilon)$ also admits an asymptotic series. The proof is simple and leverages that $\tilde{w}(t;\epsilon)$ solves an initial value problem. Indeed, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{w}(t;\epsilon) = \epsilon^{-2}\int_{-K}^{t}\left(\frac{1}{\tilde{w}^3(t',\epsilon)} - \tilde{\Omega}^2\tilde{w}(t';\epsilon)\right)\mathrm{d}t'. \tag{3.17}$$

Using that $1/z^3$ is analytic away from $z = 0$, the integrand admits an asymptotic power series. Integrating term by term, we therefore have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{w}(t;\epsilon) \sim \epsilon^{-2}\int_{-K}^{t}\left[\tilde{\Omega}^{3/2}(t')(-3\tilde{w}_2(t')\tilde{\Omega}^{1/2}(t')\epsilon^2 + \cdots) - \tilde{\Omega}^2(t')(\tilde{w}_2(t')\epsilon^2 + \cdots)\right]\mathrm{d}t' \tag{3.18}$$

$$= a_0(t) + \epsilon^2 a_2(t) + \epsilon^4 a_4(t) + \cdots . \tag{3.19}$$

Since $\tilde{w}(t;\epsilon) = \Omega_0^{-1/2} + \int_{-K}^{t} \mathrm{d}/\mathrm{d}t\,\tilde{w}(t';\epsilon)\mathrm{d}t'$, the uniqueness of asymptotic series implies that

$$\tilde{w}_0(t) + \epsilon^2\tilde{w}_2(t) + \cdots = \Omega_0^{-1/2} + \int_{-K}^{t}\left(a_0(t') + \epsilon^2 a_2(t') + \cdots\right)\mathrm{d}t'. \tag{3.20}$$

Since $\tilde{w}_n(t)$ are smooth, differentiating (3.20) gives for $t \geqslant 0$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}w(t;\epsilon) \sim \frac{\mathrm{d}}{\mathrm{d}t}w_0(t) + \epsilon^2\frac{\mathrm{d}}{\mathrm{d}t}w_2(t) + \cdots . \tag{3.21}$$

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Equation (3.21) easily extends to $t<0$ by only a trivial modification of the proof above. Hence, the asymptotic series for $w'(t;\epsilon)$ is obtained by term-by-term differentiation of the asymptotic series for $w(t;\epsilon)$.

We note that if $\int_{-\infty}^{\infty} |(\mathrm{d}^n/\mathrm{d}t^n)(\Omega'/2\Omega)| < \infty$ for all $n \geqslant 0$, then setting H to unity shows $|w(\infty,\epsilon) - w(-\infty,\epsilon)| \sim 1 + 0\epsilon + 0\epsilon^2 + \cdots$. Hence $\mathcal{M}_\epsilon$ is conserved to all orders. This is essentially the result of Littlewood (1963). More refined error bounds for the adiabatic invariance of the simple harmonic oscillator are discussed in Wasow (1973), Stengle (1977), Neishtadt (1981) and Bonet *et al.* (1998).

The Bremmer series is remarkable in that it reorders the formal power series of $\psi(t;\epsilon)$ to give a convergent solution $\psi = \psi_0 + \psi_1 + \cdots$ to Hill's equation with $|\psi_n| \in O(\epsilon^n)$. Other methods of constructing convergent WKB-like solutions can be found in Hyouguchi *et al.* (2002), Krivec *et al.* (2004) and Winitzki (2005). When $\Omega(t)$ is analytic, more refined methods based on Borel summation can be used to give exact solutions $\psi(t;\epsilon)$ with stronger asymptotic and analytic properties. We refer the interested reader to Costin *et al.* (2004), Voros (2004), Costin (2008), Takei (2017) and Nikolaev (2023).

4. Non-perturbative dynamics

In this section, we fix ϵ and study the non-perturbative dynamics of the magnetic system when $\Omega^2(t)$ is T-periodic. Using Floquet theory, we identify a form for $w(t)$ which greatly simplifies and elucidates the long-time dynamics of particles. This section illuminates some possible pitfalls of using a perturbative theory when ϵ is not so small or when long time scales are considered.

Consider Hill's equation in first-order form with

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\psi\\ \varphi\end{pmatrix} = \epsilon^{-1}\begin{pmatrix}0 & 1\\ -\Omega^2(t) & 0\end{pmatrix}\begin{pmatrix}\psi\\ \varphi\end{pmatrix} = \epsilon^{-1}M(t)\begin{pmatrix}\psi\\ \varphi\end{pmatrix}. \tag{4.1}$$

Let $\Phi(t;\epsilon^{-1}) = \mathcal{T}\exp(\epsilon^{-1}\int_0^t M(t')\mathrm{d}t')$ be the fundamental matrix of Hill's equation, and $B(\epsilon^{-1}) = \Phi(T;\epsilon^{-1})$ the monodromy matrix. Let λ_1, λ_2 be the eigenvalues of $B(\epsilon^{-1})$, and suppose $(\boldsymbol{v}_1, \boldsymbol{v}_2) \in \mathbb{C}^2 \times \mathbb{C}^2$ is a basis putting B in Jordan normal form. We define $\boldsymbol{z}_1(t) = \Phi(t;\epsilon^{-1})\boldsymbol{v}_1$ and $\boldsymbol{z}_2(t) = \Phi(t;\epsilon^{-1})\boldsymbol{v}_2$. Trivially, $\boldsymbol{z}_1(t)$ and $\boldsymbol{z}_2(t)$ are linearly independent solutions to Hill's equations obeying $\boldsymbol{z}_i(t+T) = B\boldsymbol{z}_i(t)$.

By Abel's theorem, $\det(B(\epsilon^{-1})) = 1$. If $\lambda_1 = \lambda_2^*$ are non-real, then λ_i are distinct and have unit modulus. It follows that all solutions to Hill's equation are bounded. Further, there exists a constant $c>0$ such that

$$w_{\mathrm{per}}(t) = c\left|z_1^1(t)\right|, \tag{4.2}$$

is the unique periodic solution to the Ermakov–Pinney equation. One easily computes that

$$c = \left|z_1^1(0)\right|^{-1}\left|\mathrm{Im}\left(\frac{z_1^2(0)}{z_1^1(0)}\right)\right|^{-1/2}. \tag{4.3}$$

In the case when a periodic solution of Ermakov–Pinney exists, the exact invariant $\mathcal{M}$[1] can be constructed to be time periodic. Partially compactifying the particle

[1]We drop the subscript ϵ since $w_{\mathrm{per}}(t)$ may not agree with $w(t;\epsilon)$ constructed in the previous section.

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phase space by quotienting $t \sim t+T$, one can verify the invariance of $(\mathcal{M}, \tilde{H}_\epsilon, p_\theta)$ implies the existence of action-angle variables by the Arnold–Liouville theorem. This opens the door to a renewed study of more complex magnetic geometries through, for example, Hamiltonian perturbation theory (Nekhoroshev 1971, 1977; Arnol'd 1989).

When the λ_i are strictly real, there is a constant $a > 0$ such that

$$w_{\text{can}}(t) = a|z_1(t) + iz_2(t)| \tag{4.4}$$

solves the Ermakov–Pinney equation. If $\lambda_1 = \lambda_2 = \pm 1$ and B is diagonalisable, then $w_{\text{can}}(t)$ is periodic and all solutions to Hill's equation are bounded. Otherwise, $w(t)$ grows unbounded for any choice of solution to the Ermakov–Pinney equation. It follows all particle trajectories become unbounded at a typically exponential rate. This divergence may be attributed to the particle cyclotron motion resonating with the induced electric field. In such cases, it is not possible to define action-angle variables on the partially compactified particle phase space.

Letting $\Delta(\epsilon^{-1}) = \text{tr}(B(\epsilon^{-1}))$, we have proved the long-term confinement of particles requires $|\Delta(\epsilon^{-1})| \leqslant 2$, with stability only ensured when the strict inequality holds. This fact could not have been uncovered at any order in perturbation theory. Perturbative theory predicts that μ_ϵ is time periodic to all orders. If the series for μ_ϵ converged, the limiting invariant would be time periodic as well. This would imply all particle trajectories are bounded. However, even small deviations of $\Omega(t)$ from a constant function may cause particles to become de-confined after some time. Indeed, suppose that $\Omega^2(t) = \Omega_0^2 - \epsilon^2\phi(t)$. Then Hill's equation becomes the one-dimensional Schrödinger equation

$$-\psi'' + \phi(t)\psi = E(\epsilon^{-1})\psi, \tag{4.5}$$

with $E(\epsilon^{-1}) = \Omega_0^2\epsilon^{-2}$. For any fixed $\phi(t)$, the oscillation theory of Haupt (Magnus & Shenitzer 1957) shows there exists two strictly increasing sequences $\rho_0 \leqslant \rho_2 \leqslant \ldots \to \infty$ and $\rho_1' \leqslant \rho_2' \leqslant \ldots \to \infty$ obeying the inequalities $\rho_0 < \rho_1' \leqslant \rho_2' < \rho_1 \leqslant \rho_2 < \rho_3' \leqslant \rho_4' < \rho_2 \leqslant \ldots$ such that the eigenvalues of $B(\epsilon^{-1})$ are non-real if and only if

$$E(\epsilon^{-1}) \in (\rho_0, \rho_1') \cup (\rho_2', \rho_1) \cup (\rho_2, \rho_3') \cup \ldots. \tag{4.6}$$

Equation (4.6) shows that, in general, infinitely many regions of instability are incurred as $\epsilon^{-1} \to \infty$. Regarding $\epsilon^2\phi(t)$ as a perturbation, one should expect regions of instability to form around ϵ^{-1} values resonating with the unperturbed gyrofrequency, such as in Mathieu's equation (Arnol'd 1983; Kovacic *et al.* 2018; Butikov 2018).

The stability properties of Hill's equation in the form of (4.5) is a classical and well-studied subject (Magnus & Shenitzer 1957; Magnus & Winkler 1961; Levy & Keller 1963; Weinstein & Keller 1985; Broer & Levi 1995). Work on the asymptotic stability of Hill's equation in the more relevant form (4.1) has also been conducted by Weinstein & Keller (1987). Weinstein and Keller have shown that for a certain class of analytic $\Omega(t)$, the regions where $|\Delta(\epsilon^{-1})| \geqslant 2$ grow exponentially small as $\epsilon^{-1} \to \infty$. Hence, particle orbits are bounded for most $\epsilon \ll 1$.

We remark that when $|\Delta(\epsilon^{-1})| < 2$, the existence of an exact, time-periodic invariant proves the global error in the magnetic moment is uniformly bounded. This condition can be used to correct some of the statements in Qin & Davidson (2006) when $\Omega(t)$ is periodic.

5. Numerical studies

We now use the results of the previous sections to compute and explore the accuracy of the adiabatic invariant μ_ϵ when ϵ is not so small. By direct numerical computation, and by comparison with $\mathcal{M}_\epsilon$, we find that μ_ϵ may not always be useful for studying the particle dynamics. We demonstrate how, with minimal computation, the methods developed in this work present a better alternative to the perturbative theory of confinement in time-varying magnetic fields.

For our example, we choose the 2π-periodic magnetic field

$$\Omega^{-1/2}(t) = (1 + \exp(\sin(t)). \tag{5.1}$$

As we have fully justified, the formal solution to the Ermakov–Pinney equation,

$$w(t;\epsilon) = w_0(t) + \epsilon^2 w_2(t) + O(\epsilon^4) = \Omega^{-1/2}(t) - \frac{\epsilon^2}{4\Omega^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big(\Omega^{-1/2}\big)(t) + O(\epsilon^4), \tag{5.2}$$

allows us to compute Kruskal's adiabatic invariant $\epsilon^{-1}\mu_\epsilon = \mu_1 + \epsilon\mu_2 + \epsilon^2\mu_3 + \epsilon^3\mu_4 + O(\epsilon^4)$. Namely, we have that

$$\begin{aligned}
\epsilon^{-1}\mu_1 &= \frac{p_\theta^2}{4}\left(\frac{w_0^2(t)}{r^2}\right) + \frac{1}{4}\left(\frac{r^2}{w_0^2(t)}\right) + \frac{1}{4}(p_r w_0(t))^2 - \frac{p_\theta\sigma}{2},\\
\epsilon^{-1}\mu_2 &= -\frac{p_r r}{4}\frac{\mathrm{d}w_0^2}{\mathrm{d}t}(t),\\
\epsilon^{-1}\mu_3 &= \frac{p_\theta^2}{2}\left(\frac{w_0(t)w_2(t)}{r^2}\right) - \frac{r^2 w_2(t)}{2w_0(t)^3} + \frac{r^2}{4}\left(\frac{\mathrm{d}w_0}{\mathrm{d}t}(t)\right)^2 + \frac{p_r^2}{2}w_0(t)w_2(t),\\
\epsilon^{-1}\mu_4 &= -\frac{p_r r}{2}\frac{\mathrm{d}}{\mathrm{d}t}(w_0 w_2)(t).
\end{aligned} \tag{5.3}$$

To numerically explore the error of the conservation of the nth-order invariant $\epsilon^{-1}\mu^{(n)} = \mu_1 + \cdots + \epsilon^{n-1}\mu_n$, we set $p_\theta = \sigma = 1$. We define the average L^2 error in the conservation of $\epsilon^{-1}\mu^{(n)}$ over two periods $\Delta\mu_\epsilon^{(n)} \equiv \epsilon^{-1}/4\pi\,||\mu_\epsilon^{(n)}(r(t), p_r(t)) - \mu_\epsilon^{(n)}(r(0), p_r(0))||_{L^2(0,4\pi)}$. Using the initial conditions $(r(0), p_r(0)) = (2, 0)$, corresponding to initially unmoving particles, and the initial conditions $(r, p_r) = (2, 5)$, corresponding to initially gyrating particles, we simulate the particle orbits $(r(t), p_r(t))$ using (2.1) and the optimal fourth-order symplectic integrator of McLachlan & Atela (1992). We then compute $\Delta\mu_\epsilon^{(n)}$ and plot the result against ϵ^{-1}. The results are shown in figures 1 and 2. The numerical step size is chosen proportional to ϵ^{-1} so that the integration error is asymptotically smaller than the error in the conservation of $\Delta\mu^{(n)}$ for $n \in \{1, 2, 3\}$. The constant of proportionality was ensured to be sufficiently small by recomputing the figures with significantly smaller step-sizes and observing no appreciable changes to the results.

It is interesting that $\Delta\mu_\epsilon^{(n)}$ oscillates as it tends to zero with ϵ. For any fixed, narrow range of ϵ values, the relative amplitude of these oscillations increases sharply as the accuracy of $\epsilon^{-1}\mu_\epsilon^{(n)}$ is computed over longer time intervals. This is not unexpected. When $|\Delta(\epsilon^{-1})| > 2$, the resonance between the electric field and the cyclotron motion causes particle orbits to grow unbounded exponentially fast. This implies $\lim_{t\to\infty} |\mu_\epsilon(t) - \mu_\epsilon(0)| = \infty$. For these resonant ϵ values, one therefore expects that $\epsilon^{-1}\mu_\epsilon^{(n)}$ is poorly conserved relative to the smallness of ϵ. The locations of these ill-behaved ϵ values can be estimated using the zeroth-order WKB

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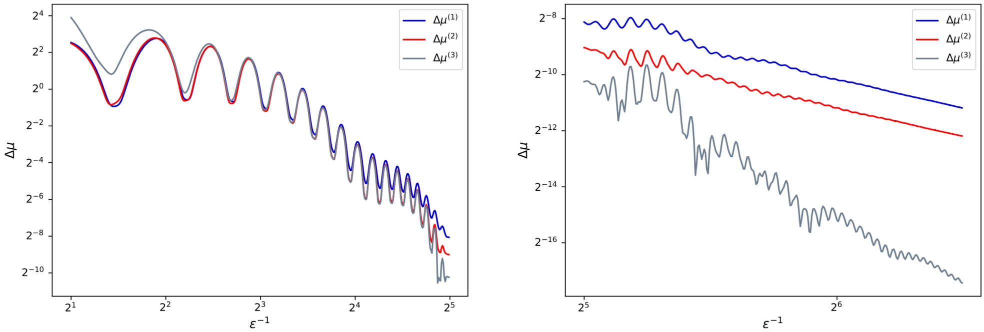

FIGURE 1. Value of $\Delta\mu^{(n)}$ vs. ϵ^{-1} for the initial conditions $(r(0), p_r(0)) = (2, 0)$. It is expected that $\Delta\mu_\epsilon^{(1)}$ and $\Delta\mu_\epsilon^{(2)}$ are both $O(\epsilon^2)$ since the particle initially has no velocity.

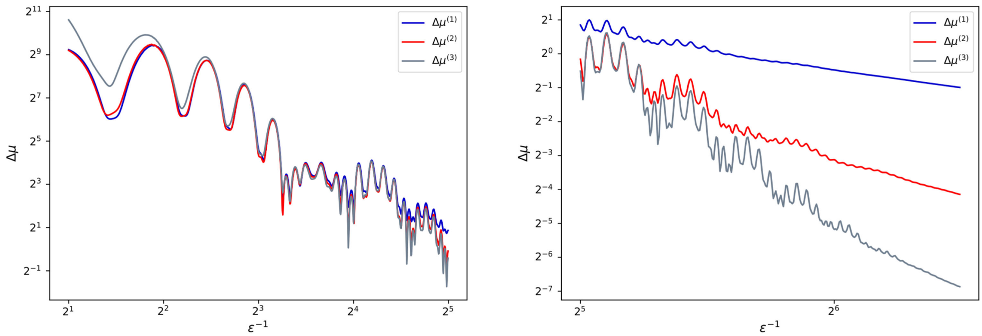

FIGURE 2. Value of $\Delta\mu_\epsilon^{(n)}$ vs. ϵ^{-1} for the initial conditions $(r(0), p_r(0)) = (2, 5)$. The typical asymptotic behaviour as $\epsilon \to 0$ is seen for $\Delta\mu_\epsilon^{(n)}$.

approximation. As a zeroth-order approximation, one has that

$$\Delta\left(\epsilon^{-1}\right) = 2A\left(\epsilon^{-1}\right)\cos\left(\epsilon^{-1}\int_0^T \varOmega(t')\,\mathrm{d}t'\right) \equiv A\left(\epsilon^{-1}\right)\Delta_{as}\left(\epsilon^{-1}\right), \tag{5.4}$$

for some function $A(\epsilon^{-1})$ limiting to unity as $\epsilon^{-1} \to \infty$. Arbitrarily long-term particle confinement may therefore fail when $\epsilon^{-1}\int_0^T \varOmega(t')\,\mathrm{d}t' \approx \pi N$ for some $N \in \mathbb{Z}$. For the present example, we compute

$$\int_0^{2\pi} \varOmega(t')\,\mathrm{d}t' = \int_0^{2\pi} (1 + \exp(\sin(t'))^{-2}\mathrm{d}t' \approx 1.7452. \tag{5.5}$$

We numerically compute $\varDelta(\epsilon^{-1})$ by finding the fundamental solutions to Hill's equation and compare the result with $\varDelta_{as}(\epsilon^{-1})$. The results are shown in figure 3. The oscillations in this plot can be directly compared with the oscillations in figure 1.

When ϵ^{-1} is not much bigger than 20, figures 1 and 2 show that $\mu^{(3)}$ does not improve $\mu^{(2)}$ by much. Up to $O(\epsilon^2)$ terms, the approximate conservation of $\mu^{(2)}$

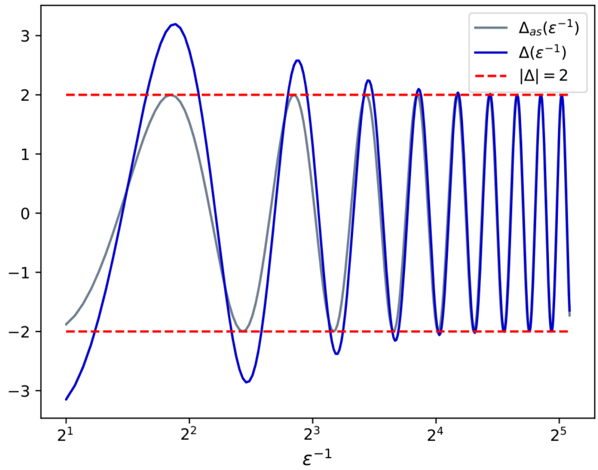

FIGURE 3. Value of $\Delta(\epsilon^{-1})$ obtained numerically and $\Delta_{\mathrm{as}}(\epsilon^{-1})$ against ϵ^{-1} for relatively large ϵ. The particle dynamics is stable when $|\Delta(\epsilon^{-1})| < 2$.

amounts to the conservation of

$$\epsilon^{-1}\tilde{\mu}^{(2)} = \frac{p_\theta^2}{4}\left(\frac{\Omega^{-1/2}(t)}{r}\right)^2 + \frac{1}{4}\left(\frac{r}{\Omega^{-1/2}(t)}\right)^2 + \frac{1}{4}\left(p_r\Omega^{-1/2}(t) - \epsilon r\frac{\mathrm{d}}{\mathrm{d}t}\Omega^{-1/2}(t)\right)^2. \tag{5.6}$$

For the present example, this should imply the 2π-flowmap $\exp(2\pi X_\epsilon)$ has the approximate invariant

$$\epsilon^{-1}\tilde{\mu}^{(2)}_{\mathrm{dis}}(\epsilon) = \frac{1}{r^2} + \frac{r^2}{16} + \frac{1}{4}\left(2p_r - \epsilon r\right)^2. \tag{5.7}$$

In general, non-perturbative theory implies that when $|\Delta(\epsilon^{-1})| < 2$, there exists an exact invariant of the 2π-flowmap of the form

$$\epsilon^{-1}\mathcal{M}_{\mathrm{dis}}(\epsilon) = \frac{1}{4}\left(\frac{a^2}{r^2}\right) + \frac{1}{4}\left(\frac{r^2}{a^2}\right) + \frac{1}{4}(p_r a - \epsilon b r)^2. \tag{5.8}$$

Here, $a = w_{\mathrm{per}}(0)$ and $b = w'_{\mathrm{per}}(0)$ are the initial conditions of the periodic solution to the Ermakov–Pinney equation. In the present example, perturbative theory gives the approximation $a \approx 2$ and $b \approx 1$. To obtain a and b exactly, one computes an eigenvector of the monodromy matrix $B(\epsilon^{-1})$ to (4.1). For example, if $\epsilon^{-1} = 20$ then

$$B(20) = \begin{bmatrix} -0.8221 & -0.9786 \\ 0.0464 & -1.1613 \end{bmatrix}. \tag{5.9}$$

Here, $B(20)$ has the eigenvector $\boldsymbol{v} = (0.9771, 0.1694 + 0.1286i)^t$ and strictly complex eigenvalues. Using (4.3) to find the correct scaling for $\boldsymbol{v}$, we have exactly that $a = 2.756$ and $b = 9.554$. This differs considerably from the perturbative prediction. Evidently, μ_ϵ fails to be well conserved. This becomes more evident when one looks at $\mu_\epsilon^{(n)}(t)$ in figure 4. The large growth of $\mu_\epsilon^{(n)}$ reflects that the eigenvalues of $B(20)$ lie near the real axis, implying $\epsilon^{-1} = 20$ is nearly resonant. Since $\epsilon^{-1} = 20$ is not

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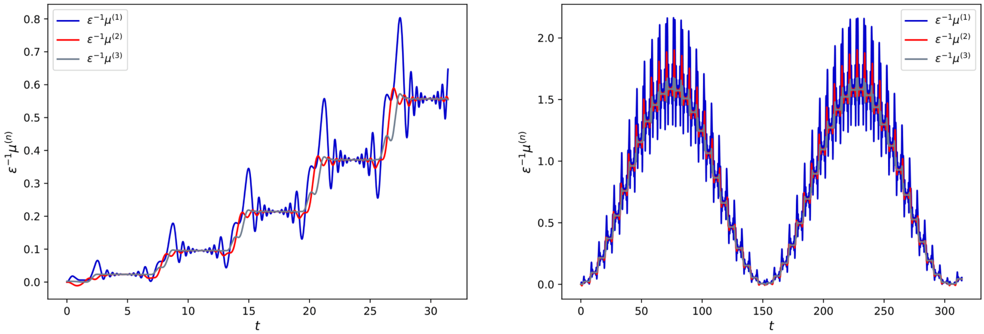

FIGURE 4. Value of $\epsilon^{-1}\mu_\epsilon^{(n)}(t)$ vs. t for the initial condition $(r(0), p_r(0)) = (2, 0)$ and $\epsilon^{-1} = 20$.

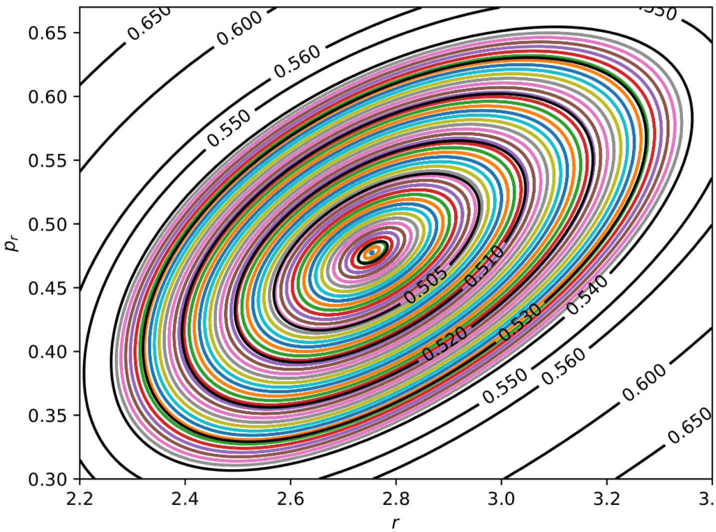

FIGURE 5. Poincaré plot of 2π-flowmap near the fixed point. Level sets of $\mathcal{M}_{\text{dis}}(\epsilon)$ are overlayed in black. Particle orbits are obtained by a full-orbit simulation over 3000 periods.

resonant, the Poincaré return theorem ensures that $\mu_\epsilon^{(n)}(t)$ returns close to its initial value.

In contrast to the perturbative invariant (5.7), the non-perturbative invariant (5.8) fully characterises the orbit structure of the 2π-flowmap. We demonstrate this by running a full-orbit simulation of $(r(t), p_r(t))$ and plotting the resulting Poincaré plot. The result is shown in figure 5 with the level sets of (5.8) overlayed. We note that a full-orbit simulation is unnecessary beyond an independent validation of our results since $(r(2\pi N), p(2\pi N))_{N\in\mathbb{Z}}$ can be readily computed from B(20).

6. Discussion

In this work, we have carefully argued that two, seemingly contradictory, conclusions can be drawn about particle motion in a homogeneous, time-dependent

https://doi.org/10.1017/S0022377826101469 Published online by Cambridge University Press

magnetic field. The first is that there exists an exact invariant of motion $\mathcal{M}_\epsilon$ asymptotic to Kruskal's adiabatic invariant μ_ϵ to all orders in ϵ. The second is that, despite this fact, the exact charged particle dynamics may differ qualitatively from the approximate charged particle dynamics. These two facts coexist because, in general, adiabatic invariants are only approximately conserved over some finite (but long) time interval. This is evident from the system considered in this work, where ϵ values for which particle trajectories grow unbounded may accumulate around $\epsilon = 0$. We therefore emphasise that appropriate care must be taken when discussing perturbative and non-perturbative invariants, especially when ϵ is not so small, or when particle orbits grow unbounded. Such subtleties have not presented in previous work into the non-perturbative guiding-centre model, since only magnetic systems for which particle trajectories lie on invariant tori have been considered. We therefore hope this work serves as a roadmap to the analysis of more general non-perturbative models.

Many questions remain about $\mathcal{M}_\epsilon$, the solutions the singularly perturbed Ermakov–Pinney equation, and non-perturbative adiabatic invariants in general. One question of interest is the global error in $\mu_\epsilon^{(n)}$ when particle motion is uniformly bounded, and how this error grows as ϵ approaches an unstable value. Related to this question is exactly how long the usual magnetic moment can be accurate when particle trajectories are unbounded. Another question is how well the solutions to the Ermakov–Pinney equation constructed using the Bremmer series approximate the periodic solution for stable ϵ values and periodic $B(t)$. We have shown that $\mathcal{M}_\epsilon$ is asymptotic to the time-periodic adiabatic invariant series μ_ϵ when the Bremmer-series solution is used to obtain $w(t;\epsilon)$, but not when the periodic solution is used. Another interesting study would be to compare the particle dynamics between stable and unstable ϵ values, and the way particle trajectories transition from bounded to unbounded. Of particular interest are the particle dynamics in a magnetic field of the form

$$B(t) = \bar{B}(t) + \tilde{B}(t), \tag{6.1}$$

with $\tilde{B}$ a small, rapidly fluctuating magnetic field and $\bar{B}(t)$ a slowly varying background magnetic field.

We comment that it is likely impossible to define an exact non-perturbative invariant for most magnetic geometries. The magnetic fields considered in this work and in Hollas *et al.* (2025) possess sufficiently many symmetries to ensure the integrability of the particle dynamics. It is currently unclear which magnetic geometries possess exact adiabatic invariants, but it is unlikely that exact non-perturbative invariants exist in most non-symmetric fields. We nevertheless conjecture that a satisfactory theory of approximate and/or non-smooth invariants may be possible. Irrespective of whether an analytical theory of approximate non-perturbative invariants can be developed, data-driven non-perturbative invariants already offer a powerful extension of guiding-centre theory. Formulas for exact non-perturbative adiabatic invariants provide a unique opportunity to benchmark the learning error in such models. In a future work, we will use the methods of Burby *et al.* (2025) to reconstruct the particle dynamics considered in this work. We will then compare the learned $\mathcal{M}_\epsilon$ against the exact $\mathcal{M}_\epsilon$.

Acknowledgments

The first author was supported under DOE Contract No. DE-AC02-09CH11466. This material is based on work supported by the U.S. Department of Energy, Office of Science, Office of Advanced Scientific Computing Research, as a part of the Mathematical Multifaceted Integrated Capability Centers program, under Award Number DE-SC0023164. It was also supported by U.S. Department of Energy grant no. DE-FG02-04ER54742.

Editor Peter Catto thanks the referees for their advice in evaluating this article.

Declaration of interests

The authors report no conflicts of interest.

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