REMARKS ON WEAK COMPACTNESS IN $L_1(\mu, X)$

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Let $(\Omega, \Sigma, \mu)$ be a finite measure space and $X$ a Banach space. Denote by $L_1(\mu, X)$ the Banach space of (equivalence classes of) $\mu$-strongly measurable $X$-valued Bochner integrable functions $f: \Omega \to X$ normed by

$$\|f\|_1 = \int_{\Omega} \|f(\omega)\| \, d\mu(\omega).$$

The problem of characterizing the relatively weakly compact subsets of $L_1(\mu, X)$ remains open. It is known that for a bounded subset of $L_1(\mu, X)$ to be relatively weakly compact it is necessary that the set be uniformly integrable; recall that $K \subseteq L_1(\mu, X)$ is uniformly integrable whenever given $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mu(E) \leq \delta$ then $\int_E \|f\| \, d\mu \leq \varepsilon$, for all $f \in K$. S. Chatterji has noted that in case $X$ is reflexive this condition is also sufficient [4]. At present unless one assumes that both $X$ and $X^*$ have the Radon–Nikodym Property (see [1]), a rather severe restriction which, for purposes of potential applicability, is tantamount to assuming reflexivity, no good sufficient conditions for weak compactness in $L_1(\mu, X)$ exist. This note puts forth such sufficient conditions; the basic tool is the recent factorization method of W. J. Davis, T. Figiel, W. B. Johnson and A. Pelczynski [3].

First, we present a method of recognizing many weakly compact sets in $L_1(\mu, X)$ provided one has a starting point. Later, we present several possible starting points.

Before presenting the first result, recall some basic facts about $L_1(\mu, X)$ and its dual space.

An additive set function $F: \Sigma \to X^*$ (the continuous dual of $X$) is said to be $\mu$-Lipschitz whenever there exists $k > 0$ such that $\|F(E)\| \leq k\mu(E)$, for all $E \in \Sigma$. The space of $\mu$-Lipschitz $X^*$-valued maps is denoted by $V_\infty(\mu, X^*)$. If $F \in V_\infty(\mu, X^*)$ then the Lipschitz norm of $F$ is given by

$$\|F\| = \inf\{k > 0 : \|F(E)\| \leq k\mu(E), \text{ for all } E \in \Sigma\}.$$

If $F \in V_\infty(\mu, X^*)$ and $f = \sum_{i=1}^{n} x_i \chi_{A_i}$ is an $X$-valued simple function modeled on $\Sigma$ then

$$\int f \, dF = \sum_{i=1}^{n} x_i F(A_i)$$

is a well-defined scalar satisfying $|\int f \, dF| \leq \|F\| \|f\|_1$. Therefore, $\int dF$ extends in a unique manner to a linear functional $\int dF$ defined on all of $L_1(\mu, X)$. It is well known and easily established that each member of $L_1(\mu, X)^*$ is thus obtained; that is $V_\infty(\mu, X^*)$ is isometrically isomorphic to $L_1(\mu, X)^*$ with the correspondence between $F \in V_\infty(\mu, X^*)$ and $\varphi \in L_1(\mu, X)^*$ given by

$$\varphi(f) = \int f \, dF.$$

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For more details regarding \( L_1(\mu, X) \) we refer the reader to the book of N. Dinculeanu [5].

**Theorem 1.** Let \( K \) be a bounded uniformly integrable subset of \( L_1(\mu, X) \). Suppose \( K \) satisfies the following condition: (*) given \( \varepsilon > 0 \) there exists a measurable set \( \Omega_\varepsilon \) such that \( \Omega_\varepsilon \setminus \Omega_\varepsilon \) has measure no more than \( \varepsilon \) and such that \( \{ f\chi_{\Omega_\varepsilon} : f \in K \} \) is relatively weakly compact in \( L_1(\mu|_{\Omega_\varepsilon}, X) \). Then \( K \) is relatively weakly compact in \( L_1(\mu, X) \).

**Proof.** Let \( (f_n) \) be a sequence of members of \( K \). By (*) there exists a set \( \Omega_1 \subseteq \Omega \) such that \( \mu(\Omega \setminus \Omega_1) < 1/2 \) and such that \( \{ f\chi_{\Omega_1} : f \in K \} \) is relatively weakly compact in \( L_1(\mu|_{\Omega_1}, X) \). Choose a subsequence \( (f_n^{(1)}) \) of \( (f_n) \) and an \( f^1 \in L_1(\mu|_{\Omega_1}, X) \) such that

\[
 f_n^{(1)} \chi_{\Omega_1} \rightarrow f^1 \text{ weakly in } L_1(\mu|_{\Omega_1}, X).
\]

We may clearly assume that \( f^1 \) is defined on all of \( \Omega \) and vanishes off \( \Omega_1 \). Now use (*) to manufacture a measurable set \( \Omega_2 \) such that \( \mu(\Omega \setminus \Omega_2) < 1/2 \) and \( \{ f\chi_{\Omega_2} : f \in K \} \) is relatively weakly compact in \( L_1(\mu|_{\Omega_2}, X) \). We may assume that \( \Omega_1 \subseteq \Omega_2 \). There then exists a subsequence \( (f_n^{(2)}) \) of \( (f_n^{(1)}) \) and a function \( f^2 \in L_1(\mu|_{\Omega_2}, X) \) such that

\[
 f_n^{(2)} \chi_{\Omega_2} \rightarrow f^2 \text{ weakly in } L_1(\mu|_{\Omega_2}, X).
\]

Clearly, we may assume \( f^2 \) is defined on all of \( \Omega \), vanishes off \( \Omega_2 \) and \( f^2(x) = f^1(x) \) (\( x \in \Omega_1 \)).

The procedure is now clear. We obtain by repeated use of (*) a sequence \( (\Omega_n) \) of measurable subsets of \( \Omega \) with \( \Omega_n \subseteq \Omega_{n+1} \) for all \( n \), \( \mu(\Omega \setminus \Omega_n) < 1/n \) and a sequence of subsequences \( (f_n^{(k)}) \) where each \( (f_n^{(k+1)}) \) is a subsequence of \( (f_n^{(k)}) \) with \( f_n^0 = f_n \) and a sequence \( f^k \) of functions defined on \( \Omega \) such that for each \( k, f^{k+1}(x) = f^k(x) \) (\( x \in \Omega_k \)), \( f^k \) vanishes off \( \Omega_k \) and

\[
 f_n^{(k)} \chi_{\Omega_k} \rightarrow f^k \chi_{\Omega_k} \text{ weakly in } L_1(\mu|_{\Omega_k}, X).
\]

Define \( f : \Omega \rightarrow X \) in the obvious fashion; that is \( f \) is the almost everywhere limit of \( f^k \)'s. We claim that \( f \in L_1(\mu, X) \) and \( f_n^{(k)} \rightarrow f \) weakly in \( L_1(\mu, X) \).

By \( K \)'s uniform integrability, \( \sup \{ f\chi_\Sigma : f \in K, \Sigma \in \Sigma \} = H \) is also uniformly integrable. By Mazur's theorem, each \( f^k \) belongs to \( H \). Therefore \( (f^k) \) is a bounded uniformly integrable sequence of functions in \( L_1(\mu, X) \) which converges almost everywhere to \( f \). It follows from Vitali's convergence theorem (see [2]) that \( f \in L_1(\mu, X) \) and \( \| f - f^k \|_1 \rightarrow 0 \).

Now let us show that \( (f_n^{(k)}) \) converges weakly to \( f \). To this end, let \( F \in V_\mu(\mu, X^*) \). From the previous paragraph \( \{ f - f_n^{(n)} : n \in N \} \) is uniformly integrable and, given \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \( \mu(\varepsilon) \leq \delta \) implies \( \int_E \| f - f_n^{(n)} \| d\mu \leq \varepsilon/2(1 + \| F \|) \). Choose \( m \) so that \( \mu(\Omega|_{\Omega_m}) \leq \delta \). Then for \( n \geq m \),

\[
 \left| \int_{\Omega|_{\Omega_m}} [ f - f_n^{(n)} ] dF \right| \leq \| F \| \int_{\Omega|_{\Omega_m}} \| f - f_n^{(n)} \| d\mu < \varepsilon/2.
\]

Now choose \( n_0 \geq m \) so that if \( n \geq n_0 \) then \( \int_{\Omega_m} f^n - f_n^{(n)} dF < \varepsilon/2 \). Then for \( n \geq n_0 \) we have
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\[ |F(f) - F(f_n^{(n)})| \leq \left| \int_{\Omega_n} [f - f_n^{(n)}] \, dF \right| + \left| \int_{\Omega \setminus \Omega_n} [f - f_n^{(n)}] \, dF \right| \]

\[ = \left| \int_{\Omega_n} [f^n - f_n^{(n)}] \, dF \right| + \left| \int_{\Omega \setminus \Omega_n} [f - f_n^{(n)}] \, dF \right| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \]

**Theorem 2.** Let $K$ be a weakly compact convex subset of $X$ and consider the set $\bar{K} = \{ f \in L_1(\mu, X) : f(\omega) \in K \text{ for almost all } \omega \in \Omega \}$. $\bar{K}$ is weakly compact in $L_1(\mu, X)$.

**Proof.** It is obvious that $\bar{K}$ is convex and closed (hence by Mazur's theorem weakly closed) in $L_1(\mu, X)$.

By [3], there exists a reflexive Banach space $Y$ and a one-to-one continuous linear operator $T : Y \to X$ such that $K$ is the image under $T$ of some weakly compact convex set $J \subseteq Y$. Note that $T$ is weakly continuous; hence $T|_J$ is a weak homeomorphism. Next, note that $T$ "lifts" in a natural way to a continuous linear operator $\tilde{T}$ from $L_1(\mu, Y)$ to $L_1(\mu, X)$. Moreover, the lifting of $T$ to $\tilde{T}$ takes $\{ g \in L_1(\mu, Y) : g(\omega) \in J \text{ for a.a. } \omega \in \Omega \} = J$ onto $\bar{K}$. (This needs proof: it is clear that $\tilde{T}$ takes $J$ into $\bar{K}$. Let $f \in \bar{K}$. Then define $g : \Omega \to J$ by $g(\omega) = T^{-1}f(\omega)$ if $f(\omega) \in K$ and $g(\omega) = 0$ otherwise; $g(\omega) \in J$ for a.a. $\omega \in \Omega$. Also, $g$ is strongly measurable. In fact, $f$ is strongly measurable and, therefore, is weakly measurable and essentially (weakly) separably valued. $T^{-1}$ is a weak homeomorphism on $K$, and so $T^{-1}f$, which coincides with $g$ ($\mu$ almost everywhere), is weakly measurable and has weakly (hence norm) separable essential range.) By Chatterji's result, $\tilde{J}$ is relatively weakly compact in $L_1(\mu, Y)$, and so $\bar{K} = \tilde{T}J$ is weakly compact.

**Remark.** The above result also holds for $1 < p < \infty$; in this case, $L_p(\mu, Y)$ is reflexive.

It follows from the Krein–Šmulian theorem that if $K$ is a relatively weakly compact set in $X$ then the closed convex hull of $K$ is weakly compact and so by Theorem 2 or the previous remark we have

\[ \bar{K}_p = \{ f \in L_p(\mu, X) : f(\omega) \in K \text{ for a.a. } \omega \in \Omega \} \]

is relatively weakly compact.

The next result follows immediately from Theorems 1 and 2.

**Corollary 3.** Let $\bar{K}$ be a bounded uniformly integrable subset of $L_1(\mu, X)$. Suppose that given $\varepsilon > 0$ there exists a measurable set $\Omega_\varepsilon$ and a weakly compact set $K_\varepsilon \subseteq X$ such that $\mu(\Omega \setminus \Omega_\varepsilon) < \varepsilon$ and for each $f \in \bar{K}$, $f(\omega) \in K_\varepsilon$ for almost all $\omega \in \Omega_\varepsilon$. Then $\bar{K}$ is a relatively weakly compact subset of $L_1(\mu, X)$.

Proceeding similarly as in Theorem 2 one can prove the next theorem. However, the proof can be given without use of factorization and so we do it in that way.

**Theorem 4.** Let $K$ be a weakly compact subset of $X$ and $J$ a bounded uniformly integrable subset of $L_1(\mu)$. Then $\bar{K} = \{ f(\cdot)x : f \in J, x \in K \}$ is a relatively weakly compact subset of $L_1(\mu, X)$.
Proof. We note the following characterization of Banach spaces $X$ possessing the Dunford–Pettis property [7]: a Banach space $X$ has the Dunford–Pettis property if and only if given any Banach space $Y$ and any weakly compact sets $K, J$ in $Y$ and $X$ respectively $K \otimes J = \{ k \otimes j : k \in K, j \in J \}$ is weakly compact in $Y \hat{\otimes} X$—the projective tensor product of $Y$ with $X$. Pertinent remarks here are that all $L_1(\mu)$ spaces have the Dunford–Pettis property [7] and Grothendieck [8] has shown that $L_1(\mu, X)$ is identifiable with $L_1(\mu) \hat{\otimes} X$ in a natural manner.

Remark. A word or two on the aforementioned characterization of the Dunford–Pettis property is appropriate. It is well known that a Banach space $X$ has the Dunford–Pettis property if and only if given weakly convergent sequences $(x_n)$ in $X$ and $(x_n^*)$ in $X^*$ one of which has limit zero, then $\lim_n x_n^* x_n = 0$ (cf. [9, pp. 263–6]). With this in mind, suppose $X$ has the Dunford–Pettis property. To prove that the tensor of weakly compact sets, one factor from each of $X$ and $Y$, is weakly compact, it suffices to show that if $(x_n)$ is weakly null in $X$ and $y_n$ is weakly null in $Y$, then $x_n \otimes y_n$ is weakly null in $X \hat{\otimes} Y$. However, the dual of $X \hat{\otimes} Y$ is identifiable with the space of continuous linear operators from $Y$ to $X^*$, where the action of such an operator $T$ on $x \otimes y$ is given by $T(y)(x)$. If $(y_n)$ is weakly null, then $(Ty_n)(x_n)$ is null by the assumption of the Dunford–Pettis property for $X$. The converse is even easier since one need only test $Y = X^*$ and evaluate the trace functional.

Corollary 5. Let $\tilde{K}$ be a bounded uniformly integrable subset of $L_1(\mu, X)$. Suppose that given $\varepsilon > 0$ there exist a measurable set $\Omega_\varepsilon$ with $\mu(\Omega \setminus \Omega_\varepsilon) < \varepsilon$, a bounded uniformly integrable subset $J_\varepsilon$ of $L_1(\Omega_\varepsilon, \mu \mid_{\Omega_\varepsilon})$ and a weakly compact set $K_\varepsilon \subset X$ such that if $f \in \tilde{K}$ then $f$ admits a representation $f(\omega) = \sum_n \lambda_n f_n(\omega) x_n$ for almost all $\omega \in \Omega_\varepsilon$, for some sequence $(\lambda_n)$ of scalars with $|\Sigma_n \lambda_n| \leq 1$, $f_n \in J_\varepsilon$, $x_n \in K_\varepsilon$. Then $\tilde{K}$ is relatively weakly compact in $L_1(\mu, X)$.

Remark. One might hope that Corollary 3 contains the sought after necessary condition for weak compactness in $L_1(\mu, X)$. This hope is destined to doom. Professor J. J. Uhl has noted that if $X$ is not reflexive but $X^*$ has the Radon–Nikodym property then proceeding as in [1], the sequence $(r_n x_n)$ tends to zero weakly (where $(r_n)$ is the sequence of Rademacher functions and $x_n$ is any bounded sequence without a weakly convergent subsequence) in $L_1(\mu, X)$, where $\mu$ is Lebesgue measure on $(0, 1)$ yet $(r_n x_n)$ does not satisfy the criteria set forth in Corollary 3.

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References


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