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ON PSEUDO-DISTRIBUTIVE NEAR-RINGS

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1. Introduction

If G is a group and N a ring, the elements of the group ring NG can be thought of either as formal sums $\sum n_g g$ or as functions $\phi:G\to N$ with finite support. If N is a near-ring, problems arise in trying to construct a group near-ring either way. In the first case, Meldrum [7] was able to exploit properties of distributively generated near-rings (N,S) to build free (N,S)-products and hence a near-ring analogue of a group ring. For the latter case, Heatherly and Ligh [3] observed that the set of functions could be made into a near-ring under multiplication given by $(\phi^*\alpha)(g)=\sum_{x\in G}\phi(x)\alpha(x^{-1}g)$, provided N satisfies

$$\text{(P1)} \quad a_1b_1+a_2b_2=a_2b_2+a_1b_1$$

and

$$\text{(P2)} \quad n\sum_1^k a_ib_i \quad =\sum_1^k na_ib_i$$

for all $a_i,b_i,n\in N$ and $k\in Z^+$. Such near-rings are called pseudo-distributive. In fact these are precisely the conditions under which the set N_k of $k\times k$ matrices over N is also a near-ring and then both NG and N_k are pseudo-distributive.

Examples are found in [3], where some restrictions are given for a near-ring to be pseudo-distributive and yet not a ring (a so-called non-ring). In particular, N should not contain an identity. On the other hand, any non-unital ring is a pseudo-distributive near-ring so the results of this paper will apply to non-unital rings (group rings and matrix rings). Moreover, we shall see that our near-rings have close connections to appropriate rings.

In the next section we develop some general facts about pseudo-distributive near-rings. In the third section we examine the structure of matrix near-rings and group near-rings. The basic reference for near-rings is [9], and in this paper all near-rings will be zero-symmetric right near-rings. For future reference we repeat the definition of an ideal: A subgroup $(I,+)$ of $(N,+)$ is an ideal of N if it satisfies

(I1) I is a normal subgroup
(I2) $n(x+a)-nx\in I$ for all $n,x\in N$, $a\in I$
(I3) $IN\subseteq I$.

We note that because of the zero symmetry $NI\subseteq I$ also, by (I2).

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2. Pseudo-distributive near-rings

As might be expected, a near-ring which is "almost abelian" (P1) and "almost distributive" (P2) has lots of rings associated with it. The proof of the following is straight-forward:

Proposition 2.1. *If N is pseudo-distributive, the following are rings:*

(a) $Na=\{na \mid n\in N\}$ *for all* $a\in N$.

(b) $N^2=\{\sum_1^k a_ib_i \mid a_i,b_i\in N\}$.

(c) $B(N)$, *the set of central idempotents (with addition given by* $e\oplus f=e+f-2ef$).

(d) N/I *for any modular ideal* I.

(e) N/A *where* $A=\mathrm{Ann}^l N=\{n \mid nN=0\}$.

Note that A is a non-zero ideal ([3]) and Na and N^2 are also N-subgroups of N. Also N^2 satisfies (I2) and (I3) so would be an ideal iff it were a normal subgroup of N. In fact, this is true of any subgroup J with $N^2\subseteq J\subseteq N$, i.e. any such normal subgroup J is an ideal.

Following [5] let $[N,N]$ be the N-commutator ideal, namely the ideal generated by $S=\{n(a+b)-na-nb\}$. Now in [5, Proposition 2.2] it was shown that the *subgroup* generated by S satisfies (I2), and clearly (I3) holds (in fact $SN=0$) so again this subgroup lacks only normality to be an ideal. Maxson [6] has defined $D(N)$, the distributor ideal, to be the ideal generated by $T=\{n(a+b)-nb-na\}$. By P2, $S=T$ in our case so $D(N)=[N,N]$. Letting N' be the commutator subgroup of $(N,+)$, we have from [2] that when N is distributive, N' is an ideal and $N'\cdot N=N\cdot N'=0$. For N pseudo-distributive, $N'\cdot N=0$ still holds.

Proposition 2.2. *If N is pseudo-distributive,* $C(N)=N'+D(N)$ *is an ideal and* N/I *is a ring iff* $I\supseteq C(N)$.

Proof. Since $N/D(N)$ is distributive, its commutator subgroup is an ideal which has the form $C(N)/D(N)$ for some ideal $C(N)$ of N. In fact $(N/D(N))'=N'+D(N)/D(N)$. Then $N/C(N)$ is isomorphic to the distributive near-ring $N/D(N)$ modulo its commutator and so is a ring. If I is an ideal of N for which N/I is a ring, then the abelian addition and distributivity of N/I shows that $N'\subseteq I$ and $D(N)\subseteq I$ respectively, and conversely.

We shall call any ideal $I\supseteq C$ a *ring ideal* of N. For example, if N is abelian then $C=D(N)$, and if N is distributive then $C=N'$. In fact in [2], Heatherly shows how nilpotent groups of class 2 can be made into distributive near-rings by defining $a\cdot b=a+b-a-b$. Then $C=N'=N^2$. In the same paper it is pointed out that S_3 has the structure of a distributive near-ring (see #29, p. 342 of [9]). In this case $C=A_3$ and, in fact, N/A_3 is a unital ring. See also Theorem 2.6.

Setting $K(N)=$set of distributive elements of N, Maxson showed ([6]) that $K(N)$ is a subnear-ring of N precisely when P1 holds for all $a_i\in K(N), b_i\in N$, and in [3, Theorem 2] $K(N)$ was shown to be a normal subgroup. In fact we have

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Proposition 2.3. *$K(N)$ is a subnear-ring and a left ideal when N is pseudo-distributive.*

Proof. If $x \in K(N)$, then for all $n_i, a_i \in N$

$$\begin{aligned}[n_1(n_2+x)-n_1n_2](a_1+a_2) &= n_1(n_2+x)(a_1+a_2)-n_1n_2(a_1+a_2) \\ &= n_1[n_2(a_1+a_2)+xa_1+xa_2]-n_1n_2(a_1+a_2) \\ &= n_1n_2(a_1+a_2)+n_1xa_1+n_1xa_2-n_1n_2(a_1+a_2) \quad \text{by P2} \\ &= n_1xa_1+n_1xa_2 \quad \text{by P1}\end{aligned}$$

On the other hand $[n_1(n_2+x)-n_1n_2]a_1+[n_1(n_2+x)-n_1n_2]a_2$ also equals $n_1xa_1+n_1xa_2$ by a straight forward calculation. Thus (I2) holds.

Note that $A \subseteq K(N)$.

In [3] it was shown that if N is a simple pseudo-distributive near-ring then either $A=N$ and $N^2=0$, or $A=N$ is the finite field of order p, or $A=(0)$ and N is a ring. In fact in the first case we also have $N=N'$ (i.e. $(N,+)$ is a perfect group) since $A=N$ implies $K(N)=N$ so $D(N)=0$ and $C(N)=N'=N$.

Recall [**9**, pp. 136–7] that in near-rings there are four "radicals" $J_i(N), i=0,1/2,1,2$ which generalize the Jacobson radical $J(R)$ of a ring. J_i is an ideal for $i=0,\ 1,2$.

Theorem 2.4. (a) *$J_i(N/I)=J_i(N)/I$ for any ideal $I \subset J_i, i=0,1,2$.*
(b) *In a pseudo-distributive near-ring N all the radicals coincide.*

Proof. (a) Let J represent any one of J_0, J_1 or J_2 and suppose I is an ideal, $I \subseteq J$. Then the canonical surjection $f: N/I \to N/J$ yields, by [**9**, 5.13(b) and 5.16], $f(J(N/I)) \subseteq J(f(N/I)) = J(N/J)=0$. Therefore $J(N/I) \subset \ker f = J/I$. On the other hand, by [**9**, 5.15c)], $J(N/I) \supseteq J/I$ so equality holds as required.

(b) Since $J_0 \subseteq J_1 \subseteq J_2$, applying (a) we have $J_i(N/J_0)=J_i(N)/J_0$ for $i=0,1,2$. Since $C(N) \subseteq A \subseteq (L:N)$ for every left ideal L, therefore $C \subseteq J_0$ and hence N/J_0 is a ring. Thus all its radicals coincide and since $J_0(N/J_0)=0$ therefore $J_i(N)=J_0$ for $i=1,2$. Since $J_{\frac{1}{2}}$, (which, in general, is only a left ideal), lies between J_0 and J_1 therefore $J_{\frac{1}{2}}=J_0$ also.

We can then characterize $J(N)$ by a kind of "quasi-regularity". Recall that in a ring R, x is right quasi-regular if $x+b+xb=0$ for some b, and $J(R)=\{x \mid xr \text{ is right quasi-regular for all } r\}$.

Corollary 2.5. *If N is pseudo-distributive, $J(N)=\{x \mid$ for all r there exists an s such that for all n, $xrn+sn+xrsn=0\}$.*

Proof. Applying Theorem 2.4(a) to $I=A \subset J$ and using Proposition 2.1(e) we see $J(N)/A$ is the set $\{x+A \mid (x+A)(r+A)$ is right quasi-regular for all $r \in N\}$. Therefore $J(N)=\{x \mid$ for all r there exists s such that $xr+s+xrs \in A\}$.

Now if N is pseudo-distributive, N/J is a ring with zero Jacobson radical. Suppose N has DCCL (descending chain condition on left ideals). Then ([**9**, 5.48]) since all radicals coincide, J is nilpotent. Since N/J also has DCCL, it is a unital ring. Thus we have the following analogue to the Artin–Wedderbun Theorem for rings:

Theorem 2.6. *If N is a pseudo-distributive near-ring with DCCL, then N/J is a unital ring which is a finite direct sum of matrix rings over skew fields.*

3. Matrix and group near-rings

Let N be pseudo-distributive and for any $S \subset N$ let S_k denote the set of matrices in N_k all of whose entries belong to S. Let $\mathscr{J}(N)$ be the set of ideals of N.

Proposition 3.1. *There is a mapping $T: \mathscr{J}(N) \to \mathscr{J}(N_k)$ and a mapping $S: \mathscr{J}(N_k) \to \mathscr{J}(N)$ such that $ST = \mathrm{id}$ (so T is 1–1 and S is onto).*

Proof. Clearly if J is an ideal in N, J_k is an ideal in N_k so $T(J) = J_k$. Conversely, if $\mathscr{J}$ is an ideal in N_k, set $S(\mathscr{J}) = \{n \mid n = a_{11}$ for some $(a_{ij}) = A \in \mathscr{J}\}$. Clearly $S(\mathscr{J})$ is a normal subgroup of N. Moreover if $n \in S(\mathscr{J})$ let $X = (\delta_{ij}x)$ and $Y = (y_{ij})$ where $y_{11} = y, y_{ij} = 0$ for $(i, j) \neq (1, 1)$. Then $x(y+n) - xy$ is the 1–1 entry of $X(Y+A) - XY \in \mathscr{J}$. Since $S(\mathscr{J})$ is clearly right N-closed, it is an ideal.

Next we have $J \subseteq ST(J)$ trivially and if n is the 1–1 entry of some matrix in J_k, clearly $n \in J$.

We note that in the case of unital rings $TS = \mathrm{id}$ also, since by using matrix units one can show that if $A = (a_{ij}) \in \mathscr{J}$ then a_{ij} is the 1–1 entry of some matrix in $\mathscr{J}$. In the present case, let X_{ij} be the matrix with entry x in position i–j and zeros elsewhere. If $A \in \mathscr{J}$, then $B = \sum_k X_{ki} A Y_{jk} \in \mathscr{J}$ and B is the matrix with $xa_{ij}y$ all along the diagonal. Thus for all i, j, and all x, y, we have $xa_{ij}y \in S(\mathscr{J})$. This also shows that every ideal in N_k intersects N non-trivially (where N embeds canonically in N_k).

Lemma 3.2. *If I is an ideal in N, $(N/I)_k$ is isomorphic to N_k/I_k.*

Proof. The map $N_k \to (N/I)_k$ sending (a_{ij}) to $(a_{ij} + I)$ is a surjection with kernel I_k.

As a result $D_k \supseteq D(N_k)$ and $C_k \supseteq C(N_k)$. We note in passing that $A_k = A(N_k)$ and $K(N_k) = K(N)_k$.

Let $J(N)$ be the Jacobson radical. For rings it is well known that $J(N_k) = J(N)_k$. Using this we can show

Theorem 3.3. *If N is pseudo-distributive, $J(N_k) = J(N)_k$.*

Proof. By the lemma, $N_k/J_k \simeq (N/J)_k$ and since N/J is a ring $J((N/J)_k) = (J(N/J))_k = 0$. Then $0 = J(N_k/J_k) \supseteq J(N_k) + J_k/J_k$ by [9, 5.15c)] so $J(N_k) \subseteq J_k$. Now consider the ring N/C. Using Theorem 2.4 and the lemma we have $J(N_k)/C_k \simeq J(N_k/C_k) \simeq J((N/C)_k) \simeq (J(N/C))_k \simeq (J(N)/C)_k \simeq J_k/C_k$. Combined with the fact $J(N_k) \subseteq J_k$, we have equality.

Turning now to group near-rings, we recall the comments made in the introduction that NG is taken to be the set of functions $\phi: G \to N$ with finite support, and multiplication defined by $(\phi^* \alpha)g = \sum_x \phi(x)\alpha(x^{-1}g)$. NG is then an N-group, and N is a subnear-ring of NG via the functions $\hat{n}$ where $\hat{n}(e) = \hat{n}$ and $\hat{n}(g) = 0$ for $g \neq e$ (we note in passing that this embedding depends on the zero-symmetry of N). N is also a normal subgroup of $(NG, +)$.

The standard theory of unital group rings RG (see eg. [1] or [8]) makes extensive use of the fact that G can also be embedded in RG using the identity of R. In particular, this allows one to say that $\phi \cdot g$ and $g \cdot \phi \in RG$ for all $\phi \in RG$, $g \in G$. In the pseudo-distributive near-ring case we can define left and right G-actions on NG as follows: $(\phi \circ g)(h) = \phi(hg^{-1})$ and $(g \circ \phi)(h) = \phi(g^{-1}h)$. Then for all α, $\beta \in NG$ and g, $g_1 \in G$ we have

$$(\alpha^* \beta) \circ g = \alpha^*(\beta \circ g) \qquad g \circ (\alpha^* \beta) = (g \circ \alpha)^* \beta$$

$$(\alpha + \beta) \circ g = \alpha \circ g + \beta \circ g \qquad g \circ (\alpha + \beta) = g \circ \alpha + g \circ \beta$$

$$\alpha \circ g g_1 = (\alpha \circ g) \circ g_1 \qquad g g_1 \circ \alpha = g \circ (g_1 \circ \alpha)$$

$$(\alpha \circ g)^* \beta = \alpha^*(g \circ \beta) \qquad g \circ (\alpha \circ g_1) = (g \circ \alpha) \circ g_1$$

Also $\hat{n} \circ g = g \circ \hat{n}$ for all $n \in N$, $g \in G$.
It follows that each $\alpha \in NG$ can be written as $\alpha = \sum_g \hat{n}_g \circ g$ where $n_g = \alpha(g)$. In what follows we shall generally indicate both * and ∘ by simple juxtaposition.

A left ideal I of NG will be called left (right) G-closed if $gI \subseteq I (Ig \subseteq I)$ for all $g \in G$. For example Ann NG is left and right G-closed (see Theorem 3.4) and since $J(NG) = \cap (L:NG)$ where L is an i-modular left ideal [9, p. 136] it follows that $J(NG)$ is right G-closed. Note that I is left and right G-closed if $\alpha \in I$ implies $\beta \in I$ for all β with range $\alpha =$ range β. Writing $\text{Supp}\, \phi = \{g \mid \phi(g) \neq 0\}$ for the support of ϕ we have

1. $\text{Supp}(g\phi) = g\, \text{Supp}\, \phi$
2. If $\text{Supp}\, \alpha \cap \text{Supp}\, \beta = \emptyset$ then
   $\phi(\alpha + \beta) = \phi\alpha + \phi\beta$ for all $\phi \in NG$
3. If $x \in \text{Supp}(\alpha\beta)$, there exist $h \in \text{Supp}\, \alpha$ and $g \in \text{Supp}\, \beta$ such that $x = hg$.

Theorem 3.4. *(a) If I is a (left) ideal of N then $IG = \{\phi \in NG \mid \phi(x) \in I$ for all $x \in G\}$ is a G-closed (left) ideal of NG. (b) If I is two-sided $NG/IG \simeq (N/I)G$ as near-rings. (c) $K(NG) = K(N)G$, $\text{Ann}(NG) = (\text{Ann}\, N)G$, $C(NG) = C(N)G$ and $D(NG) = D(N)G$. (d) The map $T: \mathscr{I}(N) \to \mathscr{I}(NG)$ given by $T(I) = IG$ and the map $S: \mathscr{I}(NG) \to \mathscr{I}(N)$ given by $S(\mathscr{I}) = \mathscr{I} \cap N$ satisfy $ST(I) = I$ (cf. Proposition 3.1).*

Proof. (a) Clearly $(IG, +)$ is a normal subgroup of NG. Moreover

$$[\alpha(\beta + \phi) - \alpha\beta](g) = \sum_x \alpha(x)(\beta + \phi)(x^{-1}g) - \sum_x \alpha(x)\beta(x^{-1}g) = \sum_x \alpha(x)(\beta(x^{-1}g) + \phi(x^{-1}g))$$
$$- \alpha(x)\beta(x^{-1}g) \qquad \text{(by P1)}$$

which is in I for all α, $\beta \in NG$, $\phi \in IG$. Also $\phi \in IG$ implies $\alpha \in IG$ for all α with range $\alpha =$ range ϕ so IG is left and right G-closed.

(b) If I is a two sided ideal and $\pi: N \to N/I$ the canonical map, define $f_1: NG \to N/I\ G$ by $f_1(\phi) = \pi \circ \phi$. Then f_1 is a near-ring surjection with kernel IG.

(c) If $\phi \in K(NG)$ then $\phi(\alpha+\beta) = \phi\alpha + \phi\beta$ for all α, β. Therefore $\sum_x \phi(x)(\alpha+\beta)(x^{-1}g) = \sum_x \phi(x)\alpha(x^{-1}g) + \phi(x)\beta(x^{-1}g)$ for all $g \in G$. In particular α and β can be chosen to have singleton support $x^{-1}g$ and arbitrary values n, $n' \in N$ so that for all $x \in G$, all $n, n' \in N, \phi(x)(n+n') = \phi(x)n + \phi(x)n'$ whence $\phi \in K(N)G$. The reverse inclusion $K(N)G \subset K(NG)$ is clear, and the proofs for $D(NG)$, $C(NG)$ and $A(NG)$ are straightforward.

(d) Is easily shown.

We next investigate the connection between subgroups of G and G-closed left ideals in NG. If H is any subgroup of G, the set of cosets G/H can be used to define an N-group $N\,G/H$ which will be a near-ring if H is normal. We first consider the case when N is a ring so $N\,G/H$ is an N-module.

Theorem 3.5. *Let N be a ring. The mapping $f_2: NG \to N\,G/H$ given by $(f_2\phi)(gH) = \sum_{h \in H} \phi(gh)$ is an N-module homomorphism whose kernel ωH is a left ideal which is left G-closed. Moreover ωH is additively generated by $S = \{\phi \circ h - \phi \mid \phi \in NG, h \in H\}$.*

Proof. The first part is a normal ring theoretic proof. ωH is left G-closed because $\sum_h (x \circ \phi)(gh) = \sum_h \phi(x^{-1}gh) = 0$. Note that if H is normal, ωH is also right G-closed. Clearly $S \subset \ker f_2$ since

$$\sum_{h_i \in H} (\phi \circ h - \phi)(gh_i) = \sum_{h_i \in H} \phi(gh_ih^{-1}) - \phi(gh_i) = 0.$$

Conversely consider the case $H = G$. Then if $\phi \in \omega G, \sum_g \phi(g) = 0$ and without loss of generality $e \in \operatorname{Supp} \phi$ so define

$$\alpha_i(e) = \phi(g_i) \quad \text{for} \quad i = 1, \ldots, k$$

$$= 0 \text{ otherwise}$$

Then

$$(\alpha_i \circ g_i - \alpha_i)(e) = -\phi(g_i)$$

$$(\alpha_i \circ g_i - \alpha_i)(g_i) = \phi(g_i)$$

and

$$(\alpha_i \circ g_i - \alpha_i)(g_j) = 0 \quad \text{for} \quad g_j \neq g_i, e$$

Thus

$$\sum_i (\alpha_i \circ g_i - \alpha_i)(x) = \phi(g_i) \quad \text{if} \quad x = g_i$$

$$= -\sum_i \phi(g_i) = \phi(e) \quad \text{if} \quad x = e$$

so $\phi = \sum_i (\alpha_i \circ g_i - \alpha_i)$ as required.

The proof for a proper subgroup H follows similarly.

The proof also shows that ωG is additively generated by $T = \{g \circ \phi - \phi \mid g \in G, \phi \in NG\}$ and therefore ωG is a right G-closed right ideal, *i.e.* an ideal. The same holds for ωH where H is any normal subgroup of G. Continuing with the assumption that N is a ring, to each left G-closed left ideal J of NG we can associate a subset $\Omega J = \{g \mid \phi \circ g - \phi \in J\}$ of G. In fact ΩJ is a subgroup since $\phi \circ gg_1 - \phi = \phi \circ g \circ g_1 - \phi \circ g + \phi \circ g - \phi$ and $\phi \circ g^{-1} - \phi = -(\alpha \circ g - \alpha)$ where α is defined by $\alpha = \phi \circ g^{-1}$. Thus

Proposition 3.6. *When N is a ring, there is a mapping ω:{Subgroups of G}$\to${G-closed left ideals of NG} and a mapping Ω in the reverse direction such that $\Omega\omega H = H$. Thus ω is* 1–1 *and Ω is onto.*

Now we consider NG for any pseudo-distributive near-ring N. If H is normal in G with canonical map $\sigma: G \to G/H$, then there is a natural map $\bar{\sigma}(\phi): N\,G/H \to NG$ given by $\bar{\sigma}(\phi) = \phi \circ \sigma$. In fact $\bar{\sigma}$ is a near-ring monomorphism by which we can identify $N\,G/H$ with a subnear-ring of NG, namely $\{\phi \in NG \mid \phi(gh) = \phi(g) \text{ for all } h \in H\}$. Also there are natural maps $\theta_H: NG \to NH$ (given by restriction) and $\rho_H: NH \to NG$ (where $\rho_H(\phi) = \phi$ on H and $\rho_H(\phi) = 0$ on $G - H$) such that $\theta_{H\rho H} = \text{id}$. In fact, ρ_H is a normal map so NH is a direct summand of NG (as N-groups).

If I is any ring ideal in N, and H is normal we have from Theorems 3.4 and 3.5 the composite maps $f_2 f_1: NG \to N/I\,G \to N/I\,G/H$ whose kernels $\omega_I H$ are ideals in NG. Taking $H = G$ we get the

Corollary. *NG contains ideals $\omega_I G$ for which $NG/\omega_I G \simeq N/I$ as rings.*

Note that $\omega_I G = \{\phi \in NG \mid \sum_{g \in G} \phi(g) \in I\}$.

This is a key result, corresponding to the fact for unital group rings that the augmentation ideal Δ of RG satisfies $RG/\Delta \simeq R$. (See also [7], Theorem 4.9). This leads to several results transferring properties from R to RG with appropriate conditions on G. If N is pseudo-distributive and abelian (eg. examples (1) and (2) in [4]) then $\delta: NG \to N$ given by $\delta(\alpha) = \sum_g \alpha(g)$ is still well defined and gives $NG/\ker\delta \simeq N$. In the more general case the above result allows some transfer of properties between NG and N/I for ring ideals I as we shall see below. Moreover, Theorem 3.4 shows how certain factors of NG are group *rings* $(N/I)G$, and indeed they may even be unital as the example following Proposition 2.2 shows. We repeat that certain conditions imposed on the pseudo-distributive near-ring NG (eg. Von Neumann regularity, the absence of non-zero nilpotent ideals) would force NG to be a ring; on the other hand some information about non-unital group rings can be obtained in this way. For example, using a proof like that for unital rings one can show that if NG is regular, then N is regular and G is locally finite. We give a sample result for general pseudo-distributive group rings:

Proposition 3.7. *If NG has DCCL, then N/I is a left artinian ring for all ring ideals I, and G is finite. Moreover, $N/J(N)$ is a unital artinian semi-simple ring. If N is abelian, N is also left artinian.*

Proof. $NG/\omega_I G \simeq N/I$ is left artinian for all ring ideals I. $NG/JG \simeq N/J\,G$ is an artinian ring and, as noted earlier, N/J has no nilpotent ideals. Since $N/J \simeq (N/J)G/\Delta$ is also left artinian, therefore N/J has an identity. Then the fact that $N/J\,G$ is artinian implies G is finite [**1**, Theorem 1].

Now suppose G is finite. Let S be the set of constant maps $\bar{n}: G \to N$ where $\bar{n}(g) = n$ for all $g \in G$. In unital group rings, S is an ideal equal to $\mathrm{Ann}^l\,\omega G$ and $\mathrm{Ann}^r\,\omega G$ ([**1**]).

Proposition 3.8. (*a*) *If N has a left cancellable element than $S = \mathrm{Ann}^r\,\omega_I G$ for all ring ideals $I \subseteq A$.* (*b*) *If N is distributive $\mathrm{Ann}^l\,\omega_c G \supseteq S$.* (*c*) *If N is a ring, S is an ideal contained in both $\mathrm{Ann}^l\,\omega_c G$ and $\mathrm{Ann}^r\,\omega_c G$, which are also ideals.*

Proof. (a) Note that if N has a right cancellable element, it is a ring [**3**]. Certainly $S \subseteq \mathrm{Ann}^r\,\omega_I G$ since $\phi \cdot \bar{n} = \sum_x \phi(x) n = [\sum_x \phi(x)]n = 0$ when $\sum_x \phi(x) \in I \subseteq A$. Conversely, if $\phi\alpha = 0$ for all $\phi \in \omega_I G$ then $\sum \phi(x)\alpha(x^{-1}g) = 0$ for all g so in particular $\sum \phi(x)\alpha(x^{-1}) = 0$. If α is not constant there exist y_1, y_2 such that $\alpha(y_1) \neq \alpha(y_2)$. Define ϕ by $\phi(y_1^{-1}) = n$, $\phi(y_2^{-1}) = -n$, $\phi(g) = 0$ for all other g. Then $\phi \in \omega_I G$ since in fact $\sum_x \phi(x) = 0$ and $n\alpha(y_1) - n\alpha(y_2) = 0$. Thus if n is left cancellable, $\alpha(y_1) = \alpha(y_2)$ which is contradiction.

(b) Since $\omega_c G = \{\sum_g \phi g - \phi + CG\}$ and since N distributive implies $C = N'$ therefore if $(\bar{n} \in S$, $\bar{n} \cdot (\phi g - \phi + \alpha)(y) = \sum_{x \in G} n[\phi \cdot g(x) - \phi(x) + \alpha(x)] = \sum_x n\phi(xg^{-1}) - n\phi(x) + \sum_x n\alpha(x)$ for all $\alpha \in CG$. The first sum is zero since as x runs through Supp ϕ so does xg^{-1} and the second sum is zero since $N \cdot N' = 0$ in the distributive case.

(c) This is straight forward.

We conclude with the following observations: When (N, S) is a distributively generated near-ring the N-groups of interest are the (N, S)-groups, i.e. those N-groups on which S acts distributively (see eg. [**9**, p. 182] and [**7**]). By analogy, when N is pseudo-distributive define an N-group A to be a p.d. N-group if

$$\text{(PD1)} \quad n_1 a_1 + n_2 a_2 = n_2 a_2 + n_1 a_1$$

$$\text{(PD2)} \quad n_1(n_2 a_2 + n_3 a_3) = n_1 n_2 a_2 + n_1 n_3 a_3 \quad \text{for}$$

all $n_i \in N$, $a_i \in A$. That is, NA is an abelian N-subgroup of A on which N acts distributively. Examples are N itself, any left ideal of N, N/I for I any left ideal containing $C(N)$, N_k, and NG.

Now classical group algebras were used to study the representation of a group as a group of matrices. For (N, S)-groups, Meldrum [**7**] was able to establish a 1–1 correspondence between representations of G as a group of (N, S)-automorphisms and representations of the d.g. group near-ring. In the pseudo-distributive case there is clearly no hope of restricting representations of NG to G since G is not identified with a subgroup of NG. However, if $\mu: G \to N_k$ is a semi-group homomorphism to the multiplicative structure of N_k, then μ induces a "representation" $\hat{\mu}: NG \to N_k$ defined by $\hat{\mu}\phi = \sum_g \phi(g)\mu(g)$. Since N_k is a p.d. N-group, $\hat{\mu}$ is a near-ring homomorphism.

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